The easiest way to the Heaviside ellipsoid

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The equation for the electromagnetic field of a point charge moving with constant velocity is derived from Maxwell's equations using the symmetry properties of the system. In contrast to conventional treatments, the derivation does not use retarded integrals or relativistic transformations.

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We are interested in a simple and concise derivation of the equation for the electromagnetic field produced by an electric charge moving with constant velocity. The textbook approaches are commonly based on the relativistic transformation of the fields, which suggests that classical electrodynamics is incomplete. However, electrodynamics is self-consistent and all its relations can be obtained from Maxwell's equations without recourse to any additional postulates.[1] In the following, a method is proposed which is more straightforward than that given in Ref. 1.

We begin with the equations for the electromagnetic potentials $\mathbf{A}$ and φ:

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\frac{4\pi}{c}\mathbf{j},\tag{1}$$

$$\nabla^2\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=-4\pi\rho,\tag{2}$$

which are obtained from Maxwell's equations in combination with the Lorentz gauge. For a source moving with velocity $\mathbf{v}$, the charge density is $\rho(\mathbf{x}-\mathbf{v}t)$, and the current density is

$$\mathbf{j}=\mathbf{v}\rho(\mathbf{x}-\mathbf{v}t).\tag{3}$$

For uniform translational motion, this form implies that the potentials are also functions of $\mathbf{x}-\mathbf{v}t$, that is, the charge and fields move as one with the same constant velocity.

Given the inhomogeneous d'Alembert equation

$$\nabla^2 f-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}=g(\mathbf{x}-\mathbf{v}t),\tag{4}$$

we explore a particular solution of the form $f(\mathbf{x}-vt)$ suggested by the mathematical structure of the source. Advantage will be taken of the system symmetries. We select the x_1-axis to be the direction of motion $\mathbf{v}=v\mathbf{i}_1$. The Galilean transformation

$$x_1''=x_1-vt\tag{5}$$

converts Eq. (4) to

$$\frac{\partial^2 f}{\partial x_1''^2}+\frac{\partial^2 f}{\partial x_2^2}+\frac{\partial^2 f}{\partial x_3^2}-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}+\frac{2v}{c^2}\frac{\partial^2 f}{\partial x_1''\partial t}-\frac{v^2}{c^2}\frac{\partial^2 f}{\partial x_1''^2}=g(x_1'',x_2,x_3).\tag{6}$$

If we recall that f has the form $f(\mathbf{x}-vt)$ and use Eq. (5), we obtain

$$\gamma^{-2}\frac{\partial^2 f}{\partial x_1''^2}+\frac{\partial^2 f}{\partial x_2^2}+\frac{\partial^2 f}{\partial x_3^2}=g(x_1'',x_2,x_3),\tag{7}$$

where

$$\gamma=1/\sqrt{1-v^2/c^2}.\tag{8}$$

The substitution

$$x_1'=\gamma x_1''\tag{9}$$

allows us to reduce Eq. (7) to the Poisson equation

$$\frac{\partial^2 f}{\partial x_1'^2}+\frac{\partial^2 f}{\partial x_2^2}+\frac{\partial^2 f}{\partial x_3^2}=g(\gamma^{-1}x_1',x_2,x_3).\tag{10}$$

Thus, Eq. (4) has been transformed into a static equation whose solution is well known.

Following this line of argument we consider the motion of a point electric charge. In this case Eqs. (1) and (2) become

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\frac{4\pi\mathbf{v}}{c}q\,\delta(\mathbf{x}-\mathbf{v}t),\tag{11}$$

$$\nabla^2\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=-4\pi q\,\delta(\mathbf{x}-\mathbf{v}t).\tag{12}$$

We use Eqs. (5) and (9) to obtain

$$\frac{\partial^2 A_1}{\partial x_1'^2}+\frac{\partial^2 A_2}{\partial x_2^2}+\frac{\partial^2 A_3}{\partial x_3^2}=-4\pi q\gamma\frac{v}{c}\,\delta(x_1',x_2,x_3),\tag{13}$$

$$\frac{\partial^2\varphi}{\partial x_1'^2}+\frac{\partial^2\varphi}{\partial x_2^2}+\frac{\partial^2\varphi}{\partial x_3^2}=-4\pi q\gamma\delta(x_1',x_2,x_3).\tag{14}$$

