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The Classical Theories of Radiation Reaction*)

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This article presents a critical review of the classical theories of radiation reaction. The renormalized Dirac theory is discussed in particular detail. It is shown that all regularization prescriptions for this theory may be derived from a general dynamical correspondence principle. It is also shown that the physical content of the theory is severely limited by a general radiation condition. Various non-local and extended electron theories are classified with the help of the Herglotz-Wildermuth "runaway" theorem. We discuss the existence of electrodynamic collective modes; in particular the radiationless and self-excited states of extended charge structures. A possible connection with the existence of the μ-meson is pointed out. In the concluding section various problems raised by the phenomenological radiation reaction theory of Ginzburg and Eidman are considered.

Contents

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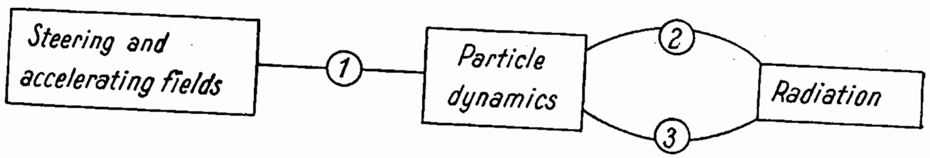
1. Introduction

The fundamental premise of all theories of radiation reaction is that there are some dynamic consequences inherently associated with every elementary act of radiation. If this concept is incorporated into the structure of ordinary electrodynamics then the resulting theory can be conveniently summarized in terms of the following block diagram:



The links (1) and (2) here represent the ordinary Lorentz force and the Liénard-Wiechert potentials. The effects of radiation reaction may in principle then be taken into account through the insertion of the "feedback" loop indicated by (3). In the overwhelming majority of practical cases the coupling to the radiation field may be considered sufficiently weak so that any iterative corrections through this feedback loop can be safely disregarded. There are however certain exceptional circumstances, for instance the self-excited states characteristic of electrodynamic collective modes (see Section 6-G), in which the cycling through the closed loop (2)⇄(3) is indeed the decisive element.

Quite apart from any direct experimental motivation, the idea of radiation reaction also enters into conventional field theories through the logical necessity of balancing the energy-momentum conservation laws. The radiation reaction theories considered in Section 2 are in fact all constructed on this basis. The essential point here is that the energy and momentum losses computed directly from the Liénard-Wiechert potentials (i. e. (2) above) may be exactly compensated through the addition of a suitable reactive force (F_{react}) to the ordinary Lorentz force law. Such a procedure has the attraction of being almost completely model-independent and also is easily adapted to various covariant generalizations. However there always remains an essential gap in this type of argument (Section 6-B) since the conservation laws of course do not uniquely specify any particular equation of motion.

A completely different line of approach, which avoids all these ambiguities, is explored in the various non-local and extended electron theories studied in Section 6. The basic point of departure in these cases is the observation that Newton's third law of action and reaction does not strictly apply to retarded action-at-a-distance theories. If for example the electron is visualized as being a sort of extended electromagnetic sponge, and if the validity of the Maxwell-Lorentz theory is extrapolated down to infinitesimal elements in its interior, then it is indeed possible to show that the net uncompensated intra-particle forces can give rise to "self-dragging" or radiation reaction effects.

Various modifications of these two major types of radiation reaction theories are discussed in several other sections of the paper. The principal conclusions of the present review are summarized in the paragraphs (i) — (iv) listed below.

(i) The Dirac Theory — Pro and Con: The basic equation of this theory is

$$F_{\text{ext}} + F_{\text{react}} = m\dot{u}\,; \quad F_{\text{react}} = \frac{2}{3}\,\frac{e^2}{c^3}\,\{\ddot{u} - u\dot{u}^2\}. \tag{1.1}$$

As is immediately evident from a consideration of the homogeneous case ($F_{\text{ext}} = 0$) this equation may contain physically unacceptable solutions. These are the so called „runaways" in which the electron self-accelerates to infinite energies. The occurrence of these non-physical solutions has given rise to a long history of controversy both about the domain of validity of (1.1), see for example [*86*], and also regarding the possibility of seeking subsidiary „regularizing" conditions that would automatically sort out the physical solutions. At the present time these controversies seem to have come to a stand-off: Every fresh objection or exception to the application of (1.1) has been met with a severer regularization method or some other stricture on the solutions. The question is therefore not so much one of isolating a mathematically well defined or physically acceptable sub-set of (1.1) but rather one of evaluating how ***rich*** the structure of this remaining theory actually is. In Section 4-B we introduce a general dynamical correspondence principle which permits a quantitative answer to this question. Given a particular F_{ext} it is immediately possible to evaluate the deviation of the "radiation reaction" trajectory [i. e. (1.1) including F_{react}] from the trajectory with $F_{\text{react}} = 0$. The extent of the permissible deviation is then directly related to the richness of the structure of the theory. Various specialized criteria yield all previously known regularizing prescriptions.

This dynamic comparison method is augmented by a general radiation condition in Section 5. This condition enforces the requirement that the radiation reaction work must (on the average) represent a transfer of energy *from* the particle *to* the radiation field. Analytically this condition can be stated in terms of a Fowler-Emden equation. A study of the characteristics of the solutions substantially verifies an old conjecture of Abraham that a proper energy transfer occurs only when the particle trajectories have a quasi-oscillatory component.

(ii) Runaway Solutions in Non-Local Electrodynamics: A number of covariant linear generalizations of the Maxwell-Lorentz theory have been proposed in which invariant form factors are introduced to eliminate the self-energy diver-

gences. (These methods are the precursors of the renormalization program in quantum electrodynamics [BOPP, STÜCKELBERG, MCMANUS, FEYNMAN].) It was first pointed out by LEHMANN that all of these form factor theories are essentially equivalent to a regularization (in the sense of PAULI and VILLARS) of ordinary electrodynamics with a continuum of negative energy meson fields. As a consequence the electrons in these theories all acquire a *negative* non-electromagnetic mass. A general result of HERGLOTZ and WILDERMUTH then shows that these form factor theories necessarily must contain runaway solutions. Similar properties have also been verified for a truncated version of quantum electrodynamics by VAN KAMPEN and NORTON and WATSON.
The original extended electron theories of ABRAHAM, LORENTZ, and SOMMERFELD on the other hand do not contain any runaway solutions. The mathematical reasons underlying this distinction are discussed in some detail in Section 6-F. It is emphasized that the occurrence of runaway difficulties in the form factor theories reopens the entire question of a proper relativistic generalization of the extended electron concept.

(iii) Collective Modes and Excited Charge States: Electrodynamics in principle contains a "strong coupling" limit corresponding to the possibility of processes dominated by the closed loop cycle $(2) \rightleftarrows (3)$ (see the block diagram above). These cycles represent transient collective modes in which radiative losses are greatly damped relative to their usual rates. Point particle theories of the kind represented by (1.1) do not contain a description of these strong coupling phenomena: The physical solutions in these theories are confined to two types of high frequency behavior — (1) either nothing at all happens as the frequencies are increased past the critical frequency $\sim mc^3/e^2$; or else (2) the solutions cease to exist altogether beyond this threshold. In order to obtain a non-trivial description of these collective modes it is necessary to introduce some kind of non-local electrodynamics. (This situation has some similarities to the "superconductor solution" problem in quantized field theories [NAMBU, GOLDSTONE].)
The basic equations of the non-local theories [e. g. (6.3) or (6.12b) of Section 6] show that these quasi-radiationless states can exist only at certain characteristic frequencies and with definite times of decay. This electromagnetic "isobar" spectrum is completely determined by the details of the particle form factors. Both classical and quantum mechanical estimates indicate that the ratio of the electromagnetic isobar energy to the ground state energy may be surprisingly large — of the order of 10^2 or greater.
Following a suggestion of BOHM and WEINSTEIN we discuss the implications of this theory for the elementary particle problem. In particular we present an electron isobar formula which predicts a mass value close to that of the μ-meson.

(iv) The Phenomenological Theory of Radiation Reaction (GINZBURG and EIDMAN): The dynamical effects of radiation reaction may become significant in connection with certain coherent radiative loss mechanisms in bulk matter: Cases of practical interest include the Cerenkov effect — in cavities as well as in matter; transition radiation, Cerenkov self- excitation, and the complex Doppler effect. The object of the Ginzburg-Eidman theory is to provide a unified description of the dynamical aspects in all these cases. In a sense this theory may be considered to be a macroscopic "boundary condition" on all the microscopic theories discussed previously. No real link between any of these theories has

however as yet been established. A point which may be of interest for future work is that the mathematical structure of the Ginzburg-Eidman theory most closely resembles that of the original Sommerfeld extended electron theory.

2. Radiation Reaction Theories Based on Conservation Laws

A. STEWART and LARMOR

The beginnings of radiation reaction go back at least as far as 1871 when BALFOUR STEWART [*1*], on the basis of qualitative thermodynamic arguments, came to the conclusion that a moving body must be subject to a retardation owing to its own radiation. Sir JOSEPH LARMOR took up the subject in an extended series of memoirs running from 1895 to 1927. Originally basing himself on a detailed aether theory and later relying on more general conservation arguments, he concluded that there should be a dissipative force accompanying radiation given by

$$\boldsymbol{F}_{\text{react}} = -\frac{R}{c^2}\,\boldsymbol{v}, \tag{2.1}$$

where R is the rate of radiation and $\boldsymbol{v}$ is velocity.¹) In its final form the reasoning was that since in general total force is given by

$$\boldsymbol{F}_{\text{total}} = \frac{d}{dt}(m\boldsymbol{v}) = m\,\frac{d\boldsymbol{v}}{dt} + \boldsymbol{v}\,\frac{dm}{dt},$$

and since in the case of a radiating body it is plausible that $dm/dt = -R/c^2$, this "extra" force component ought to be identified with a radiation reaction. LARMOR's formula will provide an interesting contrast in a subsequent discussion (Section 3-C).

B. The Radiating Oscillator

The modern work was initiated by PLANCK [*3*], ABRAHAM [*4*], and LORENTZ [*5*] with an analysis of the energy balance for a radiating oscillator. From the Liénard-Wiechert potentials it follows quite generally that the energy radiated in time T is

$$\frac{2}{3}\,\frac{e^2}{c^3}\int_t^{t+T} \ddot{x}^2\,dt. \tag{2.2}$$

The oscillator energy E is given by

$$E = \tfrac{1}{2}\,m\dot{x}^2 + \tfrac{1}{2}\,Kx^2.$$

The radiated energy loss is then presumed to be compensated by a decrease in E. Balancing the energy flow yields the requirement

$$\int_t^{t+T}\left\{\frac{dE}{dt} + \frac{2}{3}\,\frac{e^2}{c^3}\,\ddot{x}^2\right\}dt = 0. \tag{2.3}$$

¹) See his Collected Works [*2*]; especially vol. I, pp. 414—597, and vol. II, pp. 420—449.

One cannot, of course, from this alone draw any unique conclusions about the details of the oscillator dynamics. Nevertheless it is possible to get an equation of motion simply by setting the integrand of (2.3) equal to zero. This yields

$$m\ddot{x} = -Kx - \frac{2}{3}\frac{e^2}{c^3}\frac{\ddot{x}^2}{\dot{x}} \tag{2.4}$$

which is a non-linear equation. At the cost of some approximation it is also possible to obtain a linear equation. Suppose that (2.3) were rewritten in the form

$$\int_t^{t+T}\left\{\frac{d}{dt}\left(E + \frac{2}{3}\frac{e^2}{c^3}\dot{x}\ddot{x}\right) - \frac{2}{3}\frac{e^2}{c^3}\dot{x}\dddot{x}\right\}dt = 0.$$

Then imposing the restriction

$$(K/m)^{1/2} \ll \frac{mc^3}{e^2} \sim 10^{23}\,\mathrm{sec}^{-1},$$

it follows that

$$E \gg \frac{2}{3}\frac{e^2}{c^3}\dot{x}\ddot{x}.$$

Finally, neglecting this last term in the integrand, we obtain

$$m\ddot{x} = -Kx + \frac{2}{3}\frac{e^2}{c^3}\dddot{x}; \tag{2.5}$$

and this is a linear equation. (The consequences of the limit interchange are discussed in Section 3-B.)

We should like to take some care in exhibiting still another approach to this simple situation: Supposing one *insisted from the outset* that there were some force F_{react} which when added to $F = m\dot{v}$ would reproduce the radiation losses. Instead of (2.3) we should then write

$$\frac{2}{3}\frac{e^2}{c^3}\int_t^{t+T}\ddot{x}^2\,dt = -\int_t^{t+T}F_{\mathrm{react}}\,\dot{x}\,dt. \tag{2.6}$$

Integrating by parts on the left and choosing T such that $\ddot{x}$ vanishes at each end of the interval, there results

$$\int_t^{t+T}\left\{\frac{2}{3}\frac{e^2}{c^3}\dddot{x} - F_{\mathrm{react}}\right\}\dot{x}\,dt = 0.$$

Finally, in analogy with the preceding, we may set

$$F_{\mathrm{react}} \equiv \frac{2}{3}\frac{e^2}{c^3}\dddot{x} \tag{2.7}$$

to obtain a solution. Note that this is consistent with the previous result (2.4) but with the important difference that the frequency restriction $(K/m)^{1/2} \ll 10^{23}$ has fallen away.

Since the details of the nature of the oscillator do not seem to be involved in this argument, it is now inviting to assert that (2.7) is in fact a *generally valid* formula. This assertion may be formally stated in terms of two axioms:

(I) *The dynamical effects of radiation reaction may be taken into account through an augmented form of Newton's second law.*

(II) *One universal form of F_{react} is sufficient for all cases.*

The entire radiation reaction theory may then be summarized in terms of the single equation

$$F_{\mathrm{ext}} + \frac{2}{3}\frac{e^2}{c^3}\dddot{x} = m\ddot{x}. \tag{2.8}$$

C. Relativistic Generalizations

Still another reason for selecting (2.8) as the basic equation of a radiation reaction theory is its natural place in a relativistic setting. It is well known, for example, that when v/c is no longer a negligible quantity (2.2) must be replaced by

$$\frac{2}{3}\frac{e^2}{c^3}\gamma^4\int_t^{t+T}\left\{\ddot{x}^2 + \left[\frac{\gamma}{c}\dot{x}\ddot{x}\right]^2\right\}dt. \tag{2.9}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$. If one now retraces the steps leading from (2.6) to (2.7), it may be shown that the complete (vector) form of the result is (ABRAHAM[*4*])

$$\boldsymbol{F}_{\mathrm{react}} = \frac{2}{3}\frac{e^2}{c^3}\gamma^2\left\{\ddot{\boldsymbol{v}} + \left(\frac{\gamma}{c}\right)^2(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\,\boldsymbol{v} + 3\left(\frac{\gamma}{c}\right)^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\,\dot{\boldsymbol{v}} + 3\left(\frac{\gamma}{c}\right)^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\,\boldsymbol{v}\right\} \tag{2.10}$$

and this is exactly what would have been obtained by a straightforward Lorentz transformation of (2.7) (VON LAUE [*6*]). With the help of (2.10) it is then a simple matter to transcribe (2.8) into its fully covariant form. We obtain

$$F_k + \frac{2}{3}\frac{e^2}{c^3}\left\{\frac{d^2u_k}{ds^2} - u_k\left(\frac{du_i}{ds}\right)^2\right\} = m\frac{du_k}{ds} \tag{2.11}$$

where F_k and u_k are the Minkowski force and four velocity respectively. It should be noted that this involves more than the simple replacement $\dddot{x} \to d^2u/ds^2$; a point which in more formal treatments (e. g. [*7*]) requires special discussion.

