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SEPARABLE SYSTEMS OF STÄCKEL

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In 1891 Stäckel[1] showed how to determine the quantities H_i in the Hamilton-Jacobi equation

$$\text{(A)} \qquad \sum_i \frac{1}{H_i^2}\left(\frac{\partial\varphi}{\partial x_i}\right)^2 + k^2(E - V)\varphi = 0,$$

so that the variables are separable, the solution being of the form ΣX_i, where X_i is a function of x_i alone. In 1893[2] he showed that when the quadratic differential form $\Sigma H_i^2 dx_i^2$ so determined is taken as the Riemannian metric of a space V_n the equations of the geodesics of V_n admit $n - 1$ independent quadratic first integrals other than the fundamental form. In §§1, 2 we show that when this condition is satisfied, the fundamental quadratic form is of the Stäckel type.

In 1927 Robertson[3] showed that for an equation of the form

$$\text{(B)} \qquad \sum_i H \frac{\partial}{\partial x_i}\left(\frac{H}{H_i^2}\frac{\partial\varphi}{\partial x_i}\right) + k^2(E - V)\varphi = 0, \qquad H = H_1 \cdots H_n$$

to admit by separation of the variables a solution of the form ΠX_i, where X_i is a function of x_i alone, the functions H_i^2 must be of the Stäckel form and $V = \Sigma \frac{f(x_i)}{H_i^2}$, where $f(x_i)$ is an arbitrary function of x_i alone, just as in the case of equation (A) as shown by Stäckel. He found that in this case there is the additional condition

$$\text{(C)} \qquad \varphi = \Pi \frac{H_i}{\psi_i(x_i)},$$

where φ is the determinant of the Stäckel functions φ_{ij} and ψ_i is a function of x_i at most. In §2 we show that this condition is equivalent to the equations

$$R_{ij} = 0 \qquad (i \neq j)$$

in the given coordinate system, R_{ij} being the components of the Ricci tensor used by Einstein.

[1] Habilitationsschrift, Halle.

[2] Comptes Rendus, vol. 116, pp. 485–487; cf. also, Ricci et Levi-Civita, Méthodes de calcul différentiel absolu, Math. Annalen, vol. 54 (1901), pp. 183, 184.

[3] Bemerkung über separierbare Systeme in der Wellenmechanik, Math. Annalen, vol. 98, pp. 749–752.

In §§3–7 we determine the various canonical forms for euclidean 3-space and find that in each case the coordinate surfaces are confocal quadrics including the cases when one or more families consists of planes, and that every type of confocal quadrics affords a solution; only the case of real surfaces has been considered. We have thus the only orthogonal systems of coordinates in which the three dimensional Schrödinger equation can be solved by separation of the variables.

In §8 we show that similar results hold for euclidean spaces of higher order and in §9 determine the Stäckel forms for a V_3 of constant Riemannian curvature because of their bearing on the problem for euclidean 4-space.

1. **Quadratic first integrals.** A necessary and sufficient condition that

$$a_{ij}\frac{dx_i}{ds}\frac{dx_j}{ds} = \text{const.}$$

be a quadratic first integral of the equation of geodesics of a Riemannian V_n is that

$$a_{ij,k} + a_{jk,i} + a_{ki,j} = 0, \tag{1.1}$$

where a comma followed by an index indicates covariant differentiation; there is no loss in assuming that a_{ij} is symmetric in the indices.[4]

If ρ_i are the roots of the determinant equation

$$| a_{ij} - \rho g_{ij} | = 0, \tag{1.2}$$

the equations

$$(a_{ij} - \rho_h g_{ij})\lambda^i_{h|} = 0, \tag{1.3}$$

determine an orthogonal ennuple of contravariant vectors of components $\lambda^i_{h|}$, where h indicates the vector and i the component.[5] Ordinarily the vector-fields so defined are not normal in the sense that a vector field admits a family of hypersurfaces orthogonal to the vectors.

We assume that a_{ij} is such that these vector-fields are normal and that the hypersurfaces are taken as parametric; and we write the fundamental form thus

$$ds^2 = e_1H_1^2\,(dx_1)^2 + \cdots + e_nH_n^2\,(dx_n)^2, \tag{1.4}$$

where the e's are plus or minus one as the case may be. In this case $\lambda^i_{h|} = 0$ for $i \neq h$ and $a_{ij} = 0$ for $i \neq j$. Then equations (1.1) for $j = k = i$ and $j \neq i$, $k = j$ respectively reduce to

$$\begin{aligned}&\frac{\partial \log \sqrt{a_{ii}}}{\partial x_i} = \frac{\partial \log H_i}{\partial x_i},\\ &\frac{\partial a_{ii}}{\partial x_j} - 4a_{ii}\frac{\partial \log H_i}{\partial x_j} + a_{jj}\frac{1}{H_j^2}\frac{\partial H_i^2}{\partial x_j} = 0,\end{aligned} \tag{1.5}$$

[4] Eisenhart, Riemannian Geometry, p. 129. Hereafter such a reference is of the form R. G., p. 129.

[5] R. G., p. 108.

and equations (1.1) for i, j, k different are satisfied identically. From the first set of (1.5) we have

$$a_{ii} = \rho_i H_i^2, \tag{1.6}$$

where ρ_i, thus defined, is independent of x_i. The second set of (1.5) reduce to

$$\frac{\partial}{\partial x_j} \log \frac{\rho_i - \rho_j}{H_i^2} = 0, \tag{1.7}$$

from which it follows that $(\rho_i - \rho_j)/H_i^2$ is independent of x_j. Writing these results in the form

$$\frac{\partial \rho_i}{\partial x_j} = (\rho_i - \rho_j) \frac{\partial \log H_i^2}{\partial x_j}, \qquad \frac{\partial \rho_i}{\partial x_i} = 0, \tag{1.8}$$

and expressing the condition of integrability of this system of equations, we obtain

$$(\rho_i - \rho_j)\left(\frac{\partial^2 \log H_i^2}{\partial x_i \partial x_j} + \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_i}\right) = 0$$

and

$$(\rho_j - \rho_k)\left(\frac{\partial^2 \log H_i^2}{\partial x_j \partial x_k} - \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_i^2}{\partial x_k} + \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_k} + \frac{\partial \log H_i^2}{\partial x_k}\frac{\partial \log H_k^2}{\partial x_j}\right) = 0.$$

In order that (1.8) may admit a solution with all the ρ's different we must have

$$\frac{\partial^2 \log H_i^2}{\partial x_i \partial x_j} + \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_i} = 0, \tag{1.9}$$

$$\frac{\partial^2 \log H_i^2}{\partial x_j \partial x_k} - \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_i^2}{\partial x_k} + \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_k} + \frac{\partial \log H_i^2}{\partial x_k}\frac{\partial \log H_k^2}{\partial x_j} = 0. \tag{1.10}$$

Since these equations are consistent, it follows that, when they are satisfied, equations (1.8) are completely integrable. One solution is $\rho_i = \rho_j = a$, a constant. We denote by ρ_i^α (for $\alpha = 2, \cdots, n$) $n - 1$ other solutions such that the determinant of the n solutions is not zero. This may be indicated in the determinant form

$$|\rho_i^\alpha - \rho_j^\alpha| \neq 0, \tag{1.11}$$

where i is fixed, and $\alpha = 2, \cdots, n; j = 1, \cdots, n; j \neq i$. In this case the equations of the geodesics admit $n - 1$ quadratic first integrals whose coefficients are

$$a_{ii}^\alpha = \rho_i^\alpha H_i^2, \qquad a_{ij}^\alpha = 0. \tag{1.12}$$

2. The Stäckel Form. It is our purpose to show that the conditions of the preceding section determine the Stäckel form of the fundamental form of the V_n. To this end we denote by φ_{ij} n^2 functions such that their determinant φ is not zero, and we denote by φ^{ij} the cofactor of φ_{ij} in φ. We put

$$H_i^2 = \frac{\varphi}{\varphi^{i1}}, \qquad \rho_i^\alpha = \frac{\varphi^{i\alpha}}{\varphi^{i1}}, \tag{2.1}$$

and understand that the φ's are such that ρ_i^α are independent of x_i.