The following property of the δ-function was used on the right-hand side of Eqs. (13) and (14):

$$\delta(ax)=\frac{1}{a}\delta(x)\quad(a>0).\tag{15}$$

If we use the relation

$$\nabla^2\frac{1}{|\mathbf{x}|}=-4\pi\delta(\mathbf{x}),\tag{16}$$

the solutions of the static equations (13) and (14) are easily found to be

$$A_1=\gamma\frac{v}{c}\frac{q}{R},\quad A_2=0,\quad A_3=0,\tag{17}$$

$$\varphi=\gamma\frac{q}{R},\tag{18}$$

$$R=[\gamma^2(x_1-vt)^2+x_2^2+x_3^2]^{1/2}. \tag{19}$$

If we substitute Eqs. (17)–(19) with Eq. (8) into

$$\mathbf{E}=-\nabla\varphi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \tag{20}$$

we find

$$\mathbf{E}=\gamma q\frac{(x_1-vt)\mathbf{i}_1+x_2\mathbf{i}_2+x_3\mathbf{i}_3}{R^3}. \tag{21}$$

In spherical coordinates we have

$$x_1-vt=r\cos\theta, \quad x_2^2+x_3^2=r^2\sin^2\theta, \tag{22}$$

where θ is the angle between the radius vector $\mathbf{r}=(x_1-vt)\mathbf{i}_1+x_2\mathbf{i}_2+x_3\mathbf{i}_3$ and the x_1 axis. Thus, from Eq. (19) we have

$$\begin{aligned} R^2&=\gamma^2(x_1-vt)^2+x_2^2+x_3^2\\ &=\gamma^2r^2\cos^2\theta+r^2\sin^2\theta\\ &=\gamma^2r^2\left(1-\frac{v^2}{c^2}\sin^2\theta\right). \end{aligned} \tag{23}$$

If we use Eq. (23) in Eq. (21), we find

$$E=\frac{q(1-v^2/c^2)}{r^2(1-(v^2/c^2)\sin^2\theta)^{3/2}}. \tag{24}$$

Equation (24) is the famous Heaviside formula. It describes the phenomenon of ‘‘squashing’’ the electric field in the direction of motion:

$$\frac{E(0)}{E(\pi/2)}=\left(1-\frac{v^2}{c^2}\right)^{3/2}. \tag{25}$$

Finally, we derive the total electromagnetic force field generated by the moving charge q. From Eqs. (17) and (18) we have

$$\mathbf{A}=\frac{\mathbf{v}}{c}\varphi. \tag{26}$$

Hence

$$\mathbf{H}=\nabla\times\mathbf{A}=\frac{1}{c}\nabla\times(\mathbf{v}\varphi)=\frac{1}{c}(\nabla\varphi\times\mathbf{v}). \tag{27}$$

We take the vector cross product of Eq. (20) and $\mathbf{v}$ and use the result in Eq. (27) to obtain

$$\mathbf{H}=-\frac{1}{c}\left(\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right)\times\mathbf{v}=\frac{1}{c}\mathbf{v}\times\mathbf{E}. \tag{28}$$

Then using Eqs. (21) and (28), the total force on a charge q_o is given by

$$\begin{aligned} \mathbf{F}&=q_o\left[\mathbf{E}+\frac{1}{c}(\mathbf{v}\times\mathbf{H})\right]\\ &=q_o\left[\mathbf{E}-\frac{v^2}{c^2}\mathbf{E}+\frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{E})}{c^2}\right]\\ &=q_oq\frac{\gamma(x_1-vt)\mathbf{i}_1+x_2\mathbf{i}_2+x_3\mathbf{i}_3}{[\gamma^2(x_1-vt)^2+x_2^2+x_3^2]^{3/2}}\\ &=-q_oq\nabla'\frac{1}{R}=-q_o\nabla'\psi, \end{aligned} \tag{29}$$

which is just the formula for the Heaviside ellipsoid $\psi=\text{constant}$,[2] where $\psi=1/R$, R is given by Eq. (19), and the gradient ∇' is taken in the moving coordinates (5) and (9). It is thus demonstrated that the total electromagnetic field has undergone a Lorentz contraction along its direction of motion.

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[1]O. D. Jefimenko, ‘‘Direct calculation of the electric and magnetic fields of an electric point charge moving with constant velocity,’’ Am. J. Phys. **62**, 79–85 (1994).
[2]Ya. P. Terletzky and Yu. P. Rybakov, *Electrodynamics* (Vyssaja Scola, Moscow, 1990), Sec. 80.