In the case of one dimensional motion, relativistic modifications do not bring in any essentially new features. On very general grounds [*8*] it is to be expected that the change of variable

$$v = c\sinh(w/c) \tag{2.12}$$

formally reduces these equations to the non-relativistic case. Indeed substituting (2.12) into (2.11) we find that

$$F_{\text{ext}} + \frac{2}{3}\frac{e^2}{c^3}\ddot{w} = m\dot{w},$$

i. e. formal coincidence with the non-relativistic version except that now the proper time (s) has assumed the role of the independent variable.[2])

D. Dirac's Subtraction Formalism

The preceding considerations have been largely phenomenological. No direct attempt has been made to link the resulting radiation reaction equations to the Maxwell-Lorentz system; and it is in fact not even clear whether they are contained within this system or whether they involve additional assumptions. In 1938, DIRAC [*11, 12*], stimulated by the then prevailing divergence difficulties in quantum electrodynamics, attempted the construction of a relativistic radiation reaction theory from first principles. [The efforts to formulate a satisfactory classical model theory have continued since that time (e. g. [*13*], [*14*]).] There are two main points involved in DIRAC's approach:

(i) A mixing of retarded and advanced potentials to eliminate the divergences of the point electron theory.
(ii) A more careful derivation of the equations of motion from the conservation laws by means of Gauss' theorem.

We shall discuss these separately.

(i) Supposing one wanted the most direct approach to a radiation reaction equation. Within the Maxwell-Lorentz system the starting point then must be

$$e F_{kj} u_j = m \frac{d u_k}{d s} \tag{2.14}$$

since it is the only equation of motion available. The electromagnetic field tensor may be thought of as being split into two components

$$F_{kj} = F_{kj}^{\text{ext}} + F_{kj}^{\text{self}}, \tag{2.15}$$

i. e., an external field created by distant sources and a self-field due to the charge in question. For F_{kj}^{self} we may in turn make the ansatz

$$F_{kj}^{\text{self}} = F_{kj}^{\text{cc}} + F_{kj}^{\text{rad}}, \tag{2.15a}$$

where F_{kj}^{cc} represents the "convected" Coulomb field and F_{kj}^{rad} denotes that part of the field which asymptotically appears as radiation. At the position of the electron F_{kj}^{self} of course diverges and it would be meaningless to substitute (2.15) and (2.15a) back into (2.14). However we certainly know that the Coulomb part F_{kj}^{cc} has a divergence; in this case there is the hope that this is the only place where the infinities of F_{kj}^{self} appear and that F_{kj}^{rad} at the position of the charge

[2]) This reductions goes back to WESSEL [*9*]. Extensive applications may be found in PLASS [*10*].

remains finite. Since it is presumably only this additional radiation component which affects the motion, we may simply omit F_{kj}^{cc}, and tentatively assume

$$e F_{kj}^{\text{ext}} u_j + e F_{kj}^{\text{rad}} u_j = m \frac{d u_k}{d s} \tag{2.15b}$$

as the final equation. In this way the radiation reaction effects emerge as a direct and plausible consequence of the Lorentz equation (2.14).

A number of heuristic arguments are now available for evaluating F_{kj}^{rad}. For instance, if F_{kj}^{self} is assumed to be a real valued function of time with an appropriate range of definition than it is always possible to decompose it into even and odd parts, viz.

$$F_{kj}^{\text{self}} = \tfrac{1}{2}\{F_{kj}^{\text{self}}(+) + F_{kj}^{\text{self}}(-)\} + \tfrac{1}{2}\{F_{kj}^{\text{self}}(+) - F_{kj}^{\text{self}}(-)\}. \tag{2.15c}$$

$F_{kj}^{\text{self}}(+)$ corresponds to a retarded solution of Maxwell's equations and $F_{kj}^{\text{self}}(-)$ corresponds to an advanced solution. The even part of F_{kj}^{self} is then symmetric under time reversal and does not contribute to the radiation field (PAGE [*15*], NORDSTROM [*16*]). The odd part however does change its sign and this is a sufficient condition for describing energy dissipation.[3]) Identifying this term with F_{kj}^{rad} and evaluating it at the position of the electron indeed gives a finite result; *in fact the very same result which already stands in the previous relativistic formula* (2.11).

So far, so consistent. It should however be cautioned that the omission of F_{kj}^{cc} and the various field decompositions are heuristic devices which require careful justification (HAAG [*18*]) and are by no means the unique possibilities. [See for example ELIEZER [*19*], BERGMANN [*20*], and SCHÖNBERG [*21*] for variant theories.] A more systematic method of obtaining the equations of motion, without the Coulomb divergences, can be based on the conservation laws:

(ii) Consider a cylindrical region Ω of space time pierced by a thin tube of (invariant) radius r containing the world line of an electron. Assuming Ω to be source free, the divergence of the energy-momentum tensor vanishes in this region, and by Gauss' theorem

$$\int \frac{\partial T_{jk}}{\partial x_k} d\Omega = \oint T_{jk}\, d\sigma_k = 0 \tag{2.16}$$

where σ_k denotes the surface of Ω. Neglecting the contributions from remote space-like regions, (2.16) simply equates the energy-momentum flux across the tube surface with the net flow at the time-like ends of Ω. The tube contribution may be expessed in terms of a single line integral extended along its length

$$\int \left\{ e u_j [F_{kj}^{\text{ext}} + F_{kj}^{\text{rad}}] - \frac{1}{2}\frac{e^2 \dot{u}_k}{r} + O(r) \right\} ds, \tag{2.16a}$$

where the second (divergent) term represents the flux due to the Coulomb field F_{kj}^{cc}. If we impose the condition that the integrand be a perfect differential, i. e.,

$$e u_j [F_{kj}^{\text{ext}} + F_{kj}^{\text{rad}}] - \frac{1}{2}\frac{e^2 \dot{u}_k}{r} + O(r) = \dot{B}_k \tag{2.16b}$$

[3]) On this point see SCHWINGER [*17*]. The dissipative nature of the radiation reaction does not however introduce any intrinsic irreversibility into electrodynamics; see PLANCK [*3*].

for some four vector B_k, then (2.16a) depends only on the flux at the ends of Ω. However just as in the previous discussions (Section 2—B) this is not sufficient to determine an equation of motion since the conservation laws do not determine a unique choice of B_k (BHABHA [22]). A plausible selection which has the virtue of simplicity is

$$B_k = \left(m - \frac{1}{2}\frac{e^2}{r}\right) u_k. \tag{2.16c}$$

This makes possible the transition to a point electron ($\lim r \to 0$) on both sides of (2.16b) and yields an equation of motion which once again coincides with (2.11). It is apparent that this procedure makes the F^{cc}_{kj} subtraction less arbitrary and also shows clearly that it has the character of a mass renomalization. An important feature of this approach is that since no characteristic thresholds appear in the derivation, there is hope that (2.11) can in a sense be considered an *exact* equation.[4])

3. Discussion of the Solutions

A. Physical and Non-Physical Solutions

In the preceding section we have listed a number of arguments which converge on the expression

$$F_{\text{ext}} + \frac{2}{3}\frac{e^2}{c^3}\dddot{x} = m\ddot{x} \tag{3.1}$$

and its relativistic analogue as being *the exact classical equations of motion including radiation reaction.* In the present section we shall not present any further justifications of (3.1), but rather accept it as the final statement of a theory and turn to the consideration of various predictions and consequences which follow from it.

We first remark that due to the general feebleness of the effects of radiation reaction (cf. the remarks in the Introduction) it is to be expected that solutions of (3.1) should be very close to those of

$$F_{\text{ext}} = m\ddot{x}, \tag{3.1a}$$

i. e. the usual equation of motion omitting F_{react}. However, the addition of an $\dddot{x}$ term to (3.1a) not only perturbs its solutions, it also adds entirely new solutions which may or may not have a physical significance. In order to see this more clearly, let us assume for the moment that the Laplace transforms of (3.1) and (3.1a) exist. Introducing the notation

$$v' = \int_0^\infty v e^{-pt}\, dt, \quad \text{etc.,}$$

for the transforms; and setting $2e^2/(3mc^3) \equiv \tau$ we have for the Laplace transform of (3.1)

$$\frac{1}{m} F'_{\text{ext}} + p^2 \tau \ddot{v}' = p\dot{v}'. \tag{3.2}$$

[4]) This was in fact Dirac's original hope. For his second thoughts on the matter see [23]. Generalizations of the ansatz (2.16c) have been discussed by ELIEZER [24]. See also Section 5.

Solving for p, we find

$$p = \frac{\left(\dot{v}' \pm \left\{(\dot{v}')^2 - 4\frac{\tau}{m}\ddot{v}' F'_{\text{ext}}\right\}^{1/2}\right)}{(2\tau\ddot{v}')}; \tag{3.2a}$$

and this clearly shows the existence of the extra solution. In the limit $\tau \to 0$, the solution with the minus sign goes over into the equivalent of (3.1a) and therefore may be considered as the "physical" solution; the plus sign yields a solution that diverges as $\tau \to 0$ and represents a parasitic or 'runaway' solution. This convenient identification fails to extend to the general case however. If for instance F_{ext} is specified as a function of x and not of t, then (3.2) must be understood as an integral equation for $x(t)$. F'_{ext} may then acquire an explicit τ dependence. In this event

$$\lim_{\tau \to 0}\left\{-4\frac{\tau}{m}\ddot{v}' F'_{\text{ext}}\right\} \tag{3.2b}$$

is no longer necessarily zero and neither solution of (3.2a) is constrained to reduce to a solution of (3.1a). These and related features may best be discussed with the help of a number of specific examples.[5])

B. The Damped Harmonic Oscillator

This problem has been considered in some detail by LORENTZ [5], PLANCK [3], PLASS [10], and LOINGER [26]. If we set $F_{\text{ext}} = -m\omega_0^2 x$ and continue with the abbreviation $\tau = 2e^2/3mc^3$ then (3.1) becomes

$$\tau\dddot{x} - \ddot{x} - \omega_0^2 x = 0. \tag{3.3}$$

The general solution may be written as

$$x = \sum_{j=1}^{3} C_j e^{l_j t} \tag{3.3a}$$

where the l_j's are the roots of the characteristic equation

$$\tau l^3 - l^2 - \omega_0^2 = 0. \tag{3.3b}$$

This cubic always has one positive real root and two conjugate complex roots with negative real parts. For the real root, we have

$$l_1 = (A + B)^{-1} \tag{3.3c}$$

where it has been convenient to introduce the abbreviations

$$\left.\begin{matrix} A \\ B \end{matrix}\right\} = \left[\frac{\tau}{2\omega_0^2} \pm \sqrt{\left(\frac{\tau}{2\omega_0^2}\right)^2 + (27\omega_0^6)^{-1}}\right]^{1/3}. \tag{3.3d}$$

It is easily seen that in both of the limiting cases $\omega_0 \ll \tau^{-1}$ ($\Rightarrow l_1 \sim \tau^{-1}$), and $\omega_0 \gg \tau^{-1}$ ($\Rightarrow l_1 \sim [\omega_0^2/\tau]^{1/3}$); l_1 represents a catastrophically rapid "runaway"

[5]) From the mathematical point of view, these difficulties stem from the non-uniformity of the $\tau \to 0$ limit. A rigorous discussion of this aspect of the Dirac equation may be found in [87]. Formally the situation is closely analogous to the small viscosity limit of the Navier-Stokes equation [25].

solution. [This illustrates the remarks following (3.2a).] Imposing now the initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$, it is possible to *suppress* the runaway by choosing $C_1 = 0$. The complete "physically reasonable" solution of (3.3) then may be written in the form

$$x = \frac{v_0}{\omega} \cdot e^{-\alpha t} \sin \omega t \tag{3.3e}$$

with

$$\alpha = \frac{\omega_0^2}{2} (A + B)^2; \qquad \omega = \frac{\sqrt{3}\,\omega_0^2}{2\tau} (A^2 - B^2)$$

where A and B are as given in (3.3d) above. It should be noted that all the constants of integration have now been exhausted; the value of $\ddot{x}(0)$ is automatically fixed at $-2\alpha v_0$ by the choice of the other initial conditions including also the elimination of the $e^{l_1 t}$ solution.
We shall return to this point later.
The extremes of large and small ω_0 produce some simplifications in (3.3d) and (3.3e); we find:

	$\omega_0 \ll \tau^{-1}$	$\omega_0 \gg \tau^{-1}$
α	$\frac{1}{2}\tau\omega_0^2$	$\frac{1}{2}\left(\frac{\omega_0^2}{\tau}\right)^{1/3}$
ω	ω_0	$\frac{\sqrt{3}}{2}\left(\frac{\omega_0^2}{\tau}\right)^{1/3}$

(3.3f)

A remarkable feature of these solutions, which can easily be read off the structure in the extreme cases, is that *despite the presence of a natural threshold in this problem*, i. e., τ^{-1}, *the nature of the solutions does not change even for the highest* ($\omega_0 \gg \tau^{-1}$) *values of* ω_0. This is in marked contrast to the behavior in other cases, e. g. the potential step (Section 3-D), where a characteristic threshold *does* make itself felt and plays the role of an upper limiting cut-off for the validity of the classical theory. However the structure of the cubic (3.3b) is sufficiently trivial so that neither "critical" frequencies, nor fields, or distances can affect the behavior of the oscillator. Moreover one can check directly that the details of the energy balance

$$\tau m \int_0^\infty \ddot{x}\,\dot{x}\,dt = -\tfrac{1}{2}\, m v_0^2 \tag{3.3g}$$

are completely insensitive to the values assumed by ω_0.[6])
It is interesting to contrast this with the behavior of the alternative 'energy' equation (2.4) which also admits of solutions of this type. In this case the characteristic equation corresponding to (3.3b) becomes

$$\tau l^3 + l^2 + \omega_0^2 = 0.$$

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[6]) This elegant treatment of the oscillator is due to A. COSTIKAS.

This cubic has a negative real root and two conjugate complex roots with positive real parts. In the $\omega_0 \ll \tau^{-1}$ limit

$$l_1 = -\tau^{-1}; \quad l_{2,3} = \frac{\tau\omega_0^2}{2} \mp i\omega_0$$

so that the solutions represent either a tremendously overdamped decay or a slowy *increasing* oscillation.

C. Time-Dependent External Forces

If F_{ext} is specified explicitly as a function of time, then the motion may be obtained from (3.1) formally by quadratures. The general result is

$$x = x_0 + v_0 t + a_0 \tau^2 \left\{e^{t/\tau} - \left(1 + \frac{t}{\tau}\right)\right\} - \\ - \frac{e^{t/\tau}}{m\tau} \int_0^t \int_0^{t_1} \int_0^{t_2} e^{-t_1/\tau} F_{ext}(t_3)\, dt_3\, dt_2\, dt_1 \tag{3.4}$$

where x_0, v_0, and a_0 denote the position, velocity, and acceleration at $t = 0$ respectively. The 'runaway' factors $e^{t/\tau}$ signalize the presence of the unphysical solutions discussed previously [see (3.2a)]. In analogy to the treatment of the oscillator we shall see that it is possible in general to separate out the dynamically acceptable trajectories by proper adjustment of the initial conditions. Consider first the simple case of a step function force

$$F_{ext} = \begin{cases} 0 & \text{for} \quad t < 0 \\ F_0 & \text{for} \quad t \geq 0. \end{cases} \tag{3.5}$$

From the general formula (3.4) one can immediately read off the solutions (note $x_0 = 0$),

$$t \geq 0; \; x = v_0 t + \frac{F_0}{2m}\{t^2 + 2t\tau + 2\tau^2\} + \tau^2 e^{t/\tau}\left\{a_0 - \frac{F_0}{m}\right\} - a_0\tau(\tau - t) \tag{3.5a}$$

$$t < 0; \; x = v_0 t + a_0 \tau^2 \left\{e^{t/\tau} - \left(1 + \frac{t}{\tau}\right)\right\}. \tag{3.5b}$$

Clearly (3.5a) has runaway components. From the structure of the solution it is however evident that these may be cancelled out through a proper choice of a_0, that is $a_0 = F_0/m$. With this choice and the auxiliary condition $\dot{x}(\to \infty) = 0$, the dynamically reasonable portion of (3.5) may be rewritten as

$$x = \begin{cases} v_0 t + \dfrac{F_0}{2m} t^2; & t \geq 0 \quad (3.5c) \\ \dfrac{F_0}{m} \tau^2 \{e^{t/\tau} - 1\}; & t < 0. \quad (3.5d) \end{cases}$$

From a physical point of view this implies that we have managed to get rid of the divergent solutions at the cost of introducing a slight acausality in the electron's motion. Since this electron 'pre-acceleration' involves only distances comparable to the electron radius (or times of the order τ) it is actually more of a conceptual departure than an experimental difficulty. In fact this ties in very nicely with the elimination of the "extra" boundary condition (i. e., a_0) introduced by the $\dddot{x}$ term in (3.1). The proper 'phasing out' of the diverging solutions provides a natural employment for this otherwise superfluous element.