Also we put

$$b_{ij}^\alpha = \frac{\rho_i^\alpha - \rho_j^\alpha}{H_i^2} = \frac{\varphi^{j1}\varphi^{i\alpha} - \varphi^{i1}\varphi^{j\alpha}}{\varphi\varphi^{j1}}, \tag{2.2}$$

and have from §1 that b_{ij}^α are independent of x_j. We have that

$$\varphi^{j1}\varphi^{i\alpha} - \varphi^{i1}\varphi^{j\alpha} = \varphi M_{j1i\alpha},$$

where $M_{j1i\alpha}$ is the algebraic complement of $\varphi_{j1}\varphi_{i\alpha} - \varphi_{i1}\varphi_{j\alpha}$ in the determinant φ.[6] Consequently we have

$$\varphi^{j1}b_{ij}^\alpha = M_{j1i\alpha} \qquad (i, j = 1, \cdots, n; \alpha = 2, \cdots, n). \tag{2.3}$$

From the definition of $M_{j1i\alpha}$ we have

$$\varphi^{j\alpha} = \sum_{i\,(\neq j)} \varphi_{i1} M_{j1i\alpha},$$

and consequently

$$\frac{\varphi^{j\alpha}}{\varphi^{j1}} = \sum_{i\,(\neq j)} \varphi_{i1} b_{ij}^\alpha.$$

Differentiating with respect to x_j, we have

$$0 = \sum_{i\,(\neq j)} \frac{\partial \varphi_{i1}}{\partial x_j} b_{ij}^\alpha \qquad (\alpha = 2, \cdots, n; j = 1, \cdots, n).$$

For a given j the determinant of b_{ij}^α is not zero, in consequence of (1.11) and (2.2). Hence a function φ_{i1} is a function of x_i at most.

From (2.2) we have

$$\varphi^{j1}b_{ij}^\alpha = -\varphi^{i1}b_{ji}^\alpha. \tag{2.4}$$

In consequence of this result and (2.3) we have

$$\frac{M_{j1i\beta}}{M_{j1i2}} = \frac{b_{ij}^\beta}{b_{ij}^2} = \frac{b_{ji}^\beta}{b_{ji}^2} \equiv \sigma_{ij\beta} = \sigma_{ji\beta} \qquad \begin{pmatrix} \beta = 3, \cdots, n \\ i \neq j \end{pmatrix}. \tag{2.5}$$

Since the second term is independent of x_j and the third of x_i, it follows that $\sigma_{ij\beta}$ is independent of x_i and x_j. From the identities

$$\sum_{\alpha}^{2, \cdots, n} \varphi_{k\alpha} M_{j1i\alpha} = 0 \qquad (i, j, k \neq),$$

[6] Cf. Kowalewski, Einführung in die Determinantentheorie, p. 80.

we have with the aid of (2.5)

$$\varphi_{k2} + \sum_{\beta}^{3,\cdots,n} \varphi_{k\beta}\sigma_{ij\beta} = 0. \tag{2.6}$$

Differentiating with respect to x_i, we have

$$\frac{\partial \varphi_{k2}}{\partial x_i} + \sum_{\beta}^{3,\cdots,n} \frac{\partial \varphi_{k\beta}}{\partial x_i}\sigma_{ij\beta} = 0. \tag{2.7}$$

For a given i and k, there are $n - 2$ equations (2.6) satisfied by the $n - 1$ quantities $\varphi_{k2}, \cdots, \varphi_{kn}$; and these same equations are satisfied by the derivatives of these quantities with respect to x_i, hence we have

$$\frac{\partial \varphi_{k\alpha}}{\partial x_i} = \mu_{ik}\varphi_{k\alpha},$$

or

$$\frac{\partial}{\partial x_i}\left(\frac{\varphi_{k\alpha}}{\varphi_{k\gamma}}\right) = 0,$$

for $\gamma \neq \alpha$. Such equations hold for $i = 1, \cdots, n$; $i \neq k$. Hence we have

$$\varphi_{i\alpha} = e^{\nu_i}\,\psi_{i\alpha}, \tag{2.8}$$

where $\psi_{i\alpha}$ are functions of x_i at most and ν_i are to be determined.

From (2.3) we have

$$b^{\alpha}_{j1} M_{11i\alpha} = b^{\alpha}_{i1} M_{11j\alpha} \qquad (i, j = 2, \cdots, n).$$

Substituting from (2.8) we obtain

$$e^{\nu_j} b^{\alpha}_{j1} N_{1i} = e^{\nu_i} b^{\alpha}_{i1} N_{1j},$$

where N_{1i} is independent of x_1 and x_i. Differentiating with respect to x_1, we have

$$\frac{\partial}{\partial x_1}(\nu_i - \nu_j) = 0 \qquad (i, j = 2, \cdots, n).$$

Again from (2.3) and (2.4) we have

$$b^{\alpha}_{j\alpha} M_{\alpha 1\beta\beta} + b^{\beta}_{\beta\alpha} M_{j1\alpha\alpha} = 0 \qquad \left(\begin{matrix}\alpha, \beta = 2, \cdots, n; \alpha \neq \beta \\ j = 1, \cdots, n; j \neq \alpha\end{matrix}\right).$$

Substituting from (2.8) we have

$$e^{\nu_j} b^{\alpha}_{j\alpha} N_{\alpha\beta} + e^{\nu_\beta} b^{\beta}_{\beta\alpha} N_{\alpha j} = 0.$$

Differentiating with respect to x_α, we have

$$\frac{\partial}{\partial x_\alpha}(\nu_j - \nu_\beta) = 0.$$

Combining these results, we have

$$\frac{\partial(\nu_i - \nu_j)}{\partial x_k} = 0 \qquad (i, j, k = 1, \cdots, n; i, j, k \neq).$$

From the preceding equations we have

$$\nu_i - \nu_j = f_{ij}, \qquad \nu_i - \nu_k = f_{ik}, \qquad \nu_j - \nu_k = f_{jk},$$

where f_{ij} is at most a function of x_i and x_j. From these equations we have

$$f_{ij} - f_{ik} + f_{jk} = 0.$$

Differentiating with respect to x_i, we obtain

$$\frac{\partial f_{ij}}{\partial x_i} = \frac{\partial f_{ik}}{\partial x_i}.$$

Since the first term does not involve x_k and the second x_j, we have

$$f_{ij} = \sigma_i - \sigma_j, \qquad f_{ik} = \sigma_i - \sigma_k,$$

where σ_i is a function of x_i alone. Hence the above set of equations may be replaced by

$$\nu_i = \nu + \sigma_i,$$

where ν is undetermined, and (2.8) becomes

$$\varphi_{i\alpha} = e^{\nu}\theta_{i\alpha} \qquad (i = 1, \cdots, n; \alpha = 2, \cdots, n), \tag{2.9}$$

where $\theta_{i\alpha}$ are functions of x_i alone. Since we have shown that φ_{i1} is a function of x_i alone, it follows that when the above expressions are substituted in (2.1) the factor e^{ν} disappears, and consequently the general solution of the problem is obtained when each function φ_{ij} is a function of x_i alone, which is the Stäckel form for the separation of the variables. Hence we have:

A necessary and sufficient condition that the fundamental quadratic form of V_n can be given the Stäckel form is that the equations of geodesics admit $n - 1$ independent quadratic first integrals, that the roots of the characteristic equations (1.2) *for each of these integrals be simple, that* (1.11) *hold, and that the vector-fields determined by* (1.3) *be normal and be the same vector-fields for all the first integrals.* Also we have:

A necessary and sufficient condition that (1.4) *is in the Stäckel form is that equations* (1.9) *and* (1.10) *be satisfied.*[7]

We have yet to consider the condition (C) of the Introduction for the case of equation (B). To this end we observe that from the definition of φ and (2.1) we have

$$\frac{\partial \log \varphi}{\partial x_j} = \frac{1}{H_j^2}(\varphi'_{j1} + \rho_j^2 \varphi'_{j2} + \cdots + \rho_j^n \varphi'_{jn}),$$

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[7] Cf. Dall'Acqua, Le equazioni di Hamilton-Jacobi che si integrano per separazione di variabili, Rend. di Palermo, vol. 33 (1912), pp. 341–351.

where the prime indicates differentiation. In consequence of (1.8) we have

$$\frac{\partial^2 \log \varphi}{\partial x_j \partial x_k} = -\frac{1}{H_j^2}\frac{\partial \log H_j^2}{\partial x_k}\left(\varphi'_{j1} + \sum_{\alpha}^{2\cdots n} \rho_k^\alpha \varphi'_{j\alpha}\right). \tag{2.10}$$

