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Electrodynamics of Hyperbolically Accelerated Charges

IV. Energy-Momentum Conservation of Radiating Charged Particles

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The relativistic equation of motion of a radiating charge is discussed with special emphasis upon a clarification of the significance of the Schott energy for the energy-momentum conservation of the charge and the field it produces. In particular hyperbolic motion is studied. The case that a charge with constant velocity enters and leaves a region with hyperbolic motion is analysed. We find that the Schott energy is increased as the particle enters the region and that the energy it radiates while the charge moves hyperbolically comes from the Schott energy. A result of our analysis is that this energy is localized in the field close to the charge.

1. INTRODUCTION

The analysis of the energy-momentum balance of a radiating charge is usually based upon the equation of motion of a point charge. The non-relativistic version of the equation was discussed already about hundred years ago by H. A. Lorentz [1]. The relativistic generalization of the equation was originally found by M. Abraham [2] in 1904 from an analysis of conservation of energy and momentum, and rederived in 1909 by M. von Laue [3] who Lorentz transformed the non-relativistic equation from the instantaneous rest frame of the charge to an arbitrary frame. A covariant deduction of the equation was given by P. A. M. Dirac in 1938 [4]. The relativistic equation of motion of a radiating point charge shall henceforth be called the *Lorentz–Abraham–Dirac equation*, or for short, the *LAD-equation*. The uniqueness of this equation has been discussed by H. J. Bhabha [5] and E. P. G. Rowe [6].

F. Rohrlich [7, 8] has recently argued that the equation is asymmetric under time reversal. Hence, there seems to exist an arrow of time in the fundamental equations of classical electrodynamics.

There are three well known problems with the equation, both in its relativistic and non-relativistic form [4, 5]: the electromagnetic self-energy problem, the existence of "runaway solutions," and the acausal phenomenon of pre-acceleration. The first problem can be "solved" by mass renormalization with the result that only the observable physical mass appear in the Eq. (9). This problem will not be treated here. Runaway and pre-acceleration have been considered by several authors [10–18]. A thorough analysis of the LAD-equation, its problems and ways of resolving them, has recently been given by A. D. Yaghjian [19].

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A term in the equation of the energy-momentum balance of an accelerated charge and its electromagnetic field following from the equation of motion of a radiating point charge and Maxwell's field equations was noted by Schott [20] in 1915 and has been much discussed later, in particular by F. Rohrlich [21]. These matters will be considered in Sections 5, 6. About 20 years ago an advance in the understanding of the so-called Schott acceleration energy term was obtained by C. Teitelboim [22] who studied separately the near field and far field of an accelerating charge. This will be taken up in Sections 2–4. We shall make a systematic study of the energy-momentum relationships of a hyperbolically accelerated charge and its electromagnetic field, based upon this separation, and a further separation by E. G. P. Rowe [23], in Sections 5–7. In Sections 8–10 we give a detailed analysis of the evolution of different forms of energy when a charge enters and leaves a region of hyperbolic motion.

In Appendix A the electron is considered as an extended particle in an approximation [19] which is linear in the velocity and its derivatives. In Appendix B we evaluate the self force on a spherically symmetric particle in the limit of vanishing extension by summing up the internal forces in the rest frame of the charge and in the laboratory frame.

2. THE NON-RELATIVISTIC EQUATION OF MOTION OF A RADIATING CHARGE

The equation of motion of a radiating charge, Q, with physical mass m, acted upon by an external force f_{ext}, takes the form

$$m\ddot{\mathbf{R}} = \mathbf{f}_{ext} + m\tau_0\dddot{\mathbf{R}}, \qquad \tau_0 = 2Q^2/3m. \tag{2.1}$$

The general solution of the equation is

$$\ddot{\mathbf{R}}(T) = e^{T/\tau_0}\left[\ddot{\mathbf{R}}(0) - \frac{1}{m\tau_0}\int_0^T e^{-T'/\tau_0}\mathbf{f}_{ext}(T')\,dT'\right]. \tag{2.2}$$

By choosing the initial condition

$$m\tau_0\ddot{\mathbf{R}}(0) = \int_0^\infty e^{-T'/\tau_0}\mathbf{f}_{ext}(T')\,dT' \tag{2.3}$$

runaway behaviour is suppressed. Combining Eqs. (2.2) and (2.3) one obtains [24]

$$m\ddot{\mathbf{R}}(T) = \int_0^\infty e^{-s}\mathbf{f}_{ext}(T + \tau_0 s)\,ds. \tag{2.4}$$

This equation shows that the acceleration of the charge at a point of time T is determined by the future force, weighted by a decreasing exponential factor with value 1 at the time T, and a time constant τ_0; i.e., there is pre-acceleration. In the case of an electron the future time interval of significance for the present value of the acceleration has a length τ_0, which is roughly equal to the time taken by light to move a distance equal to the classical electron radius, i.e., $\tau_0 = 10^{-23}$ seconds. It has been pointed out [25] that this time interval is so short that one can hardly expect classical physics to be applicable within such time intervals.