Although this procedure is dynamically acceptable, it still does not yield a completely reasonable solution. The root of the trouble lies in (3.5c): Not only has the adjustment of a_0 removed the runaways, it has in fact removed all the traces of the radiation damping. The trajectory (3.5c) is completely identical to that given by the usual form of Newton's laws (3.1a). We have $\ddot{x} = 0$ for $t \geq 0$, and consequently the work of radiation reaction vanishes identically. If we simultaneously accept the fact that the accelerated charge radiates then it seems as if the energy balance (2.6) no longer describes the situation.

This is a non-trivial problem, having some bearing on the equivalence principle [7]), which has been discussed extensively in the literature (FULTON and ROHRLICH [28], ROHRLICH [88] and further references cited therein). There are a number of authors (e. g. DRUCKEY [29]) who have attempted to rectify this situation by an artifice resting on an integration by parts: i. e.,

$$\int_{t_1}^{t_2} F_{\text{react}}\, \dot{x}\, dt = \dot{x}\, \ddot{x}\,\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \ddot{x}^2\, dt. \tag{3.6}$$

The interpretation attached to this equation is the following: The first term on the right hand side is supposed to represent an *influx* of energy from the field in the vicinity of the particle—the so called "acceleration" or SCHOTT [30] energy—which then reappears in the second term as *radiation* to the far field zone of the particle. Since the two terms balance, F_{react} may be zero although there is still radiation. This interpretation is however clearly contrary to the essential spirit of the radiation reaction development to this point. We have consistently sought to identify the origin of radiated energy in the work done *by* the particle *on* the field: In the 'acceleration energy' argument the accelerated particle becomes merely some kind of transducer which transforms near field energy into far field energy. It should be noted further more that even the most detailed studies of the structure of the electromagnetic field in the case of uniform acceleration (SOMMERFELD [31], LINDEMANN [32]) have failed to turn up any trace of this kind of behavior.[8])

This interpretation becomes even less tenable if we consider a slight modification of (3.5), viz.,

$$F_{\text{ext}} = \begin{cases} 0 & \text{for} \quad t < 0 \\ F_0 t & \text{for} \quad t \geq 0. \end{cases} \tag{3.7}$$

In this case the solutions still show signs of radiation reaction, even after the excision of the runaways, but the modifications are in the *wrong* direction. One may easily verify that $\int F_{\text{react}}\, \dot{x}\, dt$ has now become a positive quantity so that

[7]) B. S. DE WITT and R. W. BREHME [27]; F. ROHRLICH (unpublished); S. COLEMAN, private communication.

[8]) This point has also been emphasized by ROHRLICH, [88] and private communication.

in effect there is work done *on* the particle *by* the field; the kinetic energy is increased at the expense of the field in excess of the work supplied by (3.7).

There is of course another way of meeting this problem, and that is to simply agree that $F_{\text{react}} = 0$ implies no radiation at all. (See for example [10], p. 27.) In the context of the uniform acceleration problem this may appear to be an absurd conclusion but it certainly is a consistent consequence of the radiation reaction axioms listed in Section 2-B. It is in fact precisely the conclusion that has prevailed for years with respect to the hyperbolic motion of special relativity. (VON LAUE [33], BORN [34], PAULI [35]) If one calculates the relativistic form of the radiation reaction (2.10) for the hyperbolic motion one eventually finds that $F_{\text{react}} = 0$. From this it has often been concluded that the hyperbolic motion does not give rise to radiation. From our point of view it is evident that these two problems, the relativistic and the non-relativistic constant force, are simply related by the hyperbolic transformation (2.12). It then becomes clear that such "proofs" of the radiationless character of the hyperbolic motion are about as sound as the corresponding claims for the non-relativistic case. FULTON and ROHRLICH [28] have recently given an extensive summary of this situation and by a direct recalculation of the fields, using a covariant generalization of the Poynting vector, have finally confirmed the existence of radiation. The "no radiation" hypothesis is therefore untenable and the physical acceptability of (3.5c and d) is in doubt.

By contrast, the Larmor formulation (2.1) seems to do very well in providing reasonable trajectories for (3.5): If for example the radiation rate R is assumed constant then the particle will simply follow a velocity damped trajectory. In the more realistic case that R is identified with $\ddot{x}^2$, i. e. (2.2), then (2.1) and (1.2) lead to the non-linear equation

$$\frac{1}{m} F_0 - \frac{\tau}{c^2}\, \dot{x}\, \ddot{x}^2 = \ddot{x}. \tag{3.8}$$

The variables may be separated with the result

$$\frac{dv}{dx} = -\frac{c^2}{2\tau v^2}\left\{1 \pm \left[1 + 4\,\frac{\tau F_0}{mc^2}\, v\right]^{1/2}\right\}.$$

Integrating, using the initial condition $x = 0 \Rightarrow v = v_0$; and introducing $\alpha \equiv 4 F_0 \tau / mc^2$, one finds

$$x = \frac{m}{2F_0}\left\{\frac{v^2}{2} \mp \frac{2}{15\alpha^2}\,(3\alpha v - 2)\,(1 + \alpha v)^{3/2} + (\text{terms in } v_0)\right\}. \tag{3.8a}$$

In order to identify the physical solutions we consider again the limit $\tau \to 0$. In this case (3.8a) reduces to

$$x = \frac{m}{2F_0}\begin{cases} 0 \\ v^2 - v_0^2, \end{cases}$$

and evidently the lower (+) sign is the appropriate choice. In the 'strong' radiation limit $\alpha v \gg 1$ (3.8a) becomes

$$v \sim x^{2/5}$$

which clearly exhibits the radiation damping in contrast to the usual $v \sim x^{1/3}$ behavior. From (3.8) it also follows that ultimately $R \sim x^{-2/3}$, i. e., the radiation losses slowly decrease.

D. The Potential Step

Our examples so far have illustrated various dynamical difficulties associated with the appearance of divergent runaway solutions as well as energy difficulties associated with the behavior of the radiation reaction work. There are still other difficulties latent in the $\dddot{x}$ feature of (3.1) as elegantly exhibited in an example originally due to BOPP [13].

Consider a potential step of height V_0 situated at $x = 0$. If we integrate the equation of motion

$$\ddot{x} - \tau \dddot{x} = -\frac{1}{m}\frac{dV}{dx} \tag{3.9}$$

across the potential step from left to right and assume the continuity of $\dot{x}$ and x, then we find the condition that

$$a_0(0^+) - a_0(0^-) = \frac{V_0}{m\tau v_0}$$

where $a_0(0\pm)$ represents the accelerations immediately before and after $x = 0$, and $v_0 = \dot{x}(0)$. Let $x(0) = 0$ and designate the initial velocity of the particle by $v_i[= x(-\infty)]$. With these conditions (3.4) may be rewritten as

$$x = v_i t + \tau(v_0 - v_i)(e^{t/\tau} - 1); \quad t \leq 0 \tag{3.9a}$$

$$x = \left(v_i - \frac{V_0}{mv_0}\right) t + \tau\left(v_0 - v_i + \frac{V_0}{mv_0}\right)(e^{t/\tau} - 1); \quad t > 0. \tag{3.9b}$$

If the particle is to penetrate the potential step and continue towards the right, the runaway component of (3.9b) must be eliminated. From (3.9b) it becomes clear however that this requirement will not specify a unique trajectory. Setting

$$v_0 - v_i + \frac{V_0}{mv_0} = 0$$

in order to suppress the exponential, we have

$$v_0 = \frac{v_i}{2}\left\{1 \pm \left[1 - \frac{4V_0}{mv_i^2}\right]^{1/2}\right\}, \tag{3.10}$$

and therefore two values of v_0 are compatible with all the restrictions so far. In particular, if $4V_0/mv_i^2 \ll 1$, it follows from (3.10) and (3.9b) that

$$\dot{x}(t > 0) = \begin{cases} v_i - \dfrac{V_0}{mv_i} & (3.11\text{a}) \\[2ex] \dfrac{V_0}{mv_i} & (3.11\text{b}) \end{cases}$$

are both possible final velocities of the particle. In terms of the dynamical criteria advanced so far it is impossible to distinguish one of these as being more reasonable than the other.

If we now draw on energy considerations (3.11a) appears to be a somewhat more plausible choice: One can easily verify that the kinetic energy behind the potential step is diminished essentially by V_0 with a small additional amount ($\sim V_0/mv_i^2$) being expended in radiation. (3.11b) on the other hand represents a motion in which practically all the energy is radiated away at the potential step.⁹) We note further that (3.10) imposes $\frac{1}{2}mv_i^2 \geq 2V_0$ as a minimum energy requirement.

In order to eliminate any artificialities that might be due to the abrupt rise of the potential, HAAG [18] has extended this investigation to the case of a uniformly increasing potential. He finds the interesting circumstance, reminiscent of the Klein paradox, that as long as the potential rise is a gradual one over a distance of the order of $c\tau$, the duplicity of solutions may be avoided. Here again one encounters the elusive feature of a critical distance, or "electron radius", in a formally point electron theory. WESSEL [9], STÜCKELBERG [37] and FEYNMAN [38] have in fact used this multiple solution property of (3.9) to discuss the possibility of a classical description of pair production.

Another curious feature displayed by this example is that in the case of solutions of the type (3.11b) there is an invariant upper limit to the amount of kinetic energy that may be transmitted across a potential barrier. This follows readily from (3.10) and (3.11b) since together they imply the inequality $\dot{x}(t > 0) \leq [m/4V_0]^{1/2}$. POMERANCHUK [39] has elaborated this argument into one of the few non-trivial applications of the radiation reaction theory. He showed that if the potential barrier is identified with the earth's magnetic field, then in the ultra-relativistic case any electronic "primary" cosmic ray component would be limited to energies of less than 10^{17} ev at the eath's surface—*independent* of the magnitude of the initial energy. This point is also connected with the interesting possibility of a vacuum Cerenkov effect.

E. Singular Cases

The particulary interesting possibility that *no* solution of the radiation reaction equation resembles the solutions of the corresponding equation omitting F_{react} was first discussed by ELIEZER [24]. From our previous remarks (see 3.2b *et seq*) it follows that this is equivalent to constructing an $F_{\text{ext}}(x)$ which when re-expressed as a function of time acquires a singular τ-dependence. ELIEZER showed that the one dimensional Coulomb force satisfies this condition and in particular that radiation reaction will cause two unlike changes to *repel* sufficiently so that no collision is ever possible. This highly paradoxical result has occasionally been dismissed on the grounds that the Coulomb potential is too singular to admit solutions near its origin. It has also been argued that quantum mechanical effects can be trusted to intervene at sufficiently small distances to perturb the classical idealization in the right direction. It is difficult to agree with these points of view for at least two reasons:

(1) Despite the singular character of the Coulomb potential it is only necessary to change it from an attractive to a repulsive potential in order to introduce solutions representing collisions (ZIN [40]). Solutions approaching the origin *are*

⁹) A detailed treatment of the relativistic δ-function force has been given by ASHAUER [36].

possible but just under those circumstances directly contrary to what one would ordinarily expect.

(2) The use of a quantum mechanical argument at this stage would be an inversion of the logic of the development. The difficulties underlying the classical theories are to a large measure shared by the quantized versions. On a later occasion (Section 6-F) we shall in fact use some of the insight gained from our examination of the classical theories to comment on various "runaway" features which also appear in quantum electrodynamics. (See especially footnote 21.)

A number of other singular cases involving both bounded and unbounded potentials have been discussed by WILDERMUTH [41].

4. Physical Solutions: The General Case

A. Regularization and Perturbation Methods

From the preceding it has become clear that we cannot accept all solutions of the radiation reaction equations as physically meaningful; some further procedure must be adjoined to the integrations in order to generate the physically reasonable solutions. In this connection, a number of schemes have been proposed by HAAG, WILDERMUTH, CINI, ELIEZER and others. We may group these efforts roughly as follows: (1) Regularization Procedures; and (2) Perturbation Methods.

(1) Regularization Procedures: We have already seen by means of a number of examples that parasitic runaways are persistent features of (2.8) and (2.11). We have also seen that a proper adjustment of the initial acceleration can phase out the divergent components. HAAG [18] and later PLASS [10] have described a method which produces this phasing automatically.

Consider (2.8) integrated once with respect to the time in the following way

$$\ddot{x} = e^{t/\tau} \left\{ a_0 - \frac{1}{m\tau} \int_0^t e^{-t/\tau} F_{\text{ext}}\, dt \right\}. \tag{4.1}$$

If in the limit $t \to \infty$, $\ddot{x}(t)$ is to remain finite it clearly is sufficient to choose

$$a_0 = \frac{1}{m\tau} \int_0^\infty e^{-t/\tau} F_{\text{ext}}\, dt. \tag{4.1a}$$

Entering this back into (4.1) we may rewrite it in the compact form

$$\ddot{x} = \frac{1}{m\tau} \int_t^\infty e^{-(t-t_1)/\tau} F_{\text{ext}}(t_1)\, dt_1, \tag{4.2}$$

which displays the fact that the radiation reaction makes the particle acceleration proportional to the (incomplete) Laplace transform of the force — probing ahead an interval $\sim \tau$ in time. This felicitous interpretation has encouraged some authors to advance (4.2) as the fundamental equation of the theory.[9a]) As yet,

[9a]) The relativistic generalization of (4.2) is extensively discussed in [88]; some related questions regarding the existence of solutions are examined in [87].

however, no physical argument has been given which would lead to such a result directly. It may be noted that the transition to (4.2) has an equivalent interpretation as a standard transformation in the theory of integral equations, i. e., the incorporation of boundary conditions by means of quadrature. It should not be overlooked that such a regularization procedure copes with only a part of the difficulties encountered in constructing physically meaningful solutions. Similar remarks also apply to the Fourier transform method suggested by WILDERMUTH [41] and to the asymptotic regularization of ELIEZER and MAILVAGANAM [42]. See also GORA [43].

(2) Perturbation Methods: This is in principle a less satisfactory approach than the regularization procedure since from the outset all other considerations are subordinated to the aim of manufacturing reasonable solutions for (3.1) through perturbations of (3.1a). In particular this carries the penalty that occasionally one will construct perturbation solutions *which actually have no counterparts among the solutions of the exact problem.* [See also the remarks following (3.2a).] The approximation schemes in such cases may in fact conceal important features of the underlying theory.

In the construction of these solutions one has the option of carrying out the τ-perturbations either in the differential equation itself [e. g. (2.8)] or in its solutions. STEINWEDEL [44] and BHABHA [45] have noted that the unwanted components are usually characterized by terms in $e^{t/\tau}$ which are non-analytic in τ. They have therefore suggested following the second method: The solutions are to be obtained exactly from (2.8) and then examined for analyticity about $\tau = 0$; all singular terms are then to be discarded. A slight variant of this has also been proposed by CINI [46]. ARLEY [85] has however criticized these recipes as being excessively stringent since too many otherwise well behaved functions would be excluded. Our previously acceptable "physical" solutions (3.5c and d) provide an example of this since they clearly could not survive the analyticity test: The perturbation solution here would actually converge to a discontinuous trajectory!

The other alternative, i. e. modifying directly the differential equation, is actually the simplest and most foolproof perturbation technique. The radiation reaction term $m\,\tau\,\dddot{x}$ is simply evaluated from the unperturbed equation $F_{\text{ext}} = m\ddot{x}$ and the exact equation (2.8) replaced by

$$F_{\text{ext}} + \tau F_{\text{ext}} = m\ddot{x} \tag{4.3}$$

(or its relativistic analogue). This automatically yields the perturbed trajectories and voids all infinities *ab initio*. Indeed, judging by some of the current literature, for example [47], it is in fact (4.3) and not the prickly (2.8) which is the basic equation of the theory.