If we differentiate with respect to x_j the identity

$$\varphi_{j1} + \sum_\alpha \rho_k^\alpha \varphi_{j\alpha} = 0,$$

we have in consequence of (1.8)

$$\varphi'_{j1} + \sum_\alpha \rho_k^\alpha \varphi'_{j\alpha} = \sum_\alpha \varphi_{j\alpha}(\rho_j^\alpha - \rho_k^\alpha)\frac{\partial \log H_k^2}{\partial x_j} = \left(\varphi_{j1} + \sum_\alpha \varphi_{j\alpha}\rho_j^\alpha\right)\frac{\partial \log H_k^2}{\partial x_j}$$
$$= \frac{\varphi}{\varphi^{j1}}\frac{\partial \log H_k^2}{\partial x_j} = H_j^2 \frac{\partial \log H_k^2}{\partial x_j}.$$

Consequently we have from (2.10)

$$\frac{\partial^2 \log \varphi}{\partial x_j \partial x_k} = -\frac{\partial \log H_j^2}{\partial x_k}\frac{\partial \log H_k^2}{\partial x_j}.$$

From this and (1.9) we have

$$\frac{\partial^2 \log \varphi}{\partial x_j \partial x_k} = \frac{\partial^2 \log H_j^2}{\partial x_j \partial x_k} = \frac{\partial^2 \log H_k^2}{\partial x_j \partial x_k}. \tag{2.11}$$

In order that (C) be satisfied it is necessary and sufficient that

$$\frac{\partial^2 \log}{\partial x_j \partial x_k}\frac{\Pi H_i}{\varphi} = 0 \qquad (j, k = 1, \cdots, n; j \neq k),$$

which because of (2.11) may be written

$$\frac{\partial^2 \log}{\partial x_j \partial x_k}\Pi' H_i = 0, \tag{2.12}$$

where Π' indicates the product of the H's except H_j and H_k.

In order to give an interpretation to (2.12) we consider the expression for the Riemannian symbol R_{jiik} for i, j, k different, namely[8]

$$R_{jiik} = \frac{e_i H_i^2}{4}\left[2\frac{\partial^2 \log H_i^2}{\partial x_j \partial x_k} + \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_i^2}{\partial x_k} - \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_k} - \frac{\partial \log H_i^2}{\partial x_k}\frac{\partial \log H_k^2}{\partial x_j}\right]. \tag{2.13}$$

In consequence of (1.10) this may be written

$$R_{jiik} = \frac{3}{4} e_i H_i^2 \frac{\partial^2 \log H_i^2}{\partial x_j \partial x_k}. \tag{2.14}$$

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[8] R. G., p. 119.

Since

$$g^{ii} = \frac{e_i}{H_i^2}, \qquad g^{ij} = 0 \qquad (i \neq j),$$

we have

$$R_{jk} = g^{il} R_{jilk} = \frac{3}{4} \frac{\partial^2 \log \Pi' H_i^2}{\partial x_j \partial x_k},$$

and consequently (2.12) is equivalent to

$$R_{jk} = 0 \qquad (j \neq k). \tag{2.15}$$

Consequently the condition (C) is satisfied by any Stäckel form of a euclidean space and of a space of constant Riemannian curvature. Incidentally we remark that the Schwarzschild form[9] for an Einstein 4-space satisfies the conditions (1.9) and (1.10) and in this case (C) is satisfied.

3. **Stäckel systems in euclidean space.** The components of the Riemannian tensor formed with respect to the quadratic form (1.4) are given by (2.13) and[10]

$$R_{ijkl} = 0 \tag{3.1}$$

$$R_{jiij} = e_i H_i^2 \left(\frac{\partial^2 \log H_i}{\partial x_j^2} + \frac{\partial \log H_i}{\partial x_j} \frac{\partial}{\partial x_j} \log \frac{H_i}{H_j} \right) + e_j H_j^2 \left(\frac{\partial^2 \log H_j}{\partial x_i^2} + \frac{\partial \log H_j}{\partial x_i} \frac{\partial}{\partial x_i} \log \frac{H_j}{H_i} \right) + \sum_{k(\neq i,j)} e_i e_j e_k \frac{H_i^2 H_j^2}{H_k^2} \frac{\partial \log H_i}{\partial x_k} \frac{\partial \log H_j}{\partial x_k}. \tag{3.2}$$

From (1.9) we have

$$\frac{\partial^2}{\partial x_i \partial x_j} \log \frac{H_i^2}{H_j^2} = 0,$$

from which it follows that

$$H_i^2 = \varphi_{ij}^2 \theta_{ij}, \qquad H_j^2 = \varphi_{ji}^2 \theta_{ij}, \tag{3.3}$$

where φ_{ij} is independent of x_j and φ_{ji} of x_i. Substituting in (1.9), we find that

$$\theta_{ij} = \tau_{ij} + \tau_{ji}, \tag{3.4}$$

where τ_{ij} is independent of x_j and τ_{ji} of x_i.

Equating to zero the right-hand member of (2.14), we have in consequence of (1.10)

$$\frac{\partial^2 \log H_i^2}{\partial x_j \partial x_k} = 0, \tag{3.5}$$

$$\frac{\partial \log H_i}{\partial x_j} \frac{\partial \log H_i}{\partial x_k} - \frac{\partial \log H_i}{\partial x_j} \frac{\partial \log H_j}{\partial x_k} - \frac{\partial \log H_i}{\partial x_k} \frac{\partial \log H_k}{\partial x_j} = 0 \qquad (i, j, k \neq). \tag{3.6}$$

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[9] R. G., p. 93.

[10] R. G., p. 119.

Substituting in (3.5) from (3.3) and (3.4), we find that

$$\frac{\partial \tau_{ji}}{\partial x_j} = (\tau_{ij} + \tau_{ji})\psi_{ji}(x_i, x_j), \tag{3.7}$$

and similarly

$$\frac{\partial \tau_{ij}}{\partial x_i} = (\tau_{ij} + \tau_{ji})\psi_{ij}(x_i, x_j). \tag{3.8}$$

Differentiating these equations with respect to x_i and x_j respectively, we have

$$\frac{\partial \psi_{ji}}{\partial x_i} + \psi_{ji}\psi_{ij} = 0, \qquad \frac{\partial \psi_{ij}}{\partial x_j} + \psi_{ji}\psi_{ij} = 0.$$

Accordingly we have

$$\psi_{ji} = \frac{\partial \log \alpha}{\partial x_j}, \qquad \psi_{ij} = \frac{\partial \log \alpha}{\partial x_i},$$

and we find that $\alpha = \alpha_i + \alpha_j$, where α_i and α_j are functions of x_i and x_j respectively. Then from (3.7) and (3.8) it follows that

$$\tau_{ij} + \tau_{ji} = (\alpha_i + \alpha_j)\omega_{ij},$$

where ω_{ij} is independent of x_i and x_j. In consequence of this result and (3.3) it follows that

$$H_i^2 = X_i \prod_{j\,(\neq i)} (\sigma_{ij} + \sigma_{ji}) \qquad (i = 1, \cdots, n), \tag{3.9}$$

where σ_{ij} is a function of x_i at most and σ_{ji} of x_j at most. These expressions satisfy (1.9). In order that (3.6) be satisfied we must have

$$\sigma'_{ji}\sigma'_{ki}(\sigma_{jk} + \sigma_{kj}) - \sigma'_{ji}\sigma'_{kj}(\sigma_{ki} + \sigma_{ik}) - \sigma'_{ki}\sigma'_{jk}(\sigma_{ij} + \sigma_{ji}) = 0, \tag{3.10}$$

where a prime indicates the derivative, and permuting the indices cyclically we have

$$\begin{aligned} &\sigma'_{kj}\sigma'_{ij}(\sigma_{ki} + \sigma_{ik}) - \sigma'_{kj}\sigma'_{ik}(\sigma_{ij} + \sigma_{ji}) - \sigma'_{ij}\sigma'_{ki}(\sigma_{jk} + \sigma_{kj}) = 0, \\ &\sigma'_{ik}\sigma'_{jk}(\sigma_{ij} + \sigma_{ji}) - \sigma'_{ik}\sigma'_{ji}(\sigma_{jk} + \sigma_{kj}) - \sigma'_{jk}\sigma'_{ij}(\sigma_{ki} + \sigma_{ik}) = 0. \end{aligned} \tag{3.11}$$

Equating to zero the determinant of these equations, we have

$$\sigma'_{ij}\sigma'_{jk}\sigma'_{ki} + \sigma'_{ji}\sigma'_{kj}\sigma'_{ik} = 0. \tag{3.12}$$

The same result follows, if we differentiate the above equations with respect to x_i, x_j and x_k respectively.