In his discussion of Eq. (2.1) Lorentz [1] writes:

In many cases the new force represented by the second term in Eq. (2.1) may be termed a *resistance* to the motion. This is seen, if we calculate the work of the force during an interval of time extending from $T = T_1$ to $T = T_2$. The result is

$$\frac{2Q^2}{3}\int_{T_1}^{T_2} \dot{\mathbf{a}}\cdot\mathbf{v}\,dT = \frac{2Q^2}{3}[\dot{\mathbf{a}}\cdot\mathbf{v}]_{T_1}^{T_2} - \frac{2Q^2}{3}\int_{T_1}^{T_2}\mathbf{a}^2\,dT \tag{2.5a}$$

Here the first term disappears if, in the case of a periodic motion, the integration is extended to a full period; also, if at the instants T_1 and T_2 either the velocity or the acceleration is 0. Whenever the above formula reduces to the last term, the work of the force is seen to be negative, so that the name of resistance is then justly applied.

Exchanging the left hand term and the last term on the right hand side, Drukey [25] has commented on the equation in the following way:

> The second term is the work done against the radiation reaction and vanishes for uniform acceleration, but the first term, the change in a quantity characteristic of the instantaneous state of the motion and called by Schott the acceleration energy, just accounts for the radiation previously predicted. This term, usually neglected because attention is generally confined to periodic motions or to those bounded in time, accounts for the entire energy in this problem.

P. Yi [26] gives the following interpretation:

> The total energy of the system may be split into three pieces: the kinetic energy of the charged particle, the radiation energy, and the electromagnetic energy of the Coulomb field. In effect, the last acts as a sort of energy reservoar that mediates the energy transfer from the first to the second and *in the special case of uniform acceleration provides all the radiation energy without extracting any from the charged particle*.

Thomas Erber [27] does not accept this interpretation, writing:

> The interpretation attached to this equation [Eq. (2.5)] is the following: The first term on the right hand side is supposed to represent an *influx* of energy from the field in the vicinity of the particle—the so called "acceleration" or Schott energy—which then reappears in the second term as *radiation* to the far field zone of the particle. This interpretation is however clearly contrary to the essential spirit of the radiation reaction development to this point. We have consistently sought to identify the origin of radiated energy in the work done *by* the particle *on* the field: In the "acceleration energy" argument the accelerated particle becomes merely some kind of transducer which transforms near field energy into far field energy.

We shall now go on and discuss the relativistic generalization of Eq. (2.1).

3. THE RELATIVISTIC EQUATION OF MOTION OF A RADIATING CHARGE

The relativistic generalization of the equation of motion (2.1) is [4]

$$F^{\mu}_{ext} + \Gamma^{\mu} = m_0\dot{U}^{\mu}, \tag{3.1}$$

where

$$\Gamma^{\mu} \equiv \frac{2}{3}Q^2(\dot{A}^{\mu} - A^{v}A_{v}U^{\mu}) \tag{3.2}$$

and the dot denotes differentiation with respect to the proper time of the charge. The vector Γ^{μ} is called the *Abraham four-vector*. Being orthogonal to the four-velocity of the charge, Γ^{μ} may be written [21],

$$\Gamma^{\mu} = \gamma(\mathbf{v}\cdot\boldsymbol{\Gamma}, \boldsymbol{\Gamma}), \tag{3.3}$$

where $\mathbf{\Gamma}$ is a three-dimensional force. In order to express $\mathbf{\Gamma}$ in terms of the acceleration, and the *hyperacceleration*, $\mathbf{b} = d\mathbf{a}/dT$, we need the relations

$$A^\mu = (A^0, \mathbf{A}) = (\mathbf{v} \cdot \mathbf{A}, \mathbf{A}) = (\gamma^4 \mathbf{v} \cdot \mathbf{a}, \gamma^2 \mathbf{a} + \gamma^4 (\mathbf{v} \cdot \mathbf{a})\mathbf{v}) \tag{3.4}$$

and

$$A^\upsilon A_\upsilon = \gamma^4 [a^2 + \gamma^2 (\mathbf{v} \cdot \mathbf{a})^2] = g^2, \tag{3.5}$$

where g is the magnitude of the proper acceleration of the charge. Furthermore

$$\dot{A}^\mu = \gamma \frac{dA^\mu}{dT} = \gamma^3 [\gamma^2 (\mathbf{v} \cdot \mathbf{b} + a^2) + 4\gamma^4 (\mathbf{v} \cdot \mathbf{a})^2, \mathbf{b} + 3\gamma^2 (\mathbf{v} \cdot \mathbf{a})\mathbf{a} + \gamma^2 (\mathbf{v} \cdot \mathbf{b} + a^2)\mathbf{v} + 4\gamma^4 (\mathbf{v} \cdot \mathbf{a})^2 \mathbf{v}]. \tag{3.6}$$

This leads to

$$\mathbf{\Gamma} = (2/3) Q^2 \gamma^2 [\mathbf{b} + \gamma^2 (\mathbf{v} \cdot \mathbf{b})\mathbf{v} + 3\gamma^2 (\mathbf{v} \cdot \mathbf{a})\mathbf{a} + 3\gamma^4 (\mathbf{v} \cdot \mathbf{a})^2 \mathbf{v}] \tag{3.7a}$$

$$\mathbf{v} \cdot \mathbf{\Gamma} = (2/3) Q^2 \gamma^4 [\mathbf{v} \cdot \mathbf{b} + 3\gamma^2 (\mathbf{v} \cdot \mathbf{a})^2]. \tag{3.7b}$$

The expressions (3.7) may be written in a simple and enlightening way by utilizing the transformation properties of the components Γ^μ. The components in the laboratory frame are found by Lorentz transformations from the rest frame.