Field theoretical perturbation schemes have also been discussed by STEINWEDEL [14] and PRIGOGINE and LEAF [53].

B. A Dynamical Correspondence Principle

We shall now show how most of the points of the preceding discussion can be summarized in terms of a precise mathematical argument. Consider again

$$m\ddot{x}_1 = F_{\text{ext}}, \tag{4.4a}$$

and the radiation reaction equation

$$-\tau m \dddot{x}_2 + m \ddot{x}_2 = F_{\text{ext}}, \tag{4.4b}$$

which from this point on we shall regard as its "degenerescent" partner.[10]) What we should really like to study is the extent of the correspondence between the solution-sets of these two equations.

Let us introduce a deviation function $h(t)$ defined by

$$h(t) = x_2(t) - x_1(t). \tag{4.5}$$

It is also convenient to suppose that the initial conditions of (4.4a and b) have been matched at $t = 0$ by requiring

$$x_1(0) = x_2(0) \quad \text{and} \quad \dot{x}_1(0) = \dot{x}_2(0). \tag{4.5a}$$

Introducing now the formal solutions of (4.4a and b) into the right hand side of (4.5), we find

$$h(t) = -\frac{1}{m} F^{(2)}_{\text{ext}} + a_0 \tau^2 \left\{ e^{t/\tau} - \left(1 + \frac{t}{\tau}\right) \right\} - \frac{1}{m\tau} \int_0^t e^{(t-t_1)/\tau} F^{(2)}_{\text{ext}}(t_1)\, dt_1, \tag{4.6}$$

where $F^{(2)}_{\text{ext}}$ denotes the (indefinite) repeated integral of F_{ext}, and a_0 stands for $\ddot{x}_2(0)$. The information contained in this equation is actually better displayed by inverting it to give $F^{(2)}_{\text{ext}}$ in terms of h. From this point of view (4.6) is really an integral equation. Its solution may be written as

$$F^{(2)}_{\text{ext}} = \frac{m}{\tau} \int_0^t h(t)\, dt - mh + ma_0 \frac{t^2}{2}, \tag{4.7}$$

and this we shall regard as the basic expression of the dynamic correspondence between (4.4a and b).

The principle advantage of reformulating the problem in this way is that it preserves the freedom as to what may be considered a reasonable deviation between the solutions of (4.4a) and (4.4b). In particular, with a given bound on $h(t)$, it is possible to make a value judgment as to just how *rich* the structure of the theory actually is. To illustrate this point consider the extreme case

$$|h(t)| \leq K \quad \text{for all } t. \tag{4.8}$$

From (4.7) it follows immediately that

$$|F^{(2)}_{\text{ext}}(t)| \leq K t^2. \tag{4.8a}$$

At most therefore $F^{(2)}_{\text{ext}} \sim t^2$. If we now assume in addition that $t \cdot F_{\text{ext}}$ is continuous, increasing, and positive for $t \geq 0$, then by a standard TAUBERIAN theorem

[10]) The terminology is adapted from the degeneration theory of differential equations (MINORSKY [*48*]).

(e. g. [*49*]) we have the strong conclusion that

$$F_{\text{ext}}(t) \sim 1. \tag{4.8b}$$

The uniform boundedness of h is obviously a very severe restriction.[11]) Equation (4.7) also includes most of the preceding regularization schemes. If for instance $F_{\text{ext}} = F_0$, then (4.7) may be put into the form

$$h' - \frac{1}{\tau} h = (ma_0 - F_0)\, t \tag{4.9}$$

which integrates to give

$$h = \tau (ma_0 - F_0) \left\{ e^{t/\tau} - 1 - \frac{t}{\tau} \right\} \tag{4.9a}$$

and clearly unless $a_0 = F_0/m$, that is, $h = 0$, there will be a runaway deviation. The same result could also have been obtained directly from (4.6).

A more general situation is

$$F_{\text{ext}}(t) \not\equiv 0 \quad \text{for} \quad 0 < t < t_{\max}, \quad \text{and} \quad F_{\text{ext}}(t) = 0 \quad \text{otherwise;}$$

i. e. an external force acting during a finite time interval. In this case we have

$$h' - \frac{1}{\tau} h = a_0 t - \frac{1}{m} \int_0^t F_{\text{ext}}(t)\, dt. \tag{4.9b}$$

The complete integration yields

$$h(t) = \frac{\tau}{m} \int_0^{t_{\max}} F_{\text{ext}}(t)\, dt - a_0 \tau^2 + \tau^2 e^{t/\tau} \left\{ a_0 - \frac{1}{m\tau} \int_0^{t_{\max}} e^{-t/\tau} F_{\text{ext}}(t)\, dt \right\}. \tag{4.9c}$$

In order to keep the deviation within reasonable bounds, clearly a_0 must be adjusted to

$$a_0 = \frac{1}{m\tau} \int_0^{t_{\max}} e^{-t/\tau} F_{\text{ext}}(t)\, dt, \tag{4.9d}$$

and in the limit $t_{\max} \to \infty$ this coincides exactly with the previous recipe, i. e. HAAG's regularization.

This can be contrasted with the corresponding comparison for the perturbation equation (4.3): In this case it is easy to see that the deviation function is given by

$$x_2 - x_1 = \frac{\tau}{m} \int F_{\text{ext}}(t)\, dt.$$

Clearly, for reasonable F_{ext}, runaway deviations such as those occurring in (4.7) are avoided. In particular, as $\tau \to 0$, the deviation disappears, as is consistent with the perturbation approach.

[11]) Analogous arguments can be carried through for weaker versions of (4.8).

C. Instability of Solutions

The solutions of the radiation reaction equation (2.8) have a further undesirable dynamical property and that is that *none of them are stable!*
In the case of linear forcing functions this is equivalent to a well known result which goes back to the classical Routh-Hurwitz criteria. The general situation has recently been discussed by EZEILO [50]. This result can be understood very simply in terms of our previous discussion: The key point is the delicacy of the adjustment needed to phase out the runaway solutions. If for example we consider a particular regularized solution corresponding to a given F_{ext}, we can study the effect of a small additional perturbing force $F^{(p)}$ which acts over $t_1 \leq t \leq t_2$. By a simple extension of the previous arguments it can readily be shown that the deviation from the regularized solution is of the order of

$$e^{t/\tau} \int_{t_1}^{t_2} e^{-t/\tau} F^{(p)}\, dt,$$

for $t > t_2$. In other words, small perturbations are impossible — all solutions are unstable.

5. The Radiation Condition

A. Introduction

An acceptable radiation reaction theory must give a reasonable account of the transfer of energy between the particle and the radiation field. We have already seen that this is a decisive point in the case of uniformly accelerated motion: Although the radiation reaction equation (2.8) gives a reasonable description of the dynamics in this instance (once the regularization has been included), the work of the reaction forces fails to account for the radiation losses in a natural way. In gauging the richness of the theory it is therefore essential to take both dynamic and energetic criteria into account. In the present section we shall give a quantitative formulation of a radiation condition which will assure that the work of radiation reaction represents, on the average, a loss of energy *by* the particle *to* the radiation field. It will appear that this is a very severe limitation both on the admissible external forces as well as on the character of the particle trajectories. *In particular we shall substantially confirm a conjecture originally due to* ABRAHAM [4] *that a consistent theory — based on* (2.8) *— is only possible in the case of quasi-oscillatory motions.*

B. Analytical Preliminaries

It will facilitate the subsequent discussion if we introduce some standardized nomenclature.
(a) *Essentially increasing functions:* A function f will be said to be an essentially increasing function on the interval Ω if there exists a strictly increasing function $g > 0$ and a $\delta > 0$ such that

$$|f(t) - g(t)| \leq \delta \tag{5.1}$$

for all $t \in \Omega$.

(b) *Quasi-periodic functions:* Consider a function $\tilde{f}$, also defined on Ω, which has the property that there exists an ascending sequence of points $t_0, t_1, \ldots, t_{n+1} \in \Omega$ such that $\tilde{f}(t_0) = \tilde{f}(t_1) = \ldots = \tilde{f}(t_{n+1})$. Assume further that there exists no other such sequence $t_0', \ldots, t_k'$ for which $k > n + 1$. The sequence $\{t_i\}$ is then said to be a maximally repetitive sequence. Let $\tilde{\tau}_i = t_{i+1} - t_i$ be the repetition intervals corresponding to this maximal sequence. A quasi-periodic function is then defined by the following properties:

(1) The partition $\{t_i\}$ is a fine mesh of Ω.
(2) Max $(\tilde{\tau}_i) \approx$ Min $(\tilde{\tau}_i)$. (5.2)
(3) $\tilde{f}$ is uniformly bounded in Ω.

Conditions (1) and (2) essentially guarantee that the function repeats itself sufficiently often and with enough regularity to deserve being called quasi-periodic. (3) eliminates oscillatory function with arbitrarily large amplitudes of fluctuation.
(c) *Essentially increasing quasi-periodic functions:* Combining (5.1) and (5.2) we now define an essentially increasing quasi-periodic function f in terms of the representation

$$f = \tilde{f} + g. \tag{5.3}$$

Such a function is therefore the sum of a strictly increasing function and a quasi-periodic function.
(It should be noted that the definitions (a) — (c) are not sufficiently precise to exclude a certain number of pathological cases [89]. Supplementary hypotheses will therefore have to be introduced several times in the subsequent discussion. In addition it will be convenient to assume that all functions which appear will be at least twice differentiable.)

C. The Radiation Equation

The total work of radiation reaction, in terms of the variables of (4.4b), is

$$W(t) = -m\tau \int_{t_0}^{t} \ddot{x}_2 \dot{x}_2\, dt. \tag{5.4}$$

On the average W should be an increasing function of the time: In general, of course, we cannot expect this to be a strict increase since the decomposition (3.6) suggests that there may be a fluctuating "acceleration energy" component representing a short term energy influx from the field.
We therefore make the basic assumption that in any case W must be an essentially increasing function of the time. It is then an easy corollary of (5.1) that there must be some last point of time beyond which $W(t)$ is certainly positive. It will turn out to be convenient to choose this point as the zero of time and to discuss (5.4) in terms of $h(t)$ — the deviation function between the radiation reaction equation (4.4b) and the comparison problem (4.4a) matched at this zero point of time.
We begin by transposing (4.5) to give

$$x_2 = h + x_1. \tag{5.5}$$

Scaling $\dot{x}_1(0)$ to 0, it follows that

$$\dot{x}_2 = \frac{1}{\tau} h + a_0 t, \quad \text{and} \quad \ddot{x}_2 = \frac{1}{\tau} \ddot{h}. \tag{5.6}$$

Introducing now the auxiliary variable

$$y = \frac{1}{\tau} h + a_0 t, \tag{5.7}$$

(5.4) may be rewritten in the form

$$W(t) = -m\tau \int_0^t \ddot{y} y \, dt \tag{5.8}$$

where the fluctuating portions (if any) of W for $t < 0$ have been implicitly disregarded.[11a] In this form W is finally related directly to $h(t)$ and it is possible to study the additional restrictions which arise as a consequence of W being an essentially increasing function.
It is convenient to work with the differentiated form of (5.8).

$$\ddot{y} + \frac{\dot{W}}{y} = 0. \tag{5.9}$$

This makes it clear that the radiation condition is actually equivalent to a modified Fowler-Emden equation [51], [52]. The associated boundary conditions follow from (4.7) and (5.7):

$$y(0) = 0; \quad \dot{y}(0) = a_0; \quad \ddot{y}(0) = a_0 - \frac{1}{m} F_{\text{ext}}(0). \tag{5.10}$$

In general, it is known that if $\dot{W} \sim O(t^m)$, m rational, then $y \sim O(t^k)$ for some k. More specific information will of course follow from our present restrictions on W and a number of other provisory assumptions.
It will not be an essential restriction of generality to suppose that $a_0 > 0$. By the mean value theorem it then follows that there exists an interval Ω_0 consisting of all t, $0 < t \leq t_1$ for some $t_1 > 0$, where both y and $\dot{y}$ are positive. We first show that the simplest possibility, i. e. both h and W strictly increasing functions in Ω_0, cannot occur: It is convenient to introduce the equivalent conditions $\dot{h}(t) > 0$, $\dot{W}(t) > 0$, $t \in \Omega_0$. Multiplying both sides of (5.9) by $\dot{y}$ and integrating, we have

$$\frac{1}{2} \{\dot{y}^2(t_1) - a_0\} = -\int_0^{t_1} \frac{\dot{W} \dot{y}}{y} \, dt. \tag{5.11}$$

But by (5.7) and the present hypotheses, the left hand side of (5.11) is positive while the right hand side is negative — hence there is a contradiction and this simplest case is eliminated.

[11a]) It will be convenient from here on to absorb the factor $m\tau$ into $W(t)$.

Suppose now one drops the condition on $\dot{h}$ but still maintains $\dot{W} > 0$. According to (5.9) y is then a concave function. This is a very severe restriction. It immediately implies for instance that no solutions of the form $y = kt^n$ exist; concavity restricts n to the interval $0 < n < 1$; and this range in turn is eliminated by the finiteness of $\ddot{y}(0)$ (see 5.10).
If y is concave then there is the possibility that $\dot{y}$ eventually becomes negative: From this it follows that there may be a t_2 such that y itself becomes negative for all $t > t_2$. If however $y(t_2)$ is permitted to be equal to zero, this brings with it the consequence that $\dot{W}(t_2) = 0$ because (4.7) and the presumed finiteness of F_{ext} imply that $\ddot{y}$ cannot diverge. In this event a repetition of the argument leads to the conclusion that W itself acquires an oscillatory component — i. e. the very point we are trying to establish.
Let us now go back for a moment and assume $\dot{W} > 0$ and $\dot{y} \geq 0$ in order to exhaust the alternative possibilities. It is shown in detail in Appendix 1 that under these circumstances

$$y(t) \leq a_0 t \left\{ 2 - \exp\left[\frac{\left(\int_0^t \left[\frac{W}{t}\right] dt\right)}{a_0^2 t}\right]\right\}; \tag{5.12}$$

and also

$$W(t) \leq a_0^2 t \left\{ 2 - \exp\left[\frac{\left(\int_0^t \left[\frac{W}{t}\right] dt\right)}{a_0^2 t}\right]\right\}. \tag{5.13a}$$

Furthermore

$$\int_0^t \left[\frac{W}{t}\right] dt < (a_0^2 \ln 2)\, t. \tag{5.13b}$$

These inequalities constitute severe restrictions on the nature of $W(t)$ and $y(t)$. There is in fact a less explicit although still stronger result which may be stated in terms of the following
Theorem: If $\dot{y} \geq 0$, *then all (sufficiently differentiable)* W *functions compatible with* (5.9) *must satisfy* $W \prec t$ *in the range* $t \geq 0$.[12])
Proof: Integrate (5.9) by parts,

$$a_0 - \dot{y}(t) = \frac{W}{y} + \int_0^t \frac{W \dot{y}}{y^2} \, dt. \tag{5.14}$$

The existence of the integral follows from (5.10). Since both $\dot{y} \geq 0$ and $W \geq 0$, we have the inequality

$$a_0 - \dot{y}(t) \geq \frac{W}{y}. \tag{5.15}$$

[12]) $W \prec t \Rightarrow \lim_{t\to\infty} W/t = 0$. See [49] for notation.