On the assumption that none of the terms in (3.12) is zero it follows that $\sigma'_{ij}/\sigma'_{ik}$ is a constant. Accordingly we put

$$\sigma_{ij} = a_{ij}\sigma_i,$$

where a_{ij} is a constant and σ_i involves x_i at most. The constants must satisfy the relations

$$a_{ij}a_{jk}a_{ki} + a_{ji}a_{kj}a_{ik} = 0. \tag{3.13}$$

Equations (3.10) and (3.11) are satisfied in consequence of (3.13). Hence in this case we have

$$H_i^2 = X_i \prod_{j\,(\neq i)} (a_{ij}\sigma_i + a_{ji}\sigma_j).$$

If we put

$$\sigma_i = a_{jk}a_{ki}\bar{\sigma}_i, \qquad \sigma_j = a_{kj}a_{ik}\bar{\sigma}_j,$$

in consequence of (3.13) we have $a_{ij}\sigma_i + a_{ji}\sigma_j = a_{ij}a_{jk}a_{ki}(\bar{\sigma}_i - \bar{\sigma}_j)$ and the constant factor may be absorbed in X_i. Then we have in all generality $a_{ij} = -a_{ji} = 1$, and (3.13) becomes

$$a_{jk}a_{ki} - a_{kj}a_{ik} = 0.$$

If now we put $a_{ki}\sigma_k = -a_{ik}\bar{\sigma}_k$, we have

$$a_{ki}\sigma_k + a_{ik}\bar{\sigma}_i = a_{ik}(\bar{\sigma}_i - \bar{\sigma}_k),$$

so that in all generality we may take $a_{ki} = -1$, $a_{ik} = +1$. Then $a_{jk}\bar{\sigma}_j + a_{kj}\bar{\sigma}_k = a_{jk}(\bar{\sigma}_j - \bar{\sigma}_k)$, and thus $a_{jk} = -a_{kj} = 1$, and we have

$$H_i^2 = X_i \prod_{j\,(\neq i)} (\sigma_i - \sigma_j). \tag{3.14}$$

We consider now the cases which can arise when some of the σ's are constant. Suppose that $\sigma_{ij} = a_{ij}$, where a_{ij} is a constant. From the first of (3.11) it follows that either $\sigma_{ik} = a_{ik}$, or $\sigma_{kj} = a_{kj}$, the a's being constant; we use this notation for the present. If $\sigma_{ik} = a_{ik}$, the second of (3.11) is satisfied and there remains (3.10). This is satisfied in the following cases

$$\text{(i)}\,\sigma_{ji} = a_{ji},\ \sigma_{jk} = a_{jk}; \qquad \text{(ii)}\ \sigma_{ji} = a_{ji},\ \sigma_{ki} = a_{ki}; \qquad \text{(iii)}\ \sigma_{ki} = a_{ki},\ \sigma_{kj} = a_{kj}. \tag{3.15}$$

The last follows from (i) when j and k are interchanged. If σ_{ji} and σ_{ki} are not constants, we write (3.10) in the form

$$\sigma_{jk} + \sigma_{kj} - \frac{\sigma'_{kj}}{\sigma'_{ki}}(\sigma_{ki} + a_{ik}) - \frac{\sigma'_{jk}}{\sigma'_{ji}}(\sigma_{ji} + a_{ij}) = 0. \tag{3.16}$$

From this we have

$$\sigma_{jk} - \frac{\sigma'_{jk}}{\sigma'_{ji}}(\sigma_{ji} + a_{ij}) = c, \qquad \sigma_{kj} - \frac{\sigma'_{kj}}{\sigma'_{ki}}(\sigma_{ki} + a_{ik}) = -c,$$

where c is a constant, and consequently

$$\sigma_{ji} + a_{ij} = b(\sigma_{jk} - c), \qquad \sigma_{ki} + a_{ik} = d(\sigma_{kj} + c),$$

where b and d are constants. Hence $\sigma'_{ji} = b\sigma'_{jk}$, $\sigma'_{ki} = d\sigma'_{kj}$, so that we may put $\sigma_{ji} = a_{ji}\sigma_j$, $\sigma_{jk} = a_{jk}\sigma_j$; $\sigma_{ki} = a_{ki}\sigma_k$, $\sigma_{kj} = a_{kj}\sigma_k$, and then from (3.16) we have (3.13). Thus we have three distinct types:

(3.17) $\sigma_{ij} = a_{ij}, \sigma_{ji} = a_{ji}, \sigma_{ik} = a_{ik}, \sigma_{jk} = a_{jk}$;

(3.18) $\sigma_{ij} = a_{ij}, \sigma_{ji} = a_{ji}, \sigma_{ik} = a_{ik}, \sigma_{ki} = a_{ki}$;

(3.19) $\sigma_{ij} = a_{ij}, \sigma_{ik} = a_{ik}, \sigma_{ji} = a_{ji}\sigma_j, \sigma_{jk} = a_{jk}\sigma_j, \sigma_{ki} = a_{ki}\sigma_k, \sigma_{kj} = a_{kj}\sigma_k$.

In the first two cases the a's are arbitrary, in the last case they must satisfy (3.13).

When $\sigma_{kj} = a_{kj}$ and $\sigma_{ij} = a_{ij}$, we have from (3.10) and (3.11) the case (3.15 iii), or

$$\sigma'_{ji}(\sigma_{jk} + a_{kj}) - \sigma'_{jk}(\sigma_{ji} + a_{ij}) = 0.$$

If $\sigma_{ji} = a_{ji}$, $\sigma_{jk} = a_{jk}$, we have (3.18) on interchanging i and j. Otherwise we have the type

(3.20) $$\sigma_{ij} = a_{ij}, \sigma_{kj} = a_{kj}, \sigma_{ji} = a_{ji}\sigma_j, \sigma_{jk} = a_{jk}\sigma_j,$$
$$a_{ji}a_{kj} - a_{jk}a_{ij} = 0.$$

For $n = 3$ and $i = 3, j = 2, k = 1$, we have from (3.9) and (3.17) that the coordinates can be chosen so that we have

(3.21) $$H_1 = 1, \qquad H_2 = \varphi_1, \qquad H_3 = \psi_1,$$

where φ_1 and ψ_1 are functions of x_1 at most.

For the case (3.20) we may in all generality take $a_{ij} = a_{kj} = 0$ and then from (3.9) for $i = 2, j = 1, k = 3$ by a suitable choice of coordinates we have

(3.22) $$H_1^2 = 1, \qquad H_2^2 = X_2\sigma_1(\sigma_{23} + \sigma_{32}), \qquad H_3^2 = X_3\sigma_1(\sigma_{23} + \sigma_{32}),$$

where σ_1 is a function of x_1 at most. When σ_1 is a constant, we have the case (3.18) for $i = 1, j = 2, k = 3$.

For the case (3.19) we may take $a_{ij} = a_{ik} = 0$ and for $i = 2, j = 1, k = 3$ we have in all generality

(3.23) $$H_1^2 = H_3^2 = \sigma_1 + e\sigma_3, \qquad H_2^2 = \sigma_1\sigma_3,$$

where $e = +1$ or -1, it being understood that σ_1 and σ_3 are positive.

Finally we have from (3.14) the case

(3.24) $$H_i^2 = X_i(x_i - x_j)(x_i - x_k) \qquad (i, j, k = 1, 2, 3; i, j, k \neq).$$

For the further determination of the functions appearing in (3.21), (3.22), (3.23), and (3.24) we have from (3.2) for all the e's equal to 1 the conditions

(3.25) $$\frac{1}{H_j^2}\left(2\frac{\partial^2 \log H_i^2}{\partial x_j^2} + \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial}{\partial x_j}\log\frac{H_i^2}{H_j^2}\right) + \frac{1}{H_i^2}\left(2\frac{\partial^2 \log H_j^2}{\partial x_i^2}\right.$$

$$+\frac{\partial \log H_j^2}{\partial x_i}\frac{\partial}{\partial x_i}\log\frac{H_j^2}{H_i^2}\Big) + \frac{1}{H_k^2}\frac{\partial \log H_i^2}{\partial x_k}\frac{\partial \log H_j^2}{\partial x_k} = 0 \qquad (i, j, k \neq).$$

These determinations will be made in the following sections.