We write the Abraham vector (3.2) as the sum of a Schott term Γ_S^μ and a radiation reaction term Γ_R^μ,

$$\Gamma_S^\mu = (2/3) Q^2 \dot{A}^\mu \tag{3.8a}$$

$$\Gamma_R^\mu = -(2/3) Q^2 A^\upsilon A_\upsilon U^\mu \tag{3.8b}$$

$$\Gamma^\mu = \Gamma_S^\mu + \Gamma_R^\mu. \tag{3.8c}$$

By means of Eqs. (3.5) and (3.6) we get the following component in the rest frame

$$\Gamma_S'^\mu = (2/3) Q^2 (g^2, \mathbf{b}') \tag{3.9a}$$

$$\Gamma_R'^\mu = (2/3) Q^2 (-g^2, \mathbf{0}) \tag{3.9b}$$

$$\Gamma'^\mu = (2/3) Q^2 (0, \mathbf{b}'), \tag{3.9c}$$

where the hyperacceleration in the rest frame is $\mathbf{b}' = (d\mathbf{a}/dT)'$. By a boost transformation to the laboratory frame we get

$$\Gamma_S^\mu = (2/3) Q^2 \gamma (g^2 + \mathbf{v} \cdot \mathbf{b}', g^2 \mathbf{v} + \mathbf{b}'_\parallel + \gamma^{-1} \mathbf{b}'_\perp) \tag{3.10a}$$

$$\Gamma_R^\mu = (2/3) Q^2 \gamma (-g^2, -g^2 \mathbf{v}) \tag{3.10b}$$

$$\Gamma^\mu = (2/3) Q^2 \gamma (\mathbf{v} \cdot \mathbf{b}', \mathbf{b}'_\parallel + \gamma^{-1} \mathbf{b}'_\perp), \tag{3.10c}$$

where the hyperacceleration $\mathbf{b}'$ in the rest frame of the charge is decomposed relative to the direction of $\mathbf{v}$.

Equations (3.10) give the following three-dimensional forces.

The acceleration reaction force

$$\mathbf{\Gamma}_A = (2/3)Q^2(d\mathbf{A}/dT) = (2/3)Q^2(g^2\mathbf{v} + \mathbf{b}'_{\|} + \gamma^{-1}\mathbf{b}'_{\perp}). \tag{3.11a}$$

The radiation reaction force

$$\mathbf{\Gamma}_R = -(2/3)Q^2 g^2\mathbf{v}. \tag{3.11b}$$

The field reaction force (also called *the self force* [28])

$$\mathbf{\Gamma} = \mathbf{\Gamma}_A + \mathbf{\Gamma}_R = (2/3)Q^2(\mathbf{b}'_{\|} + \gamma^{-1}\mathbf{b}'_{\perp}), \tag{3.11c}$$

where we have used the designations suggested by Rohrlich [21].

The expressions are frequently written in terms of $\dot{\mathbf{g}} \equiv d\mathbf{g}/d\tau$, where τ is the proper time of the charge. It is tempting to think of $\mathbf{b}'$ and $\dot{\mathbf{g}}$ as identical quantities since $\mathbf{g}$ is the acceleration and τ the time both referring to the rest frame. However, there is a difference with respect to the differentiation. The hyperacceleration $\mathbf{b}'$ represents the change of the acceleration in a fixed rest frame of the charge at the point where the quantities are evaluated, while $d\mathbf{g}/d\tau$ represents the rate of change of proper acceleration along the path of the charge.

Expressed in terms of laboratory quantities the proper acceleration is given by

$$\mathbf{g} = \mathbf{g}_{\|} + \mathbf{g}_{\perp} = \gamma^3\mathbf{a}_{\|} + \gamma^2\mathbf{a}_{\perp} = (\gamma^3 - \gamma^2)\mathbf{a}_{\|} + \gamma^2\mathbf{a}. \tag{3.12}$$

In order to find the relationship between $\dot{\mathbf{g}}$ and $\mathbf{b}'$ we shall need the Lorentz transformation formula of the hyperacceleration. A straightforward calculation gives

$$\mathbf{b}' = \gamma^3(\gamma\mathbf{b}_{\|} + \mathbf{b}_{\perp}) + 3\gamma^5 v a_{\|}(\gamma\mathbf{a}_{\|} + \mathbf{a}_{\perp}). \tag{3.13}$$

Differentiating Eq. (3.12) and utilizing Eq. (3.13) we obtain

$$\dot{\mathbf{g}} = \mathbf{b}' + \frac{\gamma^5}{\gamma + 1}[a_{\perp}^2\mathbf{v} - \gamma(\mathbf{v}\cdot\mathbf{a})\mathbf{a}_{\perp}]. \tag{3.14}$$

In the case of rectilinear motion this equation reduces to

$$\mathbf{b}' = \dot{\mathbf{g}}. \tag{3.15}$$

From Eqs. (3.10) it follows that for rectilinear motion

$$\Gamma^{\mu} = (2/3)Q^2\gamma(\mathbf{v}\cdot\dot{\mathbf{g}},\ \dot{\mathbf{g}}). \tag{3.16}$$

Comparing with Eq. (3.3) we see that in this case

$$\mathbf{\Gamma} = (2/3)Q^2\dot{\mathbf{g}} \tag{3.17}$$

showing that for rectilinear motion the field reaction force $\mathbf{\Gamma}$ is independent of the velocity.