Integrating once again, and introducing $W^{(1)} = \int_0^t W\,dt$, (5.15) becomes

$$0 \geq y^2 - a_0 t y + W^{(1)}. \tag{5.16}$$

In the second integration we require

$$\frac{W^{(1)}(0)}{y(0)} = 0; \quad \text{and} \quad \int_0^t \frac{W^{(1)}\dot{y}}{y^2}\,dt < \infty$$

which may be justified from the general assumptions about y and W.
For a given W, (5.16) is an integral inequality for the permissible values of y. It may be rewritten in the form

$$\frac{1}{2}\,a_0 t\left\{1 - \left[\frac{4W^{(1)}}{(a_0 t)^2}\right]^{1/2}\right\} \leq y(t) \leq \frac{1}{2}\,a_0 t\left\{1 + \left[1 - \frac{4W^{(1)}}{(a_0 t)^2}\right]^{1/2}\right\}. \tag{5.17}$$

From this it follows in turn that

$$W^{(1)} < \tfrac{1}{4}(a_0 t)^2. \tag{5.18}$$

The key point now is that one can also prove that $W^{(1)} \sim O(t^2)$ is impossible! From this statement the conclusions of the theorem will follow.
Let

$$W = \alpha t + \beta \quad \text{for} \quad t \geq t_2 > 0 \tag{5.19}$$

and assume $\alpha > 0$. Going over to the inverse functions, (5.9) is separable and integration yields

$$-\alpha \ln\left\{\frac{y(t)}{y(t_2)}\right\} + \frac{1}{2}[\dot{y}(t_2)]^2 = \frac{1}{2}[\dot{y}(t)]^2 \geq 0. \tag{5.20}$$

Therefore

$$y(t) \leq y(t_2)\,\exp\left\{\frac{[\dot{y}(t_2)]^2}{2\alpha}\right\}, \tag{5.20a}$$

which clearly shows that y is bounded from above. On the other hand (5.17) and (5.19) imply that there is a *lower* limit on the growth of y which ultimately exceeds this value as $t \to \infty$. We are forced to the conclusion that there exists a t_3 beyond which no solution to the problem exists;[13] (5.19) must therefore be amended to read $t_2 \leq t \leq t_3$, which finally implies that W cannot be $\sim O(t)$. [If α and β are replaced by smooth bounded functions of t, the argument is essentially unchanged.] Since *a priori* all pathologies have been excluded by (5.1) and (5.2) there are only the three possibilities remaining for W: Either (1) $W > t$; (2) $W \sim t$; or (3) $W < t$. But (1) and (2) have been eliminated by the preceding; so this leaves only (3), and therefore completes the proof of the theorem!
Summarizing all the work so far, one finally has the conclusion that if y is a non-decreasing function of t, then the radiation must proceed at a rate which is *less* than proportional to the time. In view of (2.2) this is clearly much too restrictive a condition. If $\dot{y}$ is permitted to become negative, then either this reduces to the case previously discussed — through y eventually crossing the t axis — or else fluctuations of $\dot{y}$, reflected in $\ddot{y}$, again force W to oscillate since $\ddot{y}$ and $\dot{W}$ are directly tied together by (5.9). The only other alternative is that both the deviation function as well as the radiation acquire oscillatory components. In this sense, with the exception of a number of highly circumscribed cases, we finally arrive back at ABRAHAM's conjecture — which now may be regarded as confirmed — that from an energy transfer standpoint the theory yields satisfactory results only in the case of quasi-periodic motions.

[13]) The equation may be completely integrated in terms of inverse error functions; t_3 then emerges from the bound of the domain of definition.

6. Non-Local and Extended Electron Theories

A. Introduction

A considerable portion of the theory of radiation reaction has been developed within the context of various non-local extensions of classical electrodynamics. Before proceeding with the detailed discussion of these features it will be useful to review some of the (contemporary) reasons for taking an interest in these developments.
(i) Non-Singular Form Factors and Runaways — A number of experimental programs are currently under way which are producing detailed information on the form factors of nucleons. If classical field theories, patterned after electrodynamics, are to continue to be of any qualitative significance in discussing such particles then clearly these new structural degrees of freedom must somehow be accommodated. Inevitably this leads to some type of non-local generalization of electrodynamics. The associated radiation reaction theories then of course must also be superposed on some type of non-local structure. In the later development it will appear that such additional constraints can actually be of help in making a proper choice among the many possible extensions of the classical theory. There are, for example, numerous "covariant form factor" generalizations of electrodynamics which can be shown to have non-physical radiation runaways despite the fact that the self-energy divergences have been successfully removed.
(ii) Phenomenological Theories of Radiation Reaction — Current progress in plasma physics and millimeter microwave generation has stimulated the development of an extensive phenomenological theory of radiation reaction. (GINZBURG and EIDMAN [*54*]) Although this theory has not as yet been put on a completely consistent foundation, it is already clear that its mathematical apparatus has only the vaguest resemblance to a point particle description. The spirit of the formulation is in fact very close to that of the classical extended electron self-force picture.
(iii) Electrodynamic Collective Modes — Non-local generalizations of electrodynamics are also interesting in that they offer certain formal possibilities which are completely lacking in the point charge theories. The most interesting of these features is the appearance — at certain critical frequencies — of self-sustaining charge oscillations. The charge oscillations have a natural interpretation as *collective* resonances of the field-matter system. In common with collec-

tive modes familiar from other areas of physics, these charge resonances are non-analytic in the coupling constant (e^2) and cannot be reached in truncated versions of the theory. The existence of these collective modes offers the formal possibility of extending electrodynamics into an "isobar" theory of excited charge states.

B. Higher Derivative Theories

The minimal intervention in the structure of the point charge theory which offers a way out of the contretemps of runaways and the non-existence of physical solutions, etc. is the introduction of higher derivatives. This avenue has been explored in detail by ELIEZER. In his first attempt [*19*] ELIEZER widened Dirac's ansatz (2.16c) by including a new constant $\varkappa$, and some higher derivative terms as follows

$$B_k = \left(m - \frac{1}{2}\frac{e^2}{r}\right) u_k + \varkappa \left\{\frac{2}{3}\frac{d^2 u_k}{ds^2} + u_k \left(\frac{d u_i}{ds}\right)^2\right\}. \qquad (6.1)$$

If the original derivation is now repeated with this new form of B_k, it is found that the resulting equations of motion contain fourth derivatives. The analysis of these equations becomes proportionately more difficult; however ELIEZER himself, and later ZIN [*40*] and HAAG [*18*] showed that the runaway solutions and related difficulties still persist in this extended formulation.

In a second attempt, ELIEZER [*24*] [*55*] introduced more drastic measures: The ansatz (2.16c) was generalized to include an infinite number of derivatives, viz.

$$B_k = \left(m - \frac{1}{2}\frac{e^2}{r}\right) u_k + \sum_{n=1}^{\infty} B_{2n} \Big\{ u_k^{(2n)} - \left[u_i u_i^{(2n)} - \ldots + (-l)^n u_i^{(n-1)} u_i^{(n+1)} + \frac{1}{2}(-l)^n u_i^{(n)} u_i^{(n)}\right] u_k \Big\} \qquad (6.2)$$

where the B_{2n} are arbitrary constants and u_k is the four-velocity. In this form the theory is actually equivalent to an extended electron model.[14]) To see this, we introduce (6.2) into (2.16b): In the non-relativistic limit the result may be written

$$m \left\{\sum_{j=0}^{\infty} c_j \mathfrak{D}^j\right\} \ddot{x} = F_{\text{ext}} \qquad (6.2\text{a})$$

where $\mathfrak{D}$ represents the derivative operator and the c_j's are constants related to the B_{2n}.

In the special case $c_j = (\bar{\tau})^j / j!$ (6.2a) simplifies still further to

$$m \ddot{x}(t + \bar{\tau}) = F_{\text{ext}} \qquad (6.2\text{b})$$

by virtue of Taylor's theorem. In this form the content of the theory is summarized in terms of a differential-difference equation; this in turn can be shown to coincide exactly with the corresponding expressions obtained from the extended electron theories. (See below.)

[14]) Note however that this approach still requires a mass renormalization.

C. Extended Electron Theories

The original versions of electrodynamics were all of course based on the concept of an extended electron. As mentioned in the Introduction, the ultimate goal of these theories was the calculation of all of the inertial and dissipative properties of the electron from a model which pictured it as a sort of electromagnetic sponge: The various parts of this sponge were supposed to be capable of interacting with each other as well as with the external fields — the whole object responding to the net sum of the "outside" and "self" forces. For the fields, the basic equations were taken to be those of Maxwell. The dynamical properties were then assumed to follow from the equation

$$\boldsymbol{F}_{\text{ext}} + \boldsymbol{F}_{\text{self}} = m_{\text{mech}} \ddot{\boldsymbol{x}} \qquad (6.3)$$

which, because of the implicit $\boldsymbol{x}(t)$ dependence of $\boldsymbol{F}_{\text{self}}$, rather than being a differential equation is actually a *functional equation*.[15]) It is this functional equation property of (6.3) which endows the classical theory with its astonishing wealth of solutions! We shall see first of all that in the limit of small accelerations (6.3) reduces to (2.8), i. e. the ordinary theory of radiation damping. Secondly we shall see that at "super-light" velocities (6.3) goes over naturally into the constant velocity radiation reaction of the Cerenkov effect. Finally it will appear that this equation also contains the radiationless charge self-oscillations discussed previously in Section 6-A (iii).

Clearly this program requires a rigorous and generally valid expression for $\boldsymbol{F}_{\text{self}}$. We begin by writing down the usual representation for the retarded potentials

$$\boldsymbol{A}(\boldsymbol{x}', t) = \int dt'' \int d\boldsymbol{x}'' \, |\boldsymbol{x}' - \boldsymbol{x}''|^{-1} \boldsymbol{j}(\boldsymbol{x}'', t'') \, \delta\left(t'' - \left[t - \frac{|\boldsymbol{x}' - \boldsymbol{x}''|}{c}\right]\right) \qquad (6.4)$$

The Lorentz force is given by

$$\mathfrak{L}(\boldsymbol{x}', t) = -\operatorname{grad}\left\{\Phi - \frac{1}{c}\,\dot{\boldsymbol{x}} \cdot \boldsymbol{A}\right\} + \frac{1}{c}\left\{(\omega \times [\boldsymbol{x}' - \boldsymbol{x}]) \times \operatorname{curl} \boldsymbol{A} - \frac{\partial \boldsymbol{A}}{\partial t}\right\} \qquad (6.5)$$

where $\boldsymbol{x}$ locates the center of the electron and ω is the angular velocity. The total self force is then

$$\boldsymbol{F}_{\text{self}}(t; \boldsymbol{x}) = \int d\boldsymbol{x}' \, \mathfrak{L}(\boldsymbol{x}', t) \, \varrho(\boldsymbol{x}', t) \qquad (6.6)$$

where the $\boldsymbol{x}$ has been inserted on the left to serve as a reminder that $\boldsymbol{F}_{\text{self}}$ is a functional of the trajectory.

The sevenfold integration implied by (6.6) can be simplified in a number of ways. For example $\boldsymbol{F}_{\text{self}}$ can be written as a Fourier transform

$$\boldsymbol{F}_{\text{self}} = -4\pi \int_0^{\infty} d\tau \int \frac{d\boldsymbol{k}}{k^2} \, |\varrho_{\boldsymbol{k}}|^2 \exp\{i\boldsymbol{k} \cdot [\boldsymbol{x}(t) - \boldsymbol{x}(t-\tau)]\} \times$$
$$\times \, [\boldsymbol{k} \times (\dot{\boldsymbol{x}}(t-\tau) \times \boldsymbol{k}) \cos(ck\tau) + ik\dot{\boldsymbol{x}}(t) \times (\dot{\boldsymbol{x}}(t-\tau) \times \boldsymbol{k}) \sin(ck\tau)] \qquad (6.6\text{a})$$

[15]) The non-electromagnetic inertial component m_{mech} is inserted as a purely phenomenological term. In a strict (classical) nucleon field theory the corresponding ansatz would be $m_{\text{mech}} \ddot{\boldsymbol{x}} \to \boldsymbol{F}^{u}_{\text{self}}$, where $\boldsymbol{F}^{u}_{\text{self}}$ represents the influence of the Yukawa field.

where

$$\varrho_k = (2\pi)^{-3/2} \int \varrho(\boldsymbol{x}', t)\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}'}\, d\boldsymbol{x}'$$

which reduces (6.6) to a fourfold integration. (See e. g. BOHM and WEINSTEIN [*56*] or GINZBURG and EIDMAN [*54*].) We shall return to this point later in connection with the phenomenological theory of radiation reaction.
The complete calculation of all the *explicit* integrals contained in this expression was carried out by SOMMERFELD [*31*] [*57*] for two special cases: The uniformly charged sphere and the infinitely thin spherical shell. For the uniformly charged sphere, in pure translation, the result may be written

$$F_{\text{self}} = -\frac{3e^2}{8r_0^4}\left\{\int_0^{\tau_1}\left[\frac{c+\dot{x}(t-\tau)}{\mathcal{T}}\frac{\partial}{\partial\mathcal{T}}f(c\tau+\mathcal{T}) - \frac{c}{\mathcal{T}^2}f(c\tau+\mathcal{T})\right]d\tau - \right.$$

$$\left. -\int_0^{\tau_2}\left[\frac{c-\dot{x}(t-\tau)}{\mathcal{T}}\frac{\partial}{\partial\mathcal{T}}f(|c\tau-\mathcal{T}|) - \frac{c}{\mathcal{T}^2}f(|c\tau-\mathcal{T}|)\right]d\tau\right\}. \qquad (6.7)$$

The auxiliary function f is defined by

$$f(z) = \int_{2r_0}^{z}\left\{4r_0\beta - 3\beta^2 + \frac{1}{4}\frac{\beta^4}{r_0^2}\right\}d\beta = \frac{1}{20}\frac{z^5}{r} - z^3 + 2r_0 z^2 - \frac{8}{5}r_0^3. \qquad (6.7\text{a})$$

$\mathcal{T}$ denotes the path length traversed by the center of the electron in time τ, i. e.

$$\mathcal{T} = x(t) - x(t-\tau). \qquad (6.7\text{b})$$

τ_1 and τ_2 are the (possibly non-unique) roots of

$$c\tau + \mathcal{T} = 2r_0 \quad \text{and} \quad |c\tau - \mathcal{T}| = 2r_0 \qquad (6.7\text{c})$$

respectively; r_0 is the charge radius.
The functional nature of F_{self} is now explicitly displayed on the right hand side of (6.7): This type of expression is actually an "integral-delay" representation because the retardations of the functions occuring in the integrand are themselves dependent on the unknown function. (See [*58*].) If this representation is carried back into (6.3), the resulting expression is evidently a functional equation in $x(t)$. It should be emphasized that as far as the rigid spherical electron is concerned, this equation is an *exact* consequence of electrodynamics — presupposing no restrictions on either the velocity or the acceleration.
It was of course one of the great triumphs of this classical theory that one could set $m_{\text{mech}} = 0$, i. e. replace (6.3) by

$$\boldsymbol{F}_{\text{ext}} = -\boldsymbol{F}_{\text{self}} \qquad (6.8)$$

and so achieve a purely electromagnetic explanation of mass. (*Pace* the stresses of Poincaré.) These remarks depend on the use of the so-called "quasi-stationary" approximation

$$\tau_0\frac{dv}{dt} \ll \frac{v}{c}\,|c-v|; \qquad (6.9)$$

which for $c \neq v$ reduces to the usual assumption that the accelerations are always small in time intervals of the order of the electron transit time (τ_0). By systematic use of this inequality, i. e. expansions of the type

$$\dot{x}(t-\tau) \approx \dot{x}(t) - \ddot{x}(t)\,\tau + \dddot{x}(t)\,\frac{\tau^2}{2} \qquad \text{etc.} \qquad (6.9\text{a})$$

the integral-delay representation (6.7) can be developed in terms of successive derivatives of $x(t)$. Carrying this expansion to the third order to include the effects of radiation reaction one finds the remarkable result

$$F_{\text{self}}(t) = -m_{\text{em}}(\dot{x}, r_0)\,\ddot{x} + \frac{2}{3}\frac{e^2}{c^3}\dddot{x} \qquad (6.9\text{b})$$

which agress exactly with the previous equation (2.8), derived on the basis of conservation laws.[16])
The significant new information contained in (6.9b) is the explicit evaluation of the electromagnetic mass in terms of the velocity and structural characteristics of the electron. SCHOTT [*30*] has carried out extensive investigations of the form of (6.9b) for a number of electron models and in particular has verified that in the case of the deformable Lorentz electron all the results of the relativistic theory are completely reproduced. Repeating this reduction from the appropriate modification of (6.7) he showed that

$$m_{\text{em}}(\dot{x}, r_0) = \frac{m_0(r_0)}{\left(1 - \dfrac{\dot{x}^2}{c^2}\right)^{1/2}}$$

in complete agreement with special relativity. Furthermore it can also be shown that the coefficient of the $\dddot{x}$ term is precisely equal to (2.10) i. e. the proper relativistic generalization of $m\tau\dddot{x}$. A very important point established by SCHOTT in this connection is that in the $v/c \ll 1$ limit the form of the radiation reaction term is *independent* of the electron model. This is basically the reason that it can be recovered at all even in the point electron limit.
Within the framework of the previous theories it was permissible — in a sense — to interpret (6.9b) as being the exact expression for the self-force. Within the present development however this appears to be merely the first approximation to the structure of a far more complicated theory. It is therefore of particular interest to study those features of (6.7) and (6.8) which are not reflected in the truncated versions. There are essentially two ways of approaching this problem — One is to remain entirely within the domain of the quasi-stationary approximation and to extend the expansion (6.9b) to include the higher terms. The other is to go beyond the quasi-stationary approximation altogether and to study the *exact* consequences of (6.7), for example its behavior in the vicinity of the velocity of light and in the "super-light" Cerenkov regime.[17])

[16]) More precisely, the quasi-stationary approximation is a necessary but not a sufficient condition for deriving (6.9b). The approximation retains its meaning and utility even if $v > c$, but then (6.9b) no longer applies.
[17]) It is easily seen that (6.9) does not necessarily hold for electrons of energies $\gtrsim 10^{13}$ ev.