When $n = 3$, equations (2.12) are (3.5). Consequently we have:

A necessary and sufficient condition that equation (B) be solvable by separation of the variables for $n = 3$ is that the H's be of one of the forms (3.21), (3.22), (3.23), (3.24).

4. **Types I.** From (3.21) and (3.25) for $i = 1, j = 2$ and $j = 3$, we find that

$$\varphi_1 = ax_1 + b, \qquad \psi_1 = cx_1 + d,$$

where a, b, c, d are constants. Substituting in (3.25) for $i = 2, j = 3$, we have $\frac{\partial \varphi_1}{\partial x_1}\frac{\partial \psi_1}{\partial x_1} = 0$. We take $a = 0, b = 1$. If $c = 0$, we have the cartesian case

$$(\mathrm{I}_1) \qquad H_i^2 = 1 \qquad (i = 1, 2, 3).$$

If $c \neq 0$, we have by a suitable choice of coordinates

$$(\mathrm{I}_2) \qquad H_1^2 = H_2^2 = 1, \qquad H_3^2 = x_1^2.$$

In this case the transformations of coordinates is

$$(4.1) \qquad x = x_1 \cos x_3, \qquad y = x_1 \sin x_3, \qquad z = x_2,$$

and the coordinate surfaces are the planes $x/y =$ const., $z =$ const. and the circular cylinders $x^2 + y^2 = x_1^2$.

The Stäckel matrices for these respective forms are

$$\begin{vmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \qquad \begin{vmatrix} 1 & -1 & -\frac{1}{x_1^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

5. **Types II.** In discussing the types (3.22) we consider first the case where σ_{32} is a constant, which may be taken equal to zero in all generality and by suitable choice of coordinates we have

$$H_1^2 = 1, \qquad H_2^2 = \varphi^2(x_1), \qquad H_3^2 = \varphi^2(x_1)\psi^2(x_2).$$

For $i = 1, j = 2, 3$ in (3.25) we have $\varphi = ax_1 + b$. For $i = 2, j = 3$ we have

$$\frac{\partial^2 \psi}{\partial x_2^2} + a^2\psi = 0.$$

If $a = 0$, we have Types I. If $a \neq 0$, we may take $a = 1, b = 0$ and obtain the case of polar coordinates

$$(\mathrm{II}_1) \qquad H_1^2 = 1, \qquad H_2^2 = x_1^2, \qquad H_3^2 = x_1^2 \sin^2 x_2.$$

If neither σ_{23} nor σ_{32} is constant, we may choose the coordinates so as to have

$$H_1^2 = 1, H_2^2 = X_2\sigma_1(x_2 - x_3), \qquad H_3^2 = X_3\sigma_1(x_2 - x_3). \tag{5.1}$$

For $i = 1, j = 2, 3$ in (3.25) we have $\sigma_1 = (ax_1 + b)^2$. For $i = 2, j = 3$ we obtain

$$2\left(\frac{1}{X_2} + \frac{1}{X_3}\right) + (x_2 - x_3)\left[\left(\frac{1}{X_3}\right)' - \left(\frac{1}{X_2}\right)'\right] - 4a^2(x_2 - x_3)^3 = 0. \tag{5.2}$$

Differentiating with respect to x_2, we obtain

$$\left(\frac{1}{X_2}\right)' + \left(\frac{1}{X_3}\right)' + (x_3 - x_2)\left(\frac{1}{X_2}\right)'' - 12a^2(x_2 - x_3)^2 = 0. \tag{5.3}$$

Differentiating again with respect to x_2, we have

$$\left(\frac{1}{X_2}\right)''' = -24a^2,$$

and consequently

$$\frac{1}{X_2} = -4a^2x_2^3 + cx_2^2 + dx_2 + e \equiv f(x_2). \tag{5.4}$$

Substituting this expression in (5.3) we have

$$\left(\frac{1}{X_3}\right)' = 12a^2x_3^2 - 2cx_3 - d,$$

and then from (5.2) we have $\frac{1}{X_3} = -f(x_3)$. There are two cases to be considered as $a = 0$ and $a \neq 0$. In the former case, as is seen on substituting the above expressions in (1.4) for $n = 3$, there is no loss in taking $b = 1$ and in the second case $a = 1, b = 0$. Hence we have the two forms

$$H_1^2 = 1, \qquad H_2^2 = \frac{x_2 - x_3}{f(x_2)}, \qquad H_3^2 = \frac{x_3 - x_2}{f(x_3)}, \tag{5.5}$$
$$f(x_i) = cx_i^2 + dx_i + e,$$

$$H_1^2 = 1, \qquad H_2^2 = \frac{x_1^2(x_2 - x_3)}{f(x_2)}, \qquad H_3^2 = \frac{x_1^2(x_3 - x_2)}{f(x_3)}, \tag{5.6}$$
$$f(x_i) = -4x_i^3 + cx_i^2 + dx_i + e.$$

If in (5.5) we assume that $f(x) = 0$ has two distinct roots, by a suitable choice of the coordinates, the form may be written $f(x) = 4(x^2 - ax)$, where $a > 0$, and $x_2 > a > x_3 > 0$; then the expressions for H_2^2 and H_3^2 are positive. If we put

$$x_2 - a/2 = \frac{1}{2}a \cosh 2\xi, \qquad x_3 - a/2 = \frac{1}{2}a \cos 2\eta, \tag{5.7}$$

we have, on replacing a by a^2

$$(II_2) \qquad ds^2 = dx_1^2 + \frac{1}{2}a^2 (\cosh 2\xi - \cos 2\eta)(d\xi^2 + d\eta^2),$$

in which case the coordinate transformation is

$$(5.8) \qquad x = x_1, \quad y = a \cosh \xi \cos \eta, \quad z = a \sinh \xi \sin \eta.$$

Hence the coordinate surfaces are the planes $x =$ const. and the confocal cylinders

$$(5.9) \qquad \frac{y^2}{\cos^2 \eta} - \frac{z^2}{\sin^2 \eta} = a^2, \qquad \frac{y^2}{\cosh^2 \xi} + \frac{z^2}{\sinh^2 \xi} = a^2.$$

No real case exists for which the two roots of $f(x) = 0$ in (5.5) are equal, nor when $f(x)$ is a constant. In case $c = 0$, we may take $f(x) = 4x$, and $x_2 > 0 > x_3$. If we put $x_2 = \xi^2$, $x_3 = -\eta^2$, we have

$$(II_3) \qquad ds^2 = dx_1^2 + (\xi^2 + \eta^2)(d\xi^2 + d\eta^2),$$

in which case the coordinate transformation is

$$x = x_1, \; y = \tfrac{1}{2}(\xi^2 - \eta^2), \; z = \xi\eta,$$

so that the coordinate surfaces are the planes $x =$ const., and the confocal parabolic cylinders

$$(5.10) \qquad z^2 = \xi^2(\xi^2 - 2y), \qquad z^2 = \eta^2(2y + \eta^2).$$

If we write $f(x)$ in (5.6) in the form $4(a - x)(b - x)(c - x)$ with $a > b > c$ and put

$$\frac{a - b}{a - c} = k^2, \qquad \frac{b - c}{a - c} = k'^2, \qquad k^2 + k'^2 = 1,$$

and

$$x_2 = a + (b - a)\,\mathrm{sn}^2(\xi, k), \qquad x_3 = c + (b - c)\,\mathrm{sn}^2(\eta, k'),$$

where sn θ is the elliptic function, the form (5.6) becomes

$$(II_4) \qquad ds^2 = dx_1^2 + x_1^2 [k^2 \,\mathrm{cn}^2(\xi, k) + k'^2 \,\mathrm{cn}^2(\eta, k')](d\xi^2 + d\eta^2).$$

The coordinate transformation is

$$(5.11) \qquad x = x_1 \,\mathrm{dn}(\xi, k)\,\mathrm{sn}(\eta, k'), \qquad y = x_1 \,\mathrm{sn}(\xi, k)\,\mathrm{dn}(\eta, k'),$$
$$z = x_1 \,\mathrm{cn}(\xi, k)\,\mathrm{cn}(\eta, k'),$$

so that the coordinate surfaces are the spheres and cones with the equations

$$(5.12) \qquad x^2 + y^2 + z^2 = x_1^2, \qquad \frac{k^2 x^2}{\mathrm{dn}^2(\xi, k)} - \frac{y^2}{\mathrm{sn}^2(\xi, k)} + \frac{z^2}{\mathrm{cn}^2(\xi, k)} = 0,$$

$$\frac{x^2}{\mathrm{sn}^2(\eta, k')} - \frac{k'^2 y^2}{\mathrm{dn}^2(\eta, k')} - \frac{z^2}{\mathrm{cn}^2(\eta, k')} = 0.$$

It is readily shown that if two of the roots of $f(x) = 0$ are equal, or all are equal, there are no real solutions of the problem.