From Eqs. (3.11) the field reaction force can generally be written as

$$\mathbf{\Gamma} = \mathbf{\Gamma}_A + \mathbf{\Gamma}_R = \frac{2}{3}Q^2\frac{d\mathbf{A}}{dT} - \frac{2}{3}Q^2 g^2\mathbf{v}. \tag{3.18}$$

According to the relativistic Larmor formula the energy radiated by the charge per unit time is

$$\mathbb{R} = (2/3)Q^2 A^v A_v = (2/3)Q^2\gamma^6[a^2 - (\mathbf{v}\times\mathbf{a})^2] = (2/3)Q^2 g^2. \tag{3.19}$$

The radiated four-momentum per unit proper time is

$$P_R^\mu = \mathbb{R}U^\mu. \tag{3.20}$$

The radiated momentum per unit time is $\mathbb{R}\mathbf{v}$, the opposite vector being just the radiation reaction force

$$\mathbf{\Gamma}_R = -\mathbb{R}\mathbf{v} \tag{3.21}$$

which always acts against the motion. It may be noted that the power of this force, $-\mathbb{R}v^2$, is not equal to minus the radiated energy per unit time. Hence, the energy loss due to the force $\mathbf{\Gamma}_R$ does not account for the energy of the radiated field.

The power due to the field reaction force $\mathbf{\Gamma}$ may, by means of Eqs. (3.2)–(3.4), be written

$$\mathbf{\Gamma}\cdot\mathbf{v} = \frac{1}{\gamma}\Gamma^0 = \frac{2}{3}Q^2\frac{d(\mathbf{A}\cdot\mathbf{v})}{dT} - \frac{2}{3}Q^2g^2 = \frac{d}{dT}\left(\frac{2}{3}Q^2\gamma^4\mathbf{v}\cdot\mathbf{a}\right) - \mathbb{R}. \tag{3.22}$$

The first term is neither the rate of change of kinetic energy of the charge nor radiated power. Schott [20] called the energy

$$E_A = (2/3)Q^2\gamma^4\mathbf{v}\cdot\mathbf{a} \tag{3.23}$$

acceleration energy because it "must be regarded as work stored in the electron in virtue of its acceleration."

Following Rohrlich [21] (except for a change of sign) the energy

$$E_S = -E_A = -\frac{2}{3}Q^2A^0 = -\frac{2}{3}Q^2\gamma^4\mathbf{v}\cdot\mathbf{a} = -\frac{2}{3}Q^2\gamma\mathbf{v}\cdot\mathbf{g} \tag{3.24}$$

shall here be called the *Schott energy*.

The power due to the field reaction force may now be written

$$\mathbf{\Gamma}\cdot\mathbf{v} = -\frac{dE_S}{dT} - \mathbb{R}. \tag{3.25}$$

From the LAD-equation (3.1) we get the energy equation

$$\gamma\mathbf{v}\cdot\mathbf{F}_{ext} = m_0\dot{U}^0 - \Gamma^0 = m_0\dot{U}^0 - \frac{2}{3}Q^2\dot{A}^0 + \frac{2}{3}Q^2\gamma g^2. \tag{3.26}$$

With our choice of sign of the Schott energy (which will later be seen to be in accordance with letting a Schott momentum and the Schott energy be components of a Schott four-momentum) the energy equation takes the form

$$\frac{dW_{ext}}{dT} = \mathbf{v}\cdot\mathbf{F}_{ext} = \frac{d}{dT}(E_K + E_S + E_R), \tag{3.27}$$

where $E_K = (\gamma - 1)m_0$ is the kinetic energy of the particle, W_{ext} is the work on the particle due to the external force, E_R is the energy of the radiation field, and

$$\frac{dE_R}{dT} = \mathbb{R} = \frac{2}{3}Q^2 g^2. \tag{3.28}$$

The Schott energy E_S is a state function of the particle being positive when the velocity decreases and negative when it increases.

The work W_{ext} performed by the external force is equal to the sum of the changes in kinetic energy, Schott energy, and radiation energy,

$$W_{ext} = \Delta E_K + \Delta E_S + \Delta E_R. \tag{3.29}$$

4. PHYSICAL CONSEQUENCES OF THE LAD-EQUATION

In 1921 Pauli [29] made the following statement: “For hyperbolic motion the radiation reaction vanishes, as it should, since no radiation takes place.”

The first part of this statement follows immediately from Eq. (3.16) for the force of “radiation reaction,” i.e., field reaction, since the criterion for hyperbolic motion is that $\dot{\mathbf{g}} = 0$. The covariant form of this criterion (that the rate of change of rest acceleration of the particle vanishes) is

$$\dot{A}^\mu - A^v A_v U^\mu = 0 \tag{4.1}$$

showing that the Abraham force (3.2) vanishes for hyperbolic motion of a charge. The equation of motion, Eq. (3.1), takes the same form as for a neutral particle. A constant external force makes a neutral particle and a charged particle move in just the same way, just as if the accelerated charged particle did not radiate.