We consider first the inclusion of the higher derivative terms: Strictly speaking this involves not only the quasi-stationary approximation but also the additional assumption that $v \ll c$, and the neglect of all the higher derivative cross terms which occur when the expressions (6.9a) are substituted into (6.7). We shall speak of these simplifications collectively as the "extended quasi-stationary" approximation. Since we have already learned from Eliezer's work (Subsection 6-B above) that the inclusion of only a limited number of higher derivatives in (6.9b) is not likely to bring about any substantial improvement, we shall immediately extend these expansions to include an infinite number of terms: Following the previous calculation, the infinite analogues of (6.9a) may be substituted back into (6.7); neglecting the products of the higher derivatives, the resulting expressions simplify sufficiently so that the remaining integrations can be performed explicitly. Formally the result is a differential equation of infinitely high order. The tedious details of this computation were actually first carried through by HERGLOTZ [*59*] (see also WILDERMUTH [*41*]) with the final result

$$F_{\text{self}} = -\frac{18e^2}{r_0 c^2}\left\{\sum_{n=0}^{\infty} A_n \mathfrak{D}^{n+1}\right\}\dot{x} \tag{6.10}$$

where

$$A_n = \frac{\left(-2\,\frac{r_0}{c}\right)^n}{([n+2]\,[n+3]\,[n+5]\,n!)}, \tag{6.10a}$$

and $\mathfrak{D}$ is the differentiation operator.
By means of the identity

$$\underset{h}{\triangle} = \exp\{h\,\mathfrak{D}\} - 1 \tag{6.11}$$

$$\underset{h}{\triangle} f(x) \equiv f(x+h) - f(x)$$

infinite order differential equations of this kind can always be transformed into difference equations. The particular equation (6.10) has however a rather complicated finite difference form (see subsection F below). It was first shown by PAGE [*15*] that this approach leads to much simpler "natural" equations if the electron model is taken to be that of a uniformly charged spherical shell. Beginning with the appropriate modification of (6.7) and again applying the extended quasi-stationary approximation one obtains in this case instead of (6.10) the expression

$$F_{\text{self}} = +\frac{e^2}{3r_0^2 c}\left\{\sum_{n=1}^{\infty}\left(-2\,\frac{r_0}{c}\right)^n \mathfrak{D}^n\right\}\dot{x}. \tag{6.12}$$

Using (6.11), this can now be transformed to

$$F_{\text{self}} = \frac{e^2}{3r_0^2 c}\left[\dot{x}\left(t - \frac{2r_0}{c}\right) - \dot{x}(t)\right]; \tag{6.12a}$$

a result which was also later derived by BOHM and WEINSTEIN [*56*]. Comparing with our original radiation reaction equation (2.8) we see that now

$$F_{\text{ext}} = \frac{e^2}{3r_0^2 c}\left[\dot{x}(t) - \dot{x}\left(t - \frac{2r_0}{c}\right)\right] \tag{6.12b}$$

in contrast to the previous

$$F_{\text{ext}} = m\,\ddot{x} - \frac{2}{3}\frac{e^2}{c^3}\dddot{x};$$

clearly the splitting into inertial and reaction terms has completely fallen away. Treating $2r_0/c$ as a small quantity, i. e., going back to the partial expansion (6.9a), the right hand side of (6.12b) is seen once more to agree with (2.8) in lowest order if one chooses

$$m = \frac{2}{3}\frac{e^2}{r_0\,c^2}. \tag{6.13a}$$

The distinction between (6.8) and (2.8) becomes even more marked if one goes beyond the quasi-stationary approximation altogether. It was for instance first discovered by SOMMERFELD that at "super-light" velocities *there is a radiation reaction even at constant velocity.* By a straightforward modification of the discussion of Appendix 2 it can be shown that if $\dot{x}(t) = v > c$, (6.7) reduces to

$$F_{\text{self}} = -\frac{9e^2}{4r_0^2}\left(1 - \frac{c^2}{v^2}\right). \tag{6.13}$$

F_{self} in this case has the interpretation of being purely a radiation reaction force. In fact with the replacement $c \to c/n$, (6.13) correctly describes the Cerenkov losses of a particle traversing a medium of index of refraction n.[18])

D. Finite Difference Theories

From a logical point of view it is possible to dispense entirely with the scaffolding leading from (6.7) to (6.12b) and to adopt the finite difference formulation as the fundamental statement of the theory. This point of view and its antecedents has been extensively reviewed by CALDIROLA [*60*]. One can show quite easily for instance that it is possible to arrive directly at (6.12b) by merely assuming an intrinsic quantization of space and time. In such a lattice manifold, clearly all the customary differential expressions have to be replaced by their finite difference analogues. In fact, carrying this approach to its logical conclusions, all non-constant physical variables must be represented by discontinuous functions. CALDIROLA himself does not go to quite such extremes but he does present an axiom system showing that if one assumes (a) the existence of a fundamental interval of time, e. g. $2r_0/c$; (b) the validity of the usual electrodynamics in the limit $r_0 \to 0$; and finally (c) covariance under Lorentz transformations, then (6.12b) emerges naturally as a non-local generalization of the point charge theory. A particulary interesting feature of this approach is that it motivates directly the covariant generalization of (6.12b), viz.

$$-\frac{e^2}{3r_0^2 c}\left\{\dot{x}_k(s - s_0) + \frac{1}{c^2}\dot{x}_k(s)\,\dot{x}_j(s - s_0)\,\dot{x}_j(s)\right\} = F_k. \tag{6.14}$$

[18]) Sommerfeld's formula (6.13) dropped into obscurity because of its apparently anti-relativistic content. It was not until 1937 that JOFFÉ (quoted in [*67*]) pointed out that it had correctly anticipated the Cerenkov Effect. The elaboration of this correspondence is essentially the basis of the phenomenological theory of radiation reaction. [Section 7]

The symbols here have their usual meaning as four-vectors; s_0 denotes the invariant interval of proper time corresponding to $2r_0/c$. This theory is therefore an example of a covariant electrodynamics which automatically includes a "fundamental length".

CALDIROLA shows that the runaway features of the point electron model do not occur in this finite difference theory (see also subsection F below) and presents other arguments showing that the solutions of these equations can always be expected to have a dynamically reasonable aspect. Nevertheless there are two broad objections which may be brought against too strong a commitment to this point of view. The first is that this formulation still does not give a satisfactory account of the energy transfer from particle to radiation field. In particular, as CALDIROLA shows, the hyperbolic motion is still a radiationless trajectory in this theory; in view of the preceding discussions on this point (see especially the remarks following (3.7)) it is difficult to accept this without reservation. It should also be emphasized that the mathematical structure of (6.12b) is not nearly as rich as that of its "parent" (6.7): The Cerenkov features and nearly all of the non-stationary cases have been completely lost. Nevertheless (6.12b) makes it particularly clear that the vanishing of the external force does *not* necessarily require the vanishing of $\ddot{x}$. The new solutions which enter here — the so-called radiationless internal modes of the electron — will be discussed further in the following sections.

E. Covariant Form Factor Theories

With the advent of special relativity, the elaborate electron models of ABRAHAM, LORENTZ, and SOMMERFELD were of course completely buried. In fact this interment was so complete that when in later years the divergence difficulties of the point electron theory were again felt to be particularly acute, the remedies were not sought among the old extended electron models but rather among far more formal devices — the covariant form factors and smearing functions. The essential idea of this new approach was to hold on to the linearity and Lorentz covariance of the classical theory but to blur the field-particle interaction sufficiently so as to smear out all singularities.[19]) Theories of this type have been studied by BOPP [*13*], FEYNMAN [*38*], MCMANUS [*61*], and especially LEHMANN [*62*]; comprehensive summaries have been given by HÖNL [*63*] and RZEWUSKI [*64*]. We shall consider here only those aspects which have a particular relevance to the radiation reaction question.

In the Hamiltonian form of electrodynamics, the usual interaction between field and particle is given by

$$S_{\text{int}} = e \iint \delta(x - z(s))\, u_k(s)\, A_k(x)\, d^4x\, ds \tag{6.15}$$

where $z(s)$ represents a space-time point on the world line of the electron, parametrized by the proper time s, and u_k is the four-velocity. In the linear non-local theories this is replaced by the ansatz

$$S_{\text{int}} = e \iint \mathfrak{F}([x - z(s)]^2)\, u_k(s)\, A_k(x)\, d^4x\, ds \tag{6.15a}$$

[19]) Several non-local and non-linear theories have also been discussed by BLOCHINZEV [*65*]. See further PAIS and UHLENBECK [*66*]. The significance of radiation reaction in these theories is not yet clear.

where $\mathfrak{F}$ is a sharply peaked function which "smears" out the divergences that would otherwise be caused by the δ-function. The equations of motion implied by (6.15a) are complicated non-linear integro-differential equations. In the non-relativistic limit they may however be reduced to the simple form

$$\boldsymbol{F}_{\text{ext}}(t) = -\frac{2}{3}\frac{e^2}{c^3}\ddot{\boldsymbol{v}}(t) + \int_{-\infty}^{+\infty} dt'\, \overline{\mathfrak{F}}([t - t']^2) \left\{\boldsymbol{v}(t') - \frac{\boldsymbol{x}(t') - \boldsymbol{x}(t)}{t' - t}\right\} \tag{6.16}$$

where $\mathfrak{F}$ is related to $\overline{\mathfrak{F}}$ essentially via a FOURIER transform. (See [*61*].) One point that is immediately evident is that the structure of (6.16) is much simpler than that of either (6.6a) or (6.7). Note in particular that the charge volume is integrated over only *once* (compare the $|\varrho_k|^2$ in (6.6a)) so that the functional dependence of $\boldsymbol{F}_{\text{self}}$ can be expected to be more rudimentary in this theory. We shall see in a moment that this implies serious drawbacks for (6.16).

F. Runaway Solutions

If the non-local or extended electron theories are to represent any improvement over the point-charge theory, then clearly they must not contain any runaway solutions. The classical investigation of this point dates back to HERGLOTZ [59]. (See also WILDERMUTH [*41*].) Consider first the case of a uniformly charged spherical electron of radius r_0. If there are no external forces then (6.3) becomes

$$F_{\text{self}} = m_{\text{mech}}\, \ddot{x}. \tag{6.17}$$

Assuming the extended quasi-stationary approximation, it is permissible to insert (6.10) for F_{self}, so that

$$m_{\text{mech}}\, \ddot{x} + \frac{18 e^2}{r_0 c^2} \sum_{n=0}^{\infty} \left\{ \frac{\left(-2\frac{r_0}{c}\right)^n}{([n+2]\,[n+3]\,[n+5]\, n!)} \right\} \mathfrak{D}^{n+2} x = 0. \tag{16.7a}$$

Clearly $\dot{x} =$ const. is a solution but it is by no means obvious nor even true that it is the only solution. To investigate this point further it is useful to make the ansatz

$$x = k \exp\left\{-\frac{c\lambda}{2r_0} t\right\}. \tag{6.17b}$$

From (6.17a) it then follows that

$$\Psi(\lambda) \exp\left\{-\frac{c\lambda}{2r_0} t\right\} = 0 \tag{6.17c}$$

where

$$\Psi(\lambda) = m_{\text{mech}} \frac{c^2}{2r_0^2} \lambda^2 + 9 \frac{e^2}{r_0^3} \Phi(\lambda) \tag{7.16d}$$

and

$$\Phi(\lambda) = [\lambda^{-1} + 4\lambda^{-2} + 4\lambda^{-3}]\, e^{\lambda} + \tfrac{1}{3} + \lambda^{-1} + 4\lambda^{-3}. \tag{6.17e}$$

Clearly the existence of any solutions of this kind hinges on whether $\Psi(\lambda)$ has any zeros. We state this important result as the

HERGLOTZ-WILDERMUTH *Theorem*: The function $\Psi(\lambda)$ defined by (6.17 d and e) above possesses *infinitely* many zeros; λ_j, $j = 1, \ldots$. These zeros have the property that

$$\mathrm{Re}\,\{\lambda_j\} \gtreqless 0 \quad \text{for} \quad m_{\text{mech}} \gtreqless 0\,.$$

The physical significance of this result is that *as long as the "mechanical" mass of the electron is not negative, there will be no runaway solutions*. The extended electron theory therefore not only can cure the runaway difficulties but also explains in a very illuminating way why they arise in the first place as $r_0 \to 0$. Basically these "unexpected" motions of the electron are controlled by two $\pm$ options: One occurs in the sign of the time — if we reverse the sign of t this corresponds to selecting an ingoing solution of Maxwell's equations (see [*3*]), i. e., energy is fed *into* the particle and it accelerates. The other option has just been revealed by the Herglotz-Wildermuth analysis: If the mechanical mass is negative, then the signs are reversed in Newton's second law and instead of the particle experiencing a damping in the face of an opposing force, it accelerates.[20]) The limit $r_0 \to 0$ enters since we have the upper bound

$$m_{\text{mech}} = m_{\text{exp}} - m_{\text{em}}\,.$$

As $r_0 \to 0$, $m_{\text{em}} \to \infty$, then clearly $m_{\text{mech}} < 0$; and this finally is seen to be the root of the runaway difficulties!

It is interesting to note that this feature has a close parallel in quantum electrodynamics. Since the quantized theory is almost always discussed in terms of a perturbation formalism the correspondence is however not usually evident. VAN KAMPEN [*68*] and NORTON and WATSON [*69*] have carried out some exact calculations with a "model" quantum-electrodynamics based on a non-relativistic harmonically bound electron coupled to the radiation field in the electric-dipole approximation. By working with an initially smeared electron, it is possible to show that as the charge distribution is permitted to shrink to a point, i. e., $\varrho(x) \to \delta(x)$, runaway continuum eigenstates of the Hamiltonian appear precisely at the point where the "mechanical" mass of the electron becomes negative. The analogy extends even further than this: If one attempts to "regularize" the Hamiltonian by altering the commutation relations to suppress this unphysical continuum, acausal effects analogous to the classical-pre-accelerations arise.[21])

We have already indicated several times our preference for drawing a distinction between the covariant form factor theories and the extended electron theories. The reason for this is that if one carries through an analysis for (6.16) analogous

[20]) As S. COLEMAN (unpublished) has pointed out this also implies that the particle kinetic energy becomes a *negative* definite quantity – even in the non-point limit. The runaways conserve energy simply because they build up the positive field energy (radiation) at the expense of negative kinetic energy. See also [*90*].