The Stäckel matrices for the forms (II_1), (II_2), (II_3) and (II_4) are respectively

$$\begin{vmatrix} 1 & -\dfrac{1}{x_1^2} & 0 \\ 0 & 1 & -\csc^2 x_2 \\ 0 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & -1 & 0 \\ 0 & a^2\cosh 2\xi & -1 \\ 0 & -a^2 \cos 2\eta & 1 \end{vmatrix},$$

$$\begin{vmatrix} 0 & -1 & 0 \\ 0 & \xi^2 & -1 \\ 0 & \eta^2 & 1 \end{vmatrix}, \quad \begin{vmatrix} x_1^2 & -1 & 0 \\ 0 & k^2 \operatorname{cn}^2(\xi, k) & -1 \\ 0 & k'^2 \operatorname{cn}^2(\eta, k') & 1 \end{vmatrix}.$$

6. **Types III.** We consider next

$$H_1^2 = \sigma_1 + e\sigma_3, \qquad H_2^2 = \sigma_1\sigma_3, \qquad H_3^2 = \sigma_1 + e\sigma_3, \tag{6.1}$$

where e is $+1$ or -1, it being understood that σ_1 and σ_3 are positive.

Substituting in (3.25) for $i = 1, j = 2$, we have

$$2\left(\sigma_1'' - \frac{\sigma_1'^2}{\sigma_1}\right) + e\sigma_3\left(2\frac{\sigma_1''}{\sigma_1} - \frac{\sigma_1'^2}{\sigma_1^2}\right) + e\frac{\sigma_3'^2}{\sigma_3} = 0. \tag{6.2}$$

Differentiating with respect to x_3, we have

$$2\frac{\sigma_1''}{\sigma_1} - \frac{\sigma_1'^2}{\sigma_1^2} + \left(\frac{\sigma_3'^2}{\sigma_3}\right)'\frac{1}{\sigma_3'} = 0.$$

Hence we have

$$2\frac{\sigma_1''}{\sigma_1} - \frac{\sigma_1'^2}{\sigma_1^2} = c, \qquad \left(\frac{\sigma_3'^2}{\sigma_3}\right)' = -c\sigma_3',$$

where c is a constant. From the second we have

$$\sigma_3'^2 = -c\sigma_3^2 + d\sigma_3, \tag{6.3}$$

where d is a constant. Then from (6.2) we have

$$\sigma_1'^2 = c\sigma_1^2 + de\sigma_1. \tag{6.4}$$

These expressions satisfy (3.25) for $i = 1, j = 3$ and $i = 2, j = 3$, without imposing any conditions on c and d.

We consider first the case when $c = 0$, which is possible only when $e = 1$ and d positive, as we understand that the coordinates are real. Then we have

$$\sqrt{\sigma_i} = \tfrac{1}{2}\sqrt{d}\, x_i + f_i.$$

By a suitable choice of coordinates we have

$$H_1^2 = H_3^2 = x_1^2 + x_3^2, \qquad H_2^2 = x_1^2 x_3^2. \tag{III$_1$}$$

The transformation is

$$(6.5)\qquad x = x_1x_3 \cos x_2, \qquad y = x_1x_3 \sin x_2, \qquad z = \tfrac{1}{2}(x_1^2 - x_3^2),$$

so that the coordinate surfaces are the planes $x/y = \text{const.}$ and

$$(6.6)\qquad x^2 + y^2 = x_1^2 (x_1^2 - 2z), \qquad x^2 + y^2 = x_3^2 (x_3^2 + 2z).$$

When $c \neq 0$, there is no loss of generality in assuming it to be positive, and replacing it by $4c^2$. We consider first the case when $e = 1$, and replace σ_1 and σ_3 by $\frac{\sigma_1 d}{4c^2}$ and $\frac{\sigma_3 d}{4c^2}$, noting from (6.3) that d is necessarily positive. Then the solution of (6.4) and (6.3) respectively is

$$\sigma_1 = \sinh^2 (cx_1 + b), \qquad \sigma_3 = \sin^2 (cx_3 + f).$$

By suitable choice of the coordinates we have

$$(\text{III}_2)\quad H_1^2 = H_3^2 = a^2(\sinh^2 x_1 + \sin^2 x_3), \qquad H_2^2 = a^2 \sinh^2 x_1 \sin^2 x_3.$$

The transformation from cartesian coordinates is

$$x = a \sinh x_1 \sin x_3 \cos x_2, \qquad y = a \sinh x_1 \sin x_3 \sin x_2,$$
$$z = a \cosh x_1 \cos x_3.$$

The coordinate surfacesare the planes $y = x \tan x_2$ and the quadrics of revolution

$$\frac{x^2 + y^2}{\sinh^2 x_1} + \frac{z^2}{\cosh^2 x_1} = a^2, \qquad \frac{z^2}{\cos^2 x_3} - \frac{x^2 + y^2}{\sin^2 x_3} = a^2.$$

When $e = -1$, we obtain in like manner

$$(\text{III}_3)\quad H_1^2 = H_3^2 = a^2 (\sinh^2 x_1 + \cos^2 x_3), \qquad H_2^2 = a^2 \cosh^2 x_1 \sin^2 x_3.$$

The coordinate transformation is

$$x = a \cosh x_1 \sin x_3 \cos x_2, \qquad y = a \cosh x_1 \sin x_1 \sin x_2,$$
$$z = a \sinh x_1 \cos x_3,$$

and the coordinate surfaces are the planes $y = x \tan x_2$ and

$$\frac{x^2 + y^2}{\cosh^2 x_1} + \frac{z^2}{\sinh^2 x_1} = a^2, \qquad \frac{x^2 + y^2}{\sin^2 x_3} - \frac{z^2}{\cos^2 x_3} = a^2.$$

The Stäckel matrices for the above forms are respectively

$$\begin{vmatrix} x_1^2 & \frac{1}{x_1^2} & 1 \\ 0 & -1 & 0 \\ x_3^2 & \frac{1}{x_3^2} & -1 \end{vmatrix}, \quad \begin{vmatrix} a^2 \sinh^2 x_1 & \operatorname{csch}^2 x_1 & 1 \\ 0 & -1 & 0 \\ a^2 \sin^2 x_3 & \csc^2 x_3 & -1 \end{vmatrix}, \quad \begin{vmatrix} a^2 \cosh^2 x_1 & \operatorname{sech}^2 x_1 & -1 \\ 0 & 1 & 0 \\ -a^2 \sin^2 x_3 & -\csc^2 x_3 & 1 \end{vmatrix}.$$

7. **Types IV.** We consider finally the type (3.24), that is

$$(7.1)\qquad H_i^2 = X_i(x_i - x_j)(x_i - x_k) \qquad (i, j, k \neq),$$

and we assume that $x_1 > x_2 > x_3$. When these expressions are substituted in (3.25) for $i = 1, j = 2$, we obtain

$$\begin{aligned}(7.2)\quad &\frac{1}{X_3} + \frac{1}{(x_2 - x_1)^2}\left\{(x_3 - x_2)^2\left[(x_1 - x_3)\left(\frac{1}{X_1}\right)' - \left(\frac{2(x_3 - x_1)}{x_2 - x_1} + 1\right)\frac{1}{X_1}\right]\right. \\ &\left. + (x_3 - x_1)^2\left[(x_2 - x_3)\left(\frac{1}{X_2}\right)' - \left(\frac{2(x_3 - x_2)}{x_1 - x_2} + 1\right)\frac{1}{X_2}\right]\right\} = 0.\end{aligned}$$

Differentiating this equation with respect to x_2, we obtain a polynomial of the third degree in x_3. Each of its coefficients must vanish. Equating to zero the coefficient of x_3^3, we have

$$(7.3)\quad (x_1 - x_2)^2\left(\frac{1}{X_2}\right)'' + 4(x_1 - x_2)\left(\frac{1}{X_2}\right)' + 6\frac{1}{X_2} + 2(x_1 - x_2)\left(\frac{1}{X_1}\right)' - 6\frac{1}{X_1} = 0.$$

Differentiating with respect to x_2 twice we have

$$\left(\frac{1}{X_2}\right)^{\text{IV}} = 0,$$

and consequently

$$(7.4)\quad \frac{1}{X_2} = a_0 x_2^3 + a_1 x_2^2 + a_2 x_2 + a_3 \equiv f(x_2).$$

Substituting in (7.3) we find $\frac{1}{X_1} = f(x_1)$, and from (7.2) we have $\frac{1}{X_3} = f(x_3)$. These expressions satisfy the three conditions (3.25).