Rohrlich [21] has given an interesting discussion of the Abraham vector (note the opposite sign of E_S in the citations compared to in the main text):

> The Abraham four-vector Γ^μ defined in (3.2) is orthogonal to the velocity, i.e., $\Gamma^\mu U_\mu = 0$. This implies that a three-vector $\mathbf{\Gamma}$ can be defined such that
>
> $$\Gamma^\mu = (\gamma \mathbf{\Gamma} \cdot \mathbf{v},\ \gamma \mathbf{\Gamma}) \tag{4.2}$$
>
> This form implies that if $\mathbf{\Gamma}$ is interpreted as a force, Γ^0 is the work done by that force per unit proper time. From (4.2) follows
>
> $$\int_{\tau_1}^{\tau_2} \Gamma^\mu d\tau = \frac{2}{3}Q^2 \int_{\tau_1}^{\tau_2} (\dot{A}^\mu - A^v A_v U^\mu) d\tau = \frac{2}{3}Q^2[A^\mu(\tau_2) - A^\mu(\tau_1)] - \int_{\tau_1}^{\tau_2} \mathbb{R} U^\mu \, d\tau \tag{4.3}$$
>
> where $\mathbb{R}$ is given in Eq. (3.19). Thus, the work done by $\mathbf{\Gamma}$ between any two points 1 and 2 on the world line is
>
> $$\int_{\tau_1}^{\tau_2} \gamma \mathbf{\Gamma} \cdot \mathbf{v} \, d\tau = [E_S(\tau_2) - E_S(\tau_1)] - \int_{\tau_1}^{\tau_2} \mathbb{R} \, dT \tag{4.4}$$
>
> where the Schott energy (also called the “acceleration energy”) is defined by
>
> $$E_S = \frac{2}{3}Q^2 A^0 = \frac{2}{3}Q^2 \gamma^4 \mathbf{v} \cdot \mathbf{a} = \frac{2}{3}Q^2 \gamma \mathbf{v} \cdot \mathbf{g} \tag{4.5}$$
>
> The meaning of Eq. (4.4) is this: The work done by $\mathbf{\Gamma}$ is in general not equal to the energy lost in the form of radiation, but differs from it by the increase in the Schott energy. However, in the special case where the two points 1 and 2 have the same four-acceleration, the Schott energies at these two points are equal, and the energy radiated is just equal to the work done by $-\mathbf{\Gamma}$ during that time.

It is this special case which alone is usually considered and from which $\mathbf{\Gamma}$ received the name of "radiation reaction force." This is obviously a misnomer, since it implies that only the field radiated away is responsible for this force. In order to show that this is not the case consider two examples: In the first example we take an instant where $A^\mu = 0$, but $\dot{A}^\mu \neq 0$. Then $\mathbb{R} = 0$ but $\Gamma^\mu = (2/3)Q^2\dot{A}^\mu \neq 0$ so that we have "radiation reaction" without having radiation. In the second example we take uniformly accelerated motion, i.e., motion for which in the rest system $u^\mu = (1, 0, 0, 0)$, $A^\mu = (0, \mathbf{a})$, $\dot{A}^\mu = (\mathbf{a}^2, 0)$ and $\mathbf{a}$ is a constant. Then $\Gamma^\mu = 0$, since it is zero in the rest system, but $\mathbb{R} \neq 0$, i.e., we have radiation but no "radiation reaction."

The rate at which energy and momentum are lost by a charge due to radiation is $\dot{P}^\mu = \mathbb{R}U^\mu$. This four-vector is parallel to U^μ whereas Γ^μ is orthogonal to U^μ. These vectors therefore cannot be proportional. A reasonable definition of radiation reaction is, therefore,

$$\Gamma_R^\mu = -\mathbb{R}U^\mu \tag{4.6}$$

This vector vanishes if and only if there is no radiation. It is related to the four-vector of Abraham by

$$\Gamma^\mu = \Gamma_S^\mu + \Gamma_R^\mu \tag{4.7}$$

where

$$\Gamma_S^\mu = (2/3)Q^2\dot{A}^\mu \tag{4.8}$$

will be called the Schott vector. This vector vanishes only for uniform motion and is, therefore, of great importance whenever accelerated motion is considered. Its time component is the rate of change of Schott energy (4.5).

An important difference between the radiation rate $\mathbb{R}$ and the rate of change of the Schott energy must be emphasized at this point. The radiation rate is always positive (or zero) and describes an *irreversible* loss of energy; the Schott energy changes in a *reversible* fashion, returning to the same value whenever the state of motion repeats itself.

The radiation reaction four-vector Γ_R^μ has the components

$$\Gamma_R^\mu = (\gamma\mathbf{\Gamma}_R \cdot \mathbf{v} - \gamma^{-1}\mathbb{R}, \gamma\mathbf{\Gamma}_R) \tag{4.9}$$

The work done on the charge by the radiation reaction force $\mathbf{\Gamma}_R$ is related to $\mathbb{R}$ as follows

$$\int_{\tau_1}^{\tau_2} \gamma\mathbf{\Gamma}_R \cdot \mathbf{v}\, d\tau = \int_{T_1}^{T_2} \mathbb{R}v^2\, dT \tag{4.10}$$

But the integral over the time component of Γ_R^μ gives

$$\int_{\tau_1}^{\tau_2} \Gamma_R^0\, d\tau = \int_{\tau_1}^{\tau_2} \gamma\mathbf{\Gamma}_R \cdot \mathbf{v}\, d\tau - \int_{T_1}^{T_2} \gamma^{-2}\mathbb{R}\, dT = -\int_{T_1}^{T_2} \mathbb{R}\, dT \tag{4.11}$$

which is exactly the total energy lost by radiation. A reasonable name for the Abraham four-vector Γ^μ would be "field reaction."