[21]) This point has also been discussed by STEINWEDEL [*14*]. In a certain sense in quantum theory the situation is even more acute: Since the interactions between the particles and the vacuum fluctuations can never be "switched off", the runaways will always persist. The classical adiabatic prescriptions, e. g. [*88*], then become nugatory. See [*90*]. Another interesting point of the quantum development is that the characteristic classical damping constant mc^3/e^2 does not get itself multiplied by factors of $e^2/\hbar c$ and therefore still remains the natural scale factor.

to that of (6.17a) one finds the surprising fact that the form factor expression (6.16) will in general contain runaway components. This has been verified in detail by BOPP [*70*] and STEINWEDEL [*14*] [*70*] for a wide class of otherwise completely successful form factor theories. Similar results have also been obtained by IRVING [*71*]. This unpleasant circumstance can best be understood in terms of the Herglotz-Wildermuth theorem: The essential point is that most of these covariant form factor theories are actually equivalent to a regularization of ordinary electrodynamics by a continuum of *negative* energy meson fields (LEHMANN [*62*]). The self energy divergences are therefore automatically compensated but the electron also acquires a component of negative non-electromagnetic mass. According to the Herglotz-Wildermuth theorem one can therefore expect runaways — and this is indeed what the computations show.[22])

G. Electrodynamic Collective Modes: The Excited States of the Electron

One of the most interesting features of the extended electron theory is its prediction of the existence of both self-oscillatory and radiationless states of motion. This is essentially due to the fact that the basic integral-delay equation $F_{\text{self}}[x(t)] = 0$ of the extended electron theory has associated with it a non-trivial eigenvalue problem in the case $x(t) \sim e^{i\omega t}$. A very simple way of seeing this is to go back to the finite difference expression (6.12a)

$$F_{\text{ext}} = \frac{e^2}{3r_0^2\,c}\left[\dot{x}(t) - \dot{x}\left(t - \frac{2r_0}{c}\right)\right]$$

which can be derived from the shell electron equivalent of (6.7) in the extended quasi-stationary approximation. Clearly any series of the type

$$x = \sum_{k=0}^{\infty}\left\{A_k \sin\left(\frac{k\pi c}{r_0}\,t\right) + B_k \cos\left(\frac{k\pi c}{r_0}\,t\right)\right\} \tag{6.18}$$

will satisfy the homogeneous problem $F_{\text{self}} = 0$. The physical meaning of this result is that an extended charge structure ought to be capable of carrying out sustained self-oscillations *even in the complete absence of any external forces*. Within the framework of the finite difference theories this is of course an exact result; in fact all the trajectories encompassed by the trigonometric representation (6.18) can be shown to be strictly non-radiating modes in this formulation [*60*]. In the Herglotz-Sommerfeld theory circumstances are somewhat more complicated but the structure is proportionately richer and more interesting. The appearance of the electrodynamic collective modes in this theory can be understood in terms of two distinct eigenvalue problems; one corresponding to the existence of radiationless modes and the other to the existence of force-free modes. We consider these in turn:

(i) *A radiationless mode* is a sustained charge oscillation which does not emit any electromagnetic radiation. The possibility of the existence of these modes was first explicitly recognized by EHRENFEST [*72*] and later by SCHOTT [*73*] although

[22]) See also [*41*]. It is interesting to note that in this respect the "local" requirements imposed by (6.16) are far more stringent than the asymptotic conditions on the Poynting vector.

the beginnings of the idea can actually be traced back to SOMMERFELD [*74*]. A concise modern formulation has been given by S. COLEMAN (unpublished). The essential point is easily exhibited by transcribing the retarded potential expression (6.4) into Fourier transform space. Ignoring singular factors, the result is

$$A_k \sim \dot{x}_k \, \varrho_k . \tag{6.19}$$

The radiationless condition is then simply

$$A_k = 0 \tag{6.19a}$$

which requires that the system oscillate at one of the zeros of ϱ_k, the Fourier transform of the charge distribution. (Note that (6.19a) applies only to the space "outside" the charge distribution.)
(ii) *A force-free mode* is a self-sustained charge oscillation which exists even in the absence of any external forces. It follows from the previous work that the frequencies characterizing the force-free modes are the eigenvalues of the functional equation

$$F_{\text{self}}(e^{i\omega t}) = 0 . \tag{6.20}$$

Roughly speaking for a charge structure of effective radius $\sim 10^{-13}$ cm these frequencies can all be expected to be of the order of 10^{23} sec^{-1} and higher.
It is important to realize that these force-free modes are not necessarily radiationless and conversely that radiationless modes are not necessarily force-free. Only if the eigenvalue problems (6.19) and (6.20) share a solution does a rigorously self-sustaining radiationless mode exist. This for example is the case for the rotational oscillations of a spherical shell electron [*31*] [*74*]. The general situation however may be still more complicated — this for instance will happen if either or both of the equations (6.19) and (6.20) have complex roots. In this event there may be damped oscillations which nevertheless formally correspond to radiationless modes and/or force-free modes which are associated with a definite time of decay.[23])
A further degree of freedom is available in the adjustment of the mechanical mass. As WILDERMUTH [*41*] has pointed out, if one insists on the auxiliary condition $\mathrm{Re}\{\lambda_j\} = 0$, i. e. no damping of the force free modes, then (6.17d) or its equivalent introduces still another eigenvalue problem which fixes the value of m_{mech}. This feature is of course especially interesting in connection with the possibility of a many-particle classical field theory. Finally it should be noted that in the entire discussion to this point we have been considering only *rigid* charge motions, i. e. the Poincaré stresses have played an entirely passive role. Surprisingly high energies may be stored in these collective modes. On a classical basis this may be simply estimated by assigning plausible values to the oscillation amplitude. For example in the case of the shell electron it can be shown that the energy stored in the circulating field components is of the order of *one hundred times* the electrostatic ground state energy. From the quantum mechanical point of view it is of course inviting to remove the amplitude ambiguity by assigning the energy via $E_{\text{exc}} = h\nu_{\text{exc}}$; the only remaining imprecision would then lie in the

[23]) These complications require close attention to mathematical details; linearized treatments, e. g. [*56*], [*41*] can often go astray. A rigorous analysis for one case is carried through in Appendix 2.

details of the charge structure. If for the sake definiteness one assumes a characteristic dimension of the order of 1 *f*, (6.20) again leads to the classical estimate $E_{\text{exc}} \sim 10^2 E_0$. Finally one may use a *model independent* scale factor, i. e. $2e^2/3mc^3$, the radiation reaction coefficient in the quasi-stationary limit, to establish a length. In this case the total energy of the first excited state turns out to be

$$mc^2\left[1 + \frac{3}{2\alpha}\right] = 105.54 \text{ Mev.} \tag{6.21}$$

It has been noted previously [*56*] [*75*] [*60*] [*76*] [*77*] that this has a suggestive resemblance to the muon rest mass

$$M_\mu = 105.65 \pm .01 \text{ Mev.}$$

7. Absorber Theories of Radiation: The Phenomenological Theory of Radiation Reaction

In the present section we shall discuss two developments which stand somewhat apart from the other theories considered thus far — the absorber theory of radiation of WHEELER and FEYNMAN [*78*] and the phenomenological radiation reaction theory of GINZBURG and EIDMAN [*54*]. The fundamental point in both of these theories is that somehow the presence of absorbers should be crucial to the elementary act of the emission of radiation. The Wheeler-Feynman approach is "fundamental" in its orientation whereas GINZBURG and EIDMAN are frankly "applied" — nevertheless these theories have a good deal of conceptual common ground albeit there is a lively contrast in some of the ideas involved. We consider first the Wheeler-Feynman approach.

A. Wheeler-Feynman Electrodynamics

The basic observation of the absorber theory of radiation is the fact that the mere presence of an absorber may produce a radiation reaction on the emitting particle. This can be most clearly demonstrated in terms of the two body problem of electrodynamics. Consider for example two objects — one far heavier than the other — bound to each other through a purely electromagnetic interaction. It was first shown by SYNGE[24]) that such a system would lose energy — the particles spiraling inwards towards a collision — *even without the factoring of the Lorentz force law into an external and a damping term* (see (2.15)). The system redistributes its energy into the radiation field simply through the action and reaction of its two bound charges. It is intuitively obvious however that such a process is slow and inefficient; indeed SYNGE finds that this system collapses at a rate of one thousands slower than one which includes explicitly the radiation reaction forces in its equations of motion.
The essential point of the Wheeler-Feynman approach now is that it is possible to amplify these reactive losses sufficiently so that they become formally identical to the ordinary radiation reaction: one simply assumes that the physical solutions of the field equation

$$\Box^2 A = j$$

[24]) See [79]. A rigorous discussion of this problem has recently been given by DRIVER [*58*].

are half the sum of the retarded and the advanced potentials. Formally this obviates the sequence of decompositions (2.15) — (2.15c) and automatically inserts the correct radiation reaction expression into the equations of motion. Physically therefore one has the picture that e. g. a charged oscillator is damped because it radiates both *forward* and *backward* in time: The radiation travelling forward in time causes electrons in the absorber to move — these in turn emit radiation both forward and backward in time. The radiation travelling backward in time reaches the oscillator at the same time that the original radiation was emitted and exerts a retarding force on the oscillator — this is the radiation reaction. The total field of oscillator and absorber combine to give the fully retarded field which corresponds to "experience".

Naturally the appearance of the acausal advanced potentials is the most startling feature of this theory, and WHEELER and FEYNMAN have gone to great lengths to show that in the case of complete absorption the results are formally identical to those of the conventional radiation reaction theory. (See also [*91*]). In particular all of the details of the absorber structure are shown to disappear from the final formulas; a result which in view of the discussion of the next section is perhaps not entirely fortunate. Moreover it is no easier in this formulation than in any other to see in detail how the radiation reaction of certain many body problems e. g. a ring charge drops to zero as one makes the transition to the continuum limit.

B. The Theory of GINZBURG and EIDMAN

The phenomenological radiation reaction theory of GINZBURG and EIDMAN [*54*] [*80*] is an attempt to give a unified description of the dynamical consequences of certain interactions of charged particles with matter. It will be helpful in fixing the ideas involved if we first give a brief qualitative summary of some representative experimental situations. It should be noted that Bremsstrahlung and other microscopic phenomena are ignored in the subsequent discussion.

(i) *The ordinary Cerenkov effect* is one of the most elementary examples of a departure from the

$$\boldsymbol{F}_{\text{react}} = \frac{2}{3}\,\frac{e^2}{c^3}\,\dddot{\boldsymbol{x}} \tag{7.0}$$

law of the point-charge vacuum theory. The correct expression for the radiation reaction in this case is

$$\boldsymbol{F}_{\text{react}} = -\frac{e^2}{c^2}\,\hat{\boldsymbol{v}} \int\limits_{v \geq c/n} \left[1 - \left(\frac{c}{v\,n(\omega)}\right)^2\right] \omega\, d\omega\,. \tag{7.1}$$

If the index of refraction n is a sufficiently slowly varying function of ω, and if the integral is cut off at $\omega \gtrsim c/r_0$, it can be shown that (7.1) reduces to the expression previously given by SOMMERFELD [see (6.13)]. It is interesting to note that such a formula may continue to apply even if the particle itself is moving in a vacuum: This for instance will be the case if the motion occurs in a narrow channel embedded in a medium with index of refraction $n > 1$. A futher extension of this description is useful for motions which occur in a waveguide or near some other periodic structure; for example, in a traveling wave tube used for the generation of millimeter microwaves.

(ii) *The complex Doppler effect* (*Cerenkov self-excitation*): Consider a system possessing a natural radiating frequency ω_0 (measured in the rest system) moving with a velocity v through a medium characterized by an index of refraction $n(\omega)$. The Doppler shifted frequency ω_d observed at an angle Θ with respect to the direction of motion is then in general determined by the equation

$$\omega = \frac{\omega_0\,[1-\beta^2]^{1/2}}{|1-\beta n(\omega)\cos\Theta|}, \quad \beta = \frac{v}{c}\,. \tag{7.2}$$

Suppose now that the auxiliary inequality

$$v\cos\Theta \geq w(\omega) \tag{7.3}$$

where w denotes the group velocity, is satisfied for a value of ω_d; it then follows automatically that for a fixed Θ there will actually be at least *two* Doppler components, i. e., $\omega_{\mathrm{d}} \to \omega_{\mathrm{d}}^{(1)}$ and $\omega_{\mathrm{d}}^{(2)}$ which are solutions of (7.2). This condition is the *Complex Doppler Effect*.[25])

Suppose furthermore that the system velocity v is sufficiently great so that

$$\beta n(\omega) > 1\,. \tag{7.4}$$

In this case, in addition to the "doubled" range of Doppler frequencies the system will also be able to emit Cerenkov radiation. Let the opening angle of the Cerenkov cone corresponding to ω_{d} be Θ_{d}, i. e.

$$\Theta_{\mathrm{d}} = \sec^{-1}[\beta n(\omega_{\mathrm{d}})]. \tag{7.5}$$

Then for all Doppler frequencies emitted into the cone $\Theta < \Theta_d$ we will have the remarkable circumstance that the emission of radiation will be accompanied by a simultaneous *excitation* of the system. (Cerenkov Self-Excitation!) Although such a result is at first sight surprising it is in fact a simple consequence of the conservation of energy and momentum for the emission of radiation from a quasi-bound (i. e. $n > 1$) system. Quantitatively this is described by

$$\triangle U = -E(\omega)\,[1-\beta n(\omega)\cos\Theta] \tag{7.6}$$

where $E(\omega)$ is the Cerenkov energy at frequence ω and $\triangle U$ is the corresponding increment of internal energy. Clearly if $\beta n(\omega)\cos\,\Theta > 1$, then $\triangle U > 0$; and this is the condition for Cerenkov self-excitation.

This self-excitation has the following dynamic consequences: Suppose that a harmonic oscillator of frequency ω_0 vibrates along the z-axis and simultaneously is in translation such that $v_{\mathrm{av}} = v_z > c/[n(\omega)\cos\Theta]$. In this event the emission of radiation to the regions *outside* the Cerenkov cone will be associated with a *damping* of the oscillator vibrations; on the other hand the radiation *into* the Cerenkov cone will be associated with an *increase* of the amplitude of oscillation. There is of course no question of a violation of the conservation of energy here since both the oscillation and the radiation occur at the expense of the kinetic energy of translation. Nevertheless this presents the novel feature of a radiation

[25]) I M. FRANK [*81*]. Necessary conditions for the occurrence of the *Complex Doppler Effect* have been derived by BARSUKOV [*82*].

reaction force formally *decreasing* the damping of a radiating oscillator. As emphasized by GINZBURG this circumstance is of practical importance in considering the beam stability of "super-light" bunches in a plasma accelerator.[26])

(iii) *The Inverse Complex Doppler and Cerenkov Effects* [84]: Radiation reaction becomes an even more complicated affair if one admits the possibility of negative group velocities. This is a matter of some practical importance in media having anisotropic and gyrotropic properties as well as in regions of high anomalous dispersion. A simple illustration is provided by an idealized medium with $\varepsilon < 0$ and $\mu < 0$. Since $n = (\varepsilon\mu)^{1/2}$ remains a positive quantity in such a medium undamped electromagnetic waves may still be propagated. However phase and group velocities will now have opposite directions since the plane wave condition requires that

$$\boldsymbol{S} = \zeta\,|E|^2\,\boldsymbol{k} \qquad (7.7)$$

and for this case $\zeta = -|\varepsilon/\mu|^{1/2}$. ($\boldsymbol{S}$, $\boldsymbol{k}$, and E have their usual significance as Poynting vector, wave vector, and electric field respectively.)

An interesting peculiarity that makes its appearance here is the necessity for using *advanced* potentials in the construction of solutions of Maxwell's equations. To preserve the correct sense of the energy flow the sign of t must be reversed in tandem with the extra minus sign acquired by $\boldsymbol{k}$ because of (7.7). Physically this means that the conventional picture of the Cerenkov cone is reflected into its mirror image, i. e. the Cerenkov cone makes an *obtuse* angle with the particle motion. A particular consequence of this is that the transition radiation emitted in the passage of fast particles from vacuum to medium will in this case go backwards into the vacuum. If now in addition there are Doppler frequencies present due to internal degress of freedom, then the *low* frequency components will be radiated *forwards* while the *high* frequency components will go *backwards*. (Inverse Doppler Effect!) Finally if we have all three conditions present simultaneously, i. e. Cerenkov effect, Doppler emission, and negative group velocities; then it may happen that the anomalous Doppler components will outweigh the normal components, so that the total emission will in fact be associated with a net *excitation* of the oscillation. The radiation reaction work in this event is a purely positive quantity. It should be noted that this is in sharp contrast to the assumptions of Section 5.