When $a_0 \neq 0$ in (7.4) and the roots of $f(x) = 0$ are distinct we write

$$(7.5)\quad f(x) = 4(\alpha - x)(\beta - x)(\gamma - x)$$

where $\alpha > \beta > \gamma$, and we have

$$(\text{IV}_1)\quad H_i^2 = \frac{(x_i - x_j)(x_i - x_k)}{f(x_i)} \qquad (i, j, k \neq).$$

This is the case of elliptic coordinates for which the transformation is

$$x^2 = \frac{(\alpha - x_1)(\alpha - x_2)(\alpha - x_3)}{(\alpha - \beta)(\alpha - \gamma)}, \qquad y^2 = \frac{(\beta - x_1)(\beta - x_2)(\beta - x_3)}{(\beta - \alpha)(\beta - \gamma)},$$

$$z^2 = \frac{(\gamma - x_1)(\gamma - x_2)(\gamma - x_3)}{(\gamma - \alpha)(\gamma - \beta)},$$

where $\alpha > x_1 > \beta > x_2 > \gamma > x_3$, the surfaces $x_i =$ const., being

$$\frac{x^2}{\alpha - x_i} + \frac{y^2}{\beta - x_i} + \frac{z^2}{\gamma - x_i} = 1.$$

It is readily shown that there is no possibility of a double or triple root of $f(x) = 0$ giving a real set of orthogonal surfaces for (7.1).

We consider next the case where $a_0 = 0$ in (7.4), and write $f(x) = 4(a - x)(b - x)$. In this case we have

$$(\text{IV}_2) \qquad H_i^2 = \frac{(x_i - x_j)(x_i - x_k)}{f(x_i)}, \qquad f(x_i) = 4(a - x_i)(b - x_i), \qquad (i, j, k \neq),$$

and assume $x_1 > b > x_2 > a > x_3$. The transformation of coordinates is

$$(7.6) \qquad x = \frac{x_1 + x_2 + x_3 - a - b}{2}, \qquad y^2 = \frac{(a - x_1)(a - x_2)(a - x_3)}{b - a}, \qquad z^2 = \frac{(b - x_1)(b - x_2)(b - x_3)}{a - b},$$

so that the coordinate surfaces are the confocal paraboloids

$$(7.7) \qquad \frac{y^2}{a - x_i} + \frac{z^2}{b - x_i} = x_i - 2x.$$

There is no real solution if the roots a, b are equal. Also there are no real solutions, when $a_0 = a_1 = 0$, and when $a_2 = 0$ also.

The Stäckel functions φ_{ij} are in these cases

$$\varphi_{i1} = \frac{x_i^2}{f(x_i)}, \qquad \varphi_{i2} = \frac{1}{f(x_i)}, \qquad \varphi_{i3} = \frac{x_i}{f(x_i)}.$$

The orthogonal systems of coordinate surfaces which afford separation of variables constitute the set of all real systems of confocal quadrics including the cases where one or more families of the systems consists of planes. Hence we have:

A necessary and sufficient condition that a triply orthogonal system of surfaces in euclidean 3*-space be a coordinate system in terms of which the fundamental quadratic form of the space is such that the variables are separated in the corresponding Hamilton-Jacobi equation and the Laplace equation is that they be any system of confocal quadrics, including the cases when one or more families of the system consists of planes.*

Also as a result of the preceding investigation we have:

Equations (1.9) *and* (1.10) *constitute a necessary and sufficient condition that the coordinate surfaces of a triply orthogonal system in euclidean* 3*-space be confocal quadrics, including the cases when one or more families of the system consists of planes.*

8. **Euclidean spaces of higher order.** When $n > 3$, we may analyze the various types as in the former discussion. We consider first the case when by a choice of x_1 we have $H_1 = 1$. In this case as follows from (3.9) we have without loss of generality, $\sigma_{j1} = 0$ for $j > 1$. From (3.9) and (3.20) for $i = 2, j = 1, k > 2$ it follows that

$$H_j^2 = X_j \sigma_{1j} \prod_k (\sigma_{jk} + \sigma_{kj}) \qquad (j, k = 2, \cdots, n; j \neq k),$$

where σ_{1j} is a function of x_1 at most. From (3.25) for $i = 1, j > 1$ it follows that $\sigma_{1j} = (a_j x_1 + b_j)^2$, where a_j and b_j are constants. For the case when all the a's are equal and all the b's, by a suitable choice of x_1 we have either $\sigma_{1j} = x_1^2$ or $\sigma_{1j} = c^2$, where c is a constant. If we substitute the above expressions in (3.2) for $i, j > 1$, we find that the hypersurfaces are of constant curvature $1/x_1^2$ in the first case and euclidean in the second. Furthermore, since $H_1 = 1$, the spaces $x_1 =$ const. are geodesically parallel. Consequently in the first case we have concentric hyperspheres and in the second case parallel hyperplanes. When $x_1 =$ const. are concentric hyperspheres, any other system of hypersurfaces consists of a pencil of planes whose axis passes through the common center of the hyperspheres or of quadric hypercones with vertices at the common center. The character of these other hypersurfaces depends upon the possible forms of $H_2^2\, dx_2^2 + \cdots + H_n^2\, dx_n^2$; in §9 we classify the Stäckel types for a V_3 of constant positive curvature. When the hyperplanes $x_1 =$ const. are parallel, any other system of hypersurfaces consists of hyperplanes orthogonal to the former or of coaxial quadric cylinders; the situation for $n = 4$ is readily obtained by giving to $H_2^2\, dx_2^2 + \cdots + H_4^2\, dx_4^2$ the various forms in §§4–7. These types for which $H_1 = 1$ are generalizations of those discussed in §§4, 5.

Generalizations of type III arise when we take $\sigma_{1j} = 0$ $(j > 1)$ and none of σ_{j1} is constant. In this case as follows from (3.19)

$$H_1^2 = \sigma_2 \cdots \sigma_n, \qquad H_j^2 = \prod_k (a_{jk}\, \sigma_j + a_{kj}\, \sigma_k)$$

$$(j, k = 2, \cdots, n; j \neq k).$$

From the equation obtained by equating to zero the right-hand member of (3.2) for $i, j = 2, \cdots, n$ we obtain a result which shows by means of (5.2) that the hypersurfaces $x_1 =$ const. are hyperplanes. Since x_1 does not appear in the H's, the sections by all these hyperplanes of each hypersurface $x_j =$ const. are the same. Analogously to types III these hyperplanes form a pencil and the hypersurfaces $x_j =$ const. are generalized spaces of revolution.

When i, j in (3.14) take the values $1, \cdots, n$, $\sigma_i = x_i$, and

$$\frac{1}{X_i} = 4(a_1 - x_i) \cdots (a_n - x_i),$$

we have the general elliptic coordinates in euclidean n-space discussed by Jacobi.[11] If $\bar{x}_i$ are cartesian coordinates, the transformation is

$$\bar{x}_i^2 = \frac{\prod_j (a_i - x_j)}{\prod_{j(\neq i)} (a_i - a_j)}.$$

[11] Vorlesungen über Dynamik, Berlin 1866, pp. 198–205.