We suggest that the four-vector Γ_R^μ given in Eq. (4.6) be called the *Rohrlich-vector*, due to the penetrating analysis of the LAD-equation given by Rohrlich [21], and that the term "radiation reaction" be reserved for the three-vector $\mathbf{\Gamma}_R$ given in Eq. (3.11b).

The time-component of the Abraham four-vector, Γ^0, represents the rate of work done by the field reaction. From Eq. (3.2) follows

$$\Gamma^0 = \frac{2}{3}Q^2\dot{A}^0 - \gamma\mathbb{R}. \tag{4.12}$$

This energy equation, and the related equation (3.27), is commented on by Grandy [30] in the following way:

It appears necessary to interpret the (indefinite) rate of change of the Schott energy as a change in the internal energy of the charge. Since $A^0 = 0$ in the rest frame, this change does not affect its rest mass. Rather, such energy must come from the velocity fields surrounding the particle and not escaping as radiation, and *not* contributing to its electromagnetic mass.

In Section 5 we shall discuss this interpretation by utilizing Teitelboim's separation [22] of the electromagnetic field into a velocity-field and a radiation-field.

Considering hyperbolic motion Rohrlich [21] writes:

> We have here an example of the importance of the Schott energy: The radiation rate is constant, but the field reaction vanishes. This is only possible if the Schott energy changes at a constant rate equal to the radiation rate (cf. (3.27)):
>
> $$\frac{dE_S}{dT} = \mathbb{R} \tag{4.13}$$
>
> The Schott contribution cannot be eliminated here, because throughout the hyperbolic motion there are no two points 1 and 2 such that $A^\mu(1) = A^\mu(2)$. Also, there is no reason for surprise that we have radiation while $\Gamma^\mu = 0$, since Γ^μ was earlier seen not to be the *radiation* reaction, but the *field* reaction.
>
> The equation of motion (3.1) then shows that *a neutral and a charged particle will fall equally fast* in a uniform gravitational field. This is at first a very surprising result, since only the charged particle will emit radiation and, consequently, will lose energy. It is therefore essential to investigate the energy balance in this case.
>
> Upon multiplication by **v**, Eq. (3.1) shows that the work done by the imposed force F^μ_{ext} is exactly equal to the change of the kinetic energy of the particle. If that is the case, what supplies the energy which is being radiated at the constant rate (3.19)?
>
> The answer to this question is to be found in the formalism we adopted; we assumed the validity of the equation of motion (3.1). This equation implies that $\Gamma^\mu = 0$ because of the balance of the Schott vector with the vector of radiation reaction. In particular, $\Gamma^0 = 0$ because the radiation energy rate equals the Schott energy rate. We are simply dealing with a special case of the Eq. (3.27) which represents the law of conservation of energy.
>
> The immediate physical interpretation which suggests itself here is that Schott's acceleration energy is part of the energy content of the moving particle, much in the same way as kinetic energy. The latter is concomitant with velocity, the former with acceleration. At any one instant the total energy content of a moving particle is given by
>
> $$E = m_0 + E_K - E_S \tag{4.14}$$
>
> where m_0 is the rest energy and E_S the Schott energy given in Eq. (4.5). Just like kinetic energy, the acceleration energy in no way affects the rest mass of the particle, its rate of change can be positive, zero or negative, and it vanishes in the instantaneous rest system, as is seen from Eq. (4.5).
>
> The main objection which might be raised against the physical picture which thus presents itself, emerges from the sign of the Schott energy. In the case of hyperbolic motion this sign causes a *decrease* of E at a constant rate $\mathbb{R} > 0$.
>
> In raising such an objection one must realize that this physical picture seems to be a necessary consequence of the equation of motion which was assumed. A modification of this equation would be necessary to avoid the Schott energy. As it stands, the equation of motion (3.1), in the spirit of the above remarks should be written in the form
>
> $$\frac{d}{d\tau}\left(m_0 U^\mu - \frac{2}{3}Q^2 A^\mu\right) = F^\mu_{ext} - \mathbb{R}U^\mu \tag{4.15}$$
>
> I would like to propose that Schott's acceleration energy should be taken seriously and put on the same level as the kinetic energy of a particle. It is to be emphasized, however, that this energy is non-vanishing only for *charged* particles which are accelerated and therefore occurs only at instances when fields are produced and radiation is emitted.

Note that with our definition (3.24) of the Schott energy, Eq. (4.14) is replaced by

$$E = m_0 + E_K + E_S. \tag{4.16}$$

In [31] Rohrlich writes:

> The physical interpretation of (4.14) is necessarily unconventional. Since according to Eqs. (4.13) and (4.14) the work done by the applied force and the increase in kinetic energy balance each other *exactly*, the source of radiation energy is mysterious by conventional ideas.
>
> The physical meaning of the term dE_S/dT in (4.13), representing the Schott energy rate, can be understood in several different ways.
>
> (a) If the Schott energy is expressed by the electromagnetic field, it would describe an energy content of the near field of the charged particle which can be changed *reversibly*. In periodic motion energy is borrowed, returned, and stored in the near field during each period. Since the time of energy measurement is usually large compared to

such a period only the average energy is of interest and that average of the Schott energy rate vanishes. Uniformly accelerated motion permits one to borrow energy from the near-field for large *macroscopic* time-intervals, and no averaging can be done because at no two points during the motion is the acceleration four-vector the same. Nobody has so far shown in detail just how the Schott energy occurs in the near field, how it is stored, borrowed, etc.