(iv) *An idealized isotropic plasma* has a refractive index

$$n = \left[1 - \frac{4\pi\, e^2 N}{m\,\omega^2}\right]^{1/2}$$

for transverse electromagnetic waves. An oscillator of natural frequency $\omega^2 < 4\pi e^2 N/m$ embedded in this plasma will therefore be unable to lose energy by radiation. This brief summary should make it clear that the construction of a phenomenological radiation reaction theory endowed with sufficient mathematical flexibility poses a formidable technical problem. The most natural idea would be of course to start from some fundamental "vacuum" theory—the $\boldsymbol{F}_{\text{react}} = (2e^2/3c^3)\,\dddot{\boldsymbol{x}}$ version for example — and then to work forwards towards a macroscopic description via suitable averaging processes. It should be realized however that this cannot simply be a matter of imitating the procedures of ordinary electrodynamics: The

[26]) If $\omega_0 = 0$, (7.3) still is relevant as a criterion for the onset of the *Complex Cerenkov Effect*, AGRANOVICH et al [83].

usual transition (Microscopic Maxwell Equations) $\Rightarrow$ (Phenomenological Maxwell Equations) merely involves the replacements $1 \Rightarrow \|\varepsilon\|$, $1 \Rightarrow \|\mu\|$ in the constitutive relations; the essential structure of the differential equations remains intact. In the radiation reaction case it should however be obvious, e. g. from the contrast between (7.0) and (7.1), that more drastic alterations will be required.[27])

GINZBURG and EIDMAN [54] meet this problem by going back to the Sommerfeld extended electron picture. The radiation reaction is once again visualized as originating in a net force resulting from the action of one part of the electron on another. It has already been shown (Section 6-C) that this approach gives rise to a mathematical structure of considerable complexity. The crucial departure in the Ginzburg-Eidman development is that the information on the macroscopic structure of the medium is inserted into the formalism by assuming the validity of the constitutive equations

$$D = \|\varepsilon\|\,E, \quad B = \|\mu\|\,H$$

in the interior of the electron. This leads precisely to the previous Fourier self-force representation (6.6a) except that now the index of refraction appears explicitly in the integrand. In a detailed discussion GINZBURG and EIDMAN show that this recipe is indeed capable of reproducing some of the behavior expected in the situations enumerated above. The spirit of the treatment is however completely pre-relativistic — an especially unpleasant feature considering the high velocities encountered in the Cerenkov effect. GINZBURG and EIDMAN attempt to minimize this embarrassment by making the transition $\varrho(r) \to \delta(r)$ as soon as feasible; the subsequent divergence is then absorbed as a mass renormalization. But this procedure is itself of course neither covariant nor model-independent. A counter example was provided long ago by SOMMERFELD [31] in his proof that the spherical shell electron would suffer *infinite* radiation reaction under Cerenkov conditions. It may be concluded that the formulation of a completely satisfactory macroscopic radiation reaction theory still remains as an unsolved problem.

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Appendix 1
Inequalities Connected With the Radiation Equation

We consider the modified Fowler-Emden equation

$$\ddot{y} = -\frac{\dot{W}}{y} \qquad (A\,1.1)$$

[27]) Another important point that arises here is the "Synge dissipation" [79]. If indeed two charges can emit radiation even in the absence of any explicit damping terms, then one must anticipate the possibility of cooperative dissipation mechanisms in a macroscopic theory. (I am indebted to Prof. Prigogine for an enlightening discussion on this point).

on the closed interval Ω_c: $0 \leq t \leq t_1$, under the assumptions

(i) $\ddot{y}(t)$ continuous on Ω_c,
(ii) $W(t)$ increasing and non-negative on Ω_c,
(iii) $y(t)$ and $\dot{y}(t)$ non-negative on Ω_c.

and with the initial conditions

(iv) $y(0) = 0$; $\dot{y}(0) = a_0 > 0$.

From (i) — (iii) it follows that $y(t)$ is a concave function. In particular from (iv) we conclude that

$$y(t) \leq a_0 t. \tag{A1.2}$$

Inserting this into (A1.1) we obtain the inequality

$$-\ddot{y} \geq \frac{\dot{W}}{a_0 t}. \tag{A1.3}$$

Integrating from 0 to t, there results

$$-\dot{y}(t) + a_0 \geq \frac{1}{a_0}\int_0^t \left(\frac{\dot{W}}{t}\right) dt; \tag{A 1.4}$$

where the existence of the integral is guaranteed by (A1.3) and (i).
But now

$$\int_0^t \left(\frac{\dot{W}}{t}\right) dt = \frac{W}{t} + \int_0^t \left(\frac{W}{t^2}\right) dt \geq \frac{W}{t} \tag{A.15}$$

which again is guaranteed by (i) and the mean value theorem.
Therefore

$$a_0 - \dot{y}(t) \geq \frac{W}{a_0 t}. \tag{A1.6}$$

Since $a_0 \geq a_0 - \dot{y}(t)$, this yields the important result

$$W(t) \leq a_0^2 t. \tag{A1.7}$$

We may now perform another quadrature: From (A1.6) it follows that

$$y(t) \leq a_0 t - \frac{\tilde{W}}{a_0}, \tag{A1.8}$$

where

$$\tilde{W} \equiv \int_0^t \left(\frac{W}{t}\right) dt \tag{A1.9}$$

and (A1.7) assures the existence of the integral.
The crucial point of the argument now becomes apparent: If the original inequality (A1.2) is taken through this cycle, then the end result (A1.8) is an inequality

which is an improvement over the input. The expectation therefore is that the entire process can be iterated with successive improvements. Let us consider the next cycle:
Putting $y_1(t) \equiv a_0 t - \tilde{W}/a_0$; then from (A1.1) it follows that

$$-\ddot{y} \geq \frac{\dot{W}}{y_1}. \tag{A1.10}$$

Integrating,

$$-\dot{y}(t) + a_0 \geq \int_0^t \left(\frac{\dot{W}}{y_1}\right) dt; \tag{A1.11}$$

where now

$$\int_0^t \left(\frac{\dot{W}}{y_1}\right) dt = \frac{W}{y_1} + \int_0^t \left(\frac{W}{y_1^2}\right) \dot{y}_1\, dt \geq \frac{W}{y_1} \tag{A1.12}$$

and we note that the boundedness of the derivatives and the mean value theorem guarantee stepwise the existence of each subsequent parts integration.
Therefore

$$W(t) \leq a_0\, y_1(t); \tag{A1.13}$$

and integrating again

$$a_0 t - y(t) \geq \int_0^t \left(\frac{W}{y_1}\right) dt. \tag{A1.14}$$

But now

$$\frac{W}{y_1} \geq \frac{W}{a_0 t}\left(1 + \frac{\tilde{W}}{a_0^2 t}\right); \tag{A1.15}$$

and

$$\int_0^t \frac{W \tilde{W}}{t^2} dt = \frac{1}{2}\frac{\tilde{W}^2}{t} + \frac{1}{2}\int_0^t \frac{\tilde{W}^2}{t^2} \geq \frac{1}{2}\frac{\tilde{W}^2}{t}. \tag{A1.16}$$

Therefore the final result of the second iteration is

$$y(t) \leq a_0 t - \frac{\tilde{W}}{a_0} - \frac{\tilde{W}^2}{2 a_0^3 t} \tag{A1.17}$$

which, as expected, is a further improvement over (A1.2). It is now clear that this pattern may be repeated n-fold. On the n^{th} iteration one obtains

$$y(t) \leq a_0 t \left\{1 - \sum_{k=1}^{n} \frac{\tilde{W}^k}{(a_0^2 t)^k\, k!}\right\};$$

or, proceeding to the limit,

$$y(t) \leq a_0 t \left\{2 - \exp\left[\frac{\tilde{W}}{a_0^2 t}\right]\right\}. \tag{A1.18}$$

This is the result stated in the text as (5.12). Since $y > 0$ for $t > 0$, (A1.18) may also be considered a transcendental inequality for $\tilde{W}$; the solution is elementary and yields (5.13b) of the text. The n^{th} analogues of (A1.7) and (A1.13) are the $W(t)$ restrictions (5.13a). It is interesting to note that this process yields *simultaneous* restrictions on y and W.

Appendix 2
Sommerfeld's Self-Force Representation

Theorem: The trajectory defined by $x(t + 2r_0/c) = x(t)$, $|\dot{x}| < c$, for a uniformly charged electron of radius r_0, is neither radiationless nor force-free.

Proof: Let

$$\varepsilon(t, \tau) = c\tau + \mathcal{T} \quad \text{where} \quad \mathcal{T} = x(t) - x(t - \tau). \tag{A2.1}$$

Then

$$\varepsilon(t, \tau_0) - 2r_0 = 0; \quad \tau_0 \equiv \frac{2r_0}{c}. \tag{A2.1a}$$

This solution is unique since $\partial\varepsilon/\partial\tau > 0$ and $\varepsilon(t, 0) = 0$.
Similarly if

$$\tilde{\varepsilon}(t, \tau) = c\tau - \mathcal{T},$$

then

$$\tilde{\varepsilon}(t, \tau) \geq 0 \quad \text{and} \quad \tilde{\varepsilon}(t, \tau_0) - 2r_0 = 0. \tag{A2.1b}$$

This establishes that the equations $c\tau + \mathcal{T} = 2r_0$; $|c\tau - \mathcal{T}| = 2r_0$ both have the unique root $\tau = \tau_0$.
The self-force representation (6.7) may therefore be rewritten in the form

$$F_{\text{self}} = -\frac{3e^2}{8r_0^2}\{I_1 + I_2 + I_3\} \tag{A2.2}$$

where

$$I_1 = -\int_0^{\tau_0} d\tau \frac{c}{\mathcal{T}^2}\{f(c\tau + \mathcal{T}) - f(c\tau - \mathcal{T})\} \tag{A2.2a}$$

$$I_2 = \int_0^{\tau_0} d\tau \frac{c}{\mathcal{T}} \frac{\partial}{\partial\mathcal{T}}\{f(c\tau + \mathcal{T}) - f(c\tau - \mathcal{T})\} \tag{A2.2b}$$

$$I_3 = \int_0^{\tau_0} d\tau \frac{\dot{x}(t - \tau)}{\mathcal{T}} \frac{\partial}{\partial\mathcal{T}}\{f(c\tau + \mathcal{T}) + f(c\tau - \mathcal{T})\}. \tag{A2.2c}$$

Inserting the explicit form of f [see (6.7a)] we have

$$I_1 = -c\int_0^{\tau_0} d\tau \left\{\mathcal{T}^{-1}\left[\frac{(c\tau)^4}{2r_0^2} - 6(c\tau)^2 + 8r_0 c\tau\right] + \right.$$
$$\left. + \mathcal{T}\left[\left(\frac{c\tau}{r_0}\right)^2 - 2\right] + \mathcal{T}^3\left[\frac{1}{(10r_0^2)}\right]\right\} \tag{A2.3a}$$

$$I_2 = c\int_0^{\tau_0} d\tau \left\{\mathcal{T}^{-1}\left[\frac{(c\tau)^4}{2r_0^2} - 6(c\tau)^2 + 8r_0(c\tau)\right] + \right.$$
$$\left. + \mathcal{T}\left[3\left(\frac{c\tau}{r_0}\right)^2 - 6\right] + \mathcal{T}^3\left[\frac{1}{(2r_0^2)}\right]\right\} \tag{A2.3b}$$

$$I_3 = \int_0^{\tau_0} d\tau\, \dot{x}(t - \tau)\left\{\left[2\frac{(c\tau)^3}{r_0^2} - 12c\tau + 8r_0\right] + \mathcal{T}^2\left[\frac{2c\tau}{r_0^2}\right]\right\}. \tag{A2.3c}$$

The $\mathcal{T}^{-1}$ terms give rise to a divergence which cancels out when the integrals are combined. It is now convenient to transform I_3 by a parts integration; this results in

$$I_3 = -c\int_0^{\tau_0} d\tau \left\{\mathcal{T}\left[6\frac{(c\tau)^2}{r_0^2} - 12\right] + \mathcal{T}^3\left[\frac{2}{(3r_0^2)}\right]\right\}. \tag{A2.3d}$$

Collecting all the terms we finally obtain

$$I_1 + I_2 + I_3 = 4c\int_0^{\tau_0} d\tau \left\{\mathcal{T}\left[2 - \frac{\tau^2}{\tau_0^2}\right] - \mathcal{T}^3\left[\frac{1}{(15r_0^2)}\right]\right\}. \tag{A2.4}$$

(Since $\mathcal{T} \sim \mathcal{O}(vr_0/c)$, a linear approximation would now be tantamount to dropping the $\mathcal{T}^3$ term.)
From (A2.4) we read off that $F_{\text{self}}(t + \tau_0) = F_{\text{self}}(t)$; $\dot{x}$ is of course also periodic with period τ_0. It is therefore sufficient to consider the work done over a period:

$$W(\tau_0) = \int_0^{\tau_0} F_{\text{self}}(t)\,\dot{x}(t)\,dt. \tag{A2.5}$$

In carrying out this computation one encounters three different types of integrals:

$$J_1 = \int_0^{\tau_0} dt \int_0^{\tau_0} d\tau\, \dot{x}\mathcal{T} \tag{A2.6a}$$

$$J_2 = \int_0^{\tau_0} dt \int_0^{\tau_0} d\tau\, \dot{x}\tau^2\mathcal{T} \tag{A2.6b}$$

$$J_3 = \int_0^{\tau_0} dt \int_0^{\tau_0} d\tau\, \dot{x}\mathcal{T}^3. \tag{A2.6c}$$

With the help of the auxiliary integrals

$$\frac{\partial}{\partial t}\int_0^{\tau_0} d\tau\, x^n(t - \tau) = 0$$

$$\int_0^{\tau_0} dt\, x^n(t)\,\dot{x}(t) = 0, \quad n \geq 0 \tag{A2.7}$$

it is easy to show that

$$J_1 = 0 = J_3. \quad \text{(A2.8)}$$

However J_2 has a non-vanishing component. Let

$$J_2 = J_2^{(1)} + J_2^{(2)} = \int_0^{\tau_0} dt \int_0^{\tau_0} d\tau [\tau^2 x(t)\,\dot{x}(t)] - \int_0^{\tau_0} dt \int_0^{\tau_0} d\tau\, [\tau^2\, x(t-\tau)\,\dot{x}(t)]. \quad \text{(A2.9)}$$

Then $J_2^{(1)} = 0$ because of (A2.7). $J_2^{(2)}$ may be transformed by successive integrations by parts; first with respect to t, then with respect to τ:

$$J_2^{(2)} = -\tau_0^2 \int_0^{\tau_0} dt\, x^2(t) + J_2^{(3)};$$

$$J_2^{(3)} = 2 \int_0^{\tau_0} dt \int_0^{\tau_0} d\tau\, [\tau x(t)\, x(t-\tau)]. \quad \text{(A2.9a)}$$

Now we note that $W(\tau_0)$ is invariant under the transformation $x(t) \to x(t) +$ const. There will therefore be no loss of generality in assuming that $\dot{y} \equiv x$ also defines a periodic function. $J_2^{(3)}$ may then again be integrated by parts just as in the transition from (A2.9) to (A2.9a).

$$J_2^{(3)} = \tau_0 \int_0^{\tau_0} dt\, y(t)\, \dot{y}(t) - \int_0^{\tau_0} dt \int_0^{\tau_0} d\tau [y(t)\, x(t-\tau)] = 0. \quad \text{(A2.9b)}$$

Finally therefore we obtain

$$W(\tau_0) = -\frac{3e^2 c}{2 r_0^2} \int_0^{\tau_0} dt\, x^2(t) < 0. \quad \text{(A2.10)}$$

A fortiori this proves that

$$F_{\text{self}} \not\equiv 0. \quad \text{(A2.11)}$$

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