9. Stäckel systems in 3-space of constant curvature $1/c^2$. If the curvature of V_3 is $1/c^2$, we have in place of (3.25)

$$\frac{1}{H_j^2}\left(2\frac{\partial^2 \log H_i^2}{\partial x_j^2} + \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial}{\partial x_j}\log\frac{H_i^2}{H_j^2}\right) \tag{9.1}$$

$$+\frac{1}{H_i^2}\left(2\frac{\partial^2 \log H_j^2}{\partial x_i^2} + \frac{\partial \log H_j^2}{\partial x_i}\frac{\partial}{\partial x_i}\log\frac{H_j^2}{H_i^2}\right) + \frac{1}{H_k^2}\frac{\partial \log H_i^2}{\partial x_k}\frac{\partial \log H_j^2}{\partial x_k} = -\frac{4}{c^2}.$$

If we write (3.21) in the form

$$H_1^2 = c^2, \qquad H_2^2 = c^2\varphi_1^2, \qquad H_3^2 = c^2\psi_1^2,$$

we have from (9.1) for $i = 1, j = 2$ and $i = 1, j = 3$ that φ_1 and ψ_1 must satisfy

$$\frac{\partial^2\theta}{\partial x_1^2} + \theta = 0,$$

and from (9.1) for $i = 2, j = 3$ we have

$$\frac{\partial\varphi_1}{\partial x_1}\frac{\partial\psi_1}{\partial x_1} = -\varphi_1\psi_1.$$

Accordingly by a suitable choice of x_1 we have

$$ds^2 = c^2(dx_1^2 + \sin^2 x_1\, dx_2^2 + \cos^2 x_1 dx_3^2). \tag{I}$$

We consider next the case

$$H_1^2 = c^2, \qquad H_2^2 = c^2\varphi^2(x_1), \qquad H_3^2 = c^2\varphi^2(x_1)\psi^2(x_2).$$

From (9.1) for $i = 1, j = 2$ and 3 we have

$$\frac{\partial^2\varphi}{\partial x_1^2} + \varphi = 0,$$

and hence

$$\varphi = a \sin(x_1 + b).$$

From (9.1) for $i = 2, j = 3$ we have

$$\frac{\partial^2\psi}{\partial x_2^2} = -a^2\psi$$

and consequently

$$\psi = d\sin(ax_2 + e).$$

By a suitable choice of coordinates we have

$$ds^2 = c^2[dx_1^2 + \sin^2 x_1(dx_2^2 + \sin^2 x_2\, dx_3^2)]. \tag{II}$$

For the case

$$H_1^2 = c^2, \qquad H_2^2 = c^2 X_2\sigma_1^2(x_2 - x_3), \qquad H_3^2 = c^2 X_3\sigma_1^2(x_2 - x_3),$$

we find that for $i = 1, j = 2, 3$, we have

$$\sigma_1 = a \sin (x_1 + b).$$

By a change of x_1 and replacing X_2 by X_2/a^2 we have

$$H_1^2 = c^2, \qquad H_2^2 = c^2 X_2 \sin^2 x_1 (x_2 - x_3), \qquad H_3^2 = c^2 X_3 \sin^2 x_1 (x_2 - x_3).$$

Then for $i = 2, j = 3$, we get (5.2) with $a = 1$. Hence we have

$$\text{(III)} \quad ds^2 = c^2 \{dx_1^2 + \sin^2 x_1 [k^2 \operatorname{cn}^2 (x_2, k) + k'^2 \operatorname{cn}^2 (x_3, k')] (dx_2^2 + dx_3^2)\}.$$

For the case

$$H_1^2 = c^2(\sigma_1 + e\sigma_3), \qquad H_2^2 = c^2\sigma_1\sigma_3, \qquad H_3^2 = c^2(\sigma_1 + e\sigma_3),$$

where e is $+1$ or -1, we have from (9.1) for $i = 1, j = 2$

$$\text{(9.2)} \quad 2\left(\sigma_1'' - \frac{\sigma_1'^2}{\sigma_1}\right) + e\sigma_3\left(2\frac{\sigma_1''}{\sigma_1} - \frac{\sigma_1'^2}{\sigma_1^2}\right) + e\frac{\sigma_3'^2}{\sigma_3} = -4(\sigma_1 + e\sigma_3)^2.$$

Differentiating with respect to x_3, we obtain

$$2\frac{\sigma_1''}{\sigma_1} - \frac{\sigma_1'^2}{\sigma_1^2} + \left(\frac{\sigma_3'^2}{\sigma_3}\right)'\frac{1}{\sigma_3'} = -8(\sigma_1 + e\sigma_3).$$

Hence we have

$$2\frac{\sigma_1''}{\sigma_1} - \frac{\sigma_1'^2}{\sigma_1^2} + 8\sigma_1 = 4d, \qquad \left(\frac{\sigma_3'^2}{\sigma_3}\right)'\frac{1}{\sigma_3'} + 8e\sigma_3 = -4d,$$

where d is a constant, from which and (9.2) we have

$$\sigma_1'^2 = 4\sigma_1(f + d\sigma_1 - \sigma_1^2), \qquad \sigma_3'^2 = 4\sigma_3(ef - d\sigma_3 - e\sigma_3^2).$$

These expressions satisfy (9.1) for $i = 1, j = 3$ and $i = 2, j = 3$.

For $e = 1$, we have, on putting

$$f = a^2 b^2, \qquad d = b^2 - a^2, \qquad k = \frac{b}{\sqrt{a^2 + b^2}}, \qquad k' = \frac{a}{\sqrt{a^2 + b^2}},$$

$$\sigma_1 = b^2 \operatorname{cn}^2 (\sqrt{a^2 + b^2}\, x_1, k), \qquad \sigma_3 = a^2 \operatorname{cn}^2 (\sqrt{a^2 + b^2}\, x_3, k').$$

A real solution does not exist when $e = -1$. Hence we have by a proper choice of x_1 and x_3

$$\text{(IV)} \quad H_1^2 = H_3^2 = c^2 [k^2 \operatorname{cn}^2 (x_1, k) + k'^2 \operatorname{cn}^2 (x_3, k')],$$
$$H_2^2 = c^2 a^2 b^2 \operatorname{cn}^2 (x_1, k) \operatorname{cn}^2 (x_3, k').$$

For the case

$$H_1^2 = c^2 X_1 (x_1 - x_2)(x_1 - x_3), \qquad H_2^2 = c^2 X_2 (x_2 - x_1)(x_2 - x_3),$$
$$H_3^2 = c^2 X_3 (x_3 - x_1)(x_3 - x_2),$$

equation (9.1) for $i = 1, j = 2$ becomes

$$\frac{1}{X_3} + \frac{1}{(x_1 - x_2)^2}\Big\{(x_3 - x_2)^2\Big[(x_1 - x_3)\Big(\frac{1}{X_1}\Big)' - \Big(\frac{2(x_3 - x_1)}{x_2 - x_1} + 1\Big)\frac{1}{X_1}\Big]$$

$$(9.3) \qquad + (x_3 - x_1)^2\Big[(x_2 - x_3)\Big(\frac{1}{X_2}\Big)' - \Big(\frac{2(x_3 - x_2)}{x_1 - x_2} + 1\Big)\frac{1}{X_2}\Big]\Big\}$$

$$+ 4(x_3 - x_1)^2 (x_3 - x_2)^2 = 0.$$

Differentiating with respect to x_2, we obtain a polynomial of the third degree in x_3. Each of its coefficients must vanish. Equating to zero the coefficient of x_3^3, we have

$$(9.4) \qquad (x_1 - x_2)^2\Big(\frac{1}{X_2}\Big)'' + 4(x_1 - x_2)\Big(\frac{1}{X_2}\Big)' + 6\frac{1}{X_2} + 2(x_1 - x_2)\Big(\frac{1}{X_1}\Big)' - 6\frac{1}{X_1}$$

$$+ 8(x_1 - x_2)^4 = 0.$$

Differentiating twice with respect to x_2, we have

$$\Big(\frac{1}{X_2}\Big)^{\mathrm{IV}} + 96 = 0,$$

are consequently

$$\frac{1}{X_2} = -4x_2^4 + a_1 x_2^3 + a_2 x_2^2 + a_3 x_2 + a_4 \equiv f(x_2).$$

Thus from (9.4) we have $1/X_1 = f(x_1)$ and from (9.3) $1/X_3 = f(x_3)$. Hence we have finally

$$(\mathrm{V}) \qquad H_i^2 = \frac{c^2 (x_i - x_j)(x_i - x_k)}{f(x_i)} \qquad (i, j, k \neq),$$

$$f(x_i) = -4x_i^4 + a_1 x_i^3 + a_2 x_i^2 + a_3 x_i + a_4.$$

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