(b) One can take a dynamical approach and regard the term in question as an inertial term, writing the Dirac equation in the form (4.15). Thus, in addition to the *inertia of the mass*, expressed by the rate of change of momentum and kinetic energy, one has an *inertia of the charge*, expressed by the rate of change of the term $(2/3)Q^2a$ and of the Schott energy. This interpretation puts heavy emphasis on the Dirac equation, which is known to be incomplete without an asymptotic condition.

An analysis of the Schott energy of the type that Rohrlich has noted is lacking will be given in the following sections.

Rohrlich [31] goes on and proposes a modified equation of motion. In the present text we shall, however, mainly discuss the Maxwell–Lorentz–Dirac theory.

Fulton and Rohrlich [32] offer the following interpretation of Eq. (3.28):

The rate of work done by the external force equals the rate of increase of kinetic energy *minus* the rate of work done by the radiation reaction. The latter consists of two parts, a reversible rate, dE_S/dT, which can be positive or negative, and an irreversible rate, $-\mathbb{R}$, which is never positive. The sum $dE_S/dT - \mathbb{R}$ in general does not vanish. Since $\mathbb{R}$ is exactly the radiation rate, one sees that the energy lost in the form of radiation is entirely accounted for by part of the work done by the radiation reaction. On the other hand, the remaining part of this work also supplies an additional energy E_S which may be positive or negative. Apparently, E_S is to be interpreted as part of the internal energy of the charged particle. Like its kinetic energy it can be decreased or increased.

Schott has considered hyperbolic motion and comments [20]:

We see that the energy radiated by the electron is derived entirely from its acceleration energy; there is as it were an internal compensation amongst the different parts of the radiation pressure, which causes its resultant effect [on the motion of the electron] to vanish.

The interpretation of Fulton and Rohrlich is:

In the case of uniform acceleration, Γ^0 is zero, i.e., the total work done by the radiation reaction vanishes. Therefore $\mathbb{R}$ in Eq. (4.12) is positive. The internal energy of the electron, $m - E_S$, therefore decreases while energy is being radiated.

This result seems to lead to a very unphysical picture: The accelerated electron decreases its “internal energy,” transforming it into radiation. Does this mean that the rest mass of the electron decreases? An observer for whom the electron is momentarily at rest ($\mathbf{v} = 0$) will also find $A^0 = \gamma \mathbf{v} \cdot \mathbf{g} = 0$ and therefore $E_S = 0$. Thus we obtain the comforting result that the change in internal energy of the particle does not affect its rest mass. Rather, the radiation energy is compensated by a decrease of that part of the field surrounding the charge, which does not escape to infinity (in the form of radiation) and which does not contribute to the (electromagnetic) mass of the particle.

Grandy [30] gives the following interpretation:

The particle radiates irreversibly into the far field because $\mathbb{R}_{IN}$ is nonzero, but all of the radiated energy is supplied by the Schott term. It is the interaction energy between bound and radiated fields, both of which are defined locally, that provides all the radiated energy in this case, and the energy provided by the external force is converted completely into particle kinetic energy.

Fulton and Rohrlich leave the following question open:

If one accepts the equation of motion (3.1), (3.2) as correct, what is the physical meaning of the acceleration energy and the apparently arbitrarily large depletion of the charge’s internal energy by radiation in the course of its motion?

In order to exhibit some of the contents in the equation of motion (3.1) we shall consider motion of a charged particle through a limited electrical field, specified by

$$\begin{aligned} E(\tau) &= 0, && \tau < 0 \\ E(\tau) &= E_0, && 0 < \tau < \tau_1 \\ E(\tau) &= 0, && \tau > \tau_1 \end{aligned} \tag{4.17}$$

as a function of the proper time τ of the particle. This was considered in the non-relativistic limit by R. Haag [33] and G. N. Plass [34], and generalized to the relativistic case by T. C. Bradbury [35].

Defining the rapidity α by

$$\tanh\alpha = v \tag{4.18}$$

so that $\gamma = \cosh\alpha$ and $\gamma v = \sinh\alpha$, the four-velocity U^μ and the four acceleration, $A^\mu = \dot{U}^\mu$ where the dot denotes differentiation with respect to the proper time of the particle, can be written

$$U^\mu = (\cosh\alpha, \sinh\alpha, 0, 0), \qquad A^\mu = \dot{\alpha}(\sinh\alpha, \cosh\alpha, 0, 0). \tag{4.19}$$

Thus the rest acceleration of the charge is $\sqrt{A^\mu A_\mu} = \dot{\alpha}$. Substituting the expressions (4.19) into the Lorentz–Dirac equation (3.1) one obtains

$$(2/3)Q^2\ddot{\alpha} + QE = m_0\dot{\alpha}. \tag{4.20}$$

Bradbury [35] then writes:

> The equation of motion (4.20) can be expressed in the form
>
> $$QEv = \frac{d}{dT}(m_0\cosh\alpha) - \frac{2}{3}Q^2 v\ddot{\alpha} \tag{4.21}$$
>
> where use is made of $d\tau = dT/\cosh\alpha$. Equation (4.21) is an expression of conservation of energy. Let us consider a case where the external field is confined to a limited region such as that given by (4.17). If (4.21) is integrated between any two limits, the result is
>
> $$Q\int E\,dX = \Delta E_K - \frac{2}{3}Q^2\int \ddot{\alpha}v\,dT \tag{4.22}$$
>
> where use is made of $v\,dT = dX$. If both limits lie outside the region of the field E, then $\dot{\alpha} = 0$ at both limits of integration, and the term representing radiation loss can be integrated by parts to give
>
> $$-\int \ddot{\alpha}v\,dT = -\int \ddot{\alpha}\sinh\alpha\,d\tau = +\int \dot{\alpha}^2\cosh\alpha\,d\tau = +\int \dot{\alpha}^2\,dT \tag{4.23}$$
>
> The above result shows that the over-all energy loss can be accounted for either by the conventional radiation rate $(2/3)Q^2\dot{\alpha}^2$ or the radiation four-force $(2/3)Q^2\ddot{\alpha}$. Since we are considering a case when the electron always moves the same distance X in the driving field, it emerges with less kinetic energy than it would have if radiation were absent. Still, the motion in the region of the field is accurately hyperbolic—i.e., the same as if radiation were neglected. The important thing is the inclusion of the points where $\ddot{\alpha}$ comes into play, i.e., where the charged particle enters and leaves the field.

This is commented on by D. W. Sciama *et al.* [36, 37] in the following way:

> The radiation reaction force acts, during the initial and final periods of *non*uniform acceleration, in just such a way as to ensure that the total work done by the agency accelerating the charge is equal to the sum of the change in the charge's kinetic energy and the total amount of energy radiated to infinity. This statement amounts to the assertion that the time integral of the rate at which work is done against the radiation-reaction force is equal to the total amount of energy radiated, which is assured provided the motion is inertial at sufficiently early and sufficiently late times.

The authors further write:

> Although there is in reality no difficulty posed by overall conservation of energy, the fact that the force of radiative reaction vanishes during the period of uniform acceleration seems counterintuitive, especially in light of

the observation that the energy loss suffered by an accelerating charge can be equivalently viewed as either the action of the self-field of the charge on itself, or as the effect of fluctuations of the electromagnetic field.

It is instructive to consider the absence of radiative reaction on a uniformly accelerated charge in the light of this duality between the fluctuation and radiative-reaction pictures. It is a standard calculation in classical electrodynamics [38] to show that the electromagnetic field of an accelerated charge acts back upon the charge with a force that is proportional to the time derivative of the acceleration. Thus, during hyperbolic motion the field arranges itself so that there is no radiation reaction upon the charge.

What we seek here is an understanding of this fact in terms of fluctuations. This is provided by the observation first made by Unruh [39] that to a uniformly accelerated observer whose acceleration is a the Minkowski vacuum takes on the appearance of a thermal mixture of temperature $a/2\pi$. We might say that the charge perceives the vacuum fluctuations as comoving and comprising a thermal bath. Thus if the charge is constrained to move with constant acceleration *there can be no net transfer of energy or momentum* between the charge and the vacuum as seen in the accelerated frame.

V. L. Ginzburg [40, 41] noted that the application of Poynting's theorem simplifies and sheds new light upon the contents of the principle of conservation of energy in connection with the hyperbolically moving charge and its field. He writes:

The presence of radiation in the absence of a radiation-deceleration force is paradoxical. The paradox arising in connection with the radiation from a uniformly accelerated charge is connected with the incorrect identification of the energy flux with the work of the radiation force.

The electromagnetic field equations yield the following relation (the Poynting theorem)

$$\frac{1}{8\pi}\frac{\partial}{\partial T}(\mathrm{E}^2+\mathrm{B}^2) = -\mathbf{j}\cdot\mathbf{E} - \nabla\cdot\mathbf{S} \tag{4.24}$$

where $\mathbf{S}$ is the Poynting vector and $\mathbf{j}$ the current density. We confine ourselves to the case of vacuum and consider the motion of a point charge. After integrating over a volume bounded by a surface σ, we get

$$\frac{dE_{em}}{dT} = -Q\mathbf{v}\cdot\mathbf{E} - \int \mathbf{S}\cdot d\boldsymbol{\sigma}, \qquad E_{em} = \frac{1}{8\pi}\int (\mathrm{E}^2+\mathrm{B}^2)\,dV \tag{4.25}$$

On the other hand, in the Newtonian limit we get from the time component of the equation of motion (3.1), (3.2)

$$\frac{dE_K}{dT} = Q\mathbf{v}\cdot\mathbf{E}_{ext} + \frac{2}{3}Q^2\frac{da}{dT} - \mathbb{R} \tag{4.26}$$

where $\mathbb{R}$ is given in Eq. (3.19). This equation can be written

$$\frac{dE_K}{dT} = Q\mathbf{v}\cdot\mathbf{E}_{ext} + \mathbf{v}\cdot\mathbf{f} \tag{4.27}$$

and the radiation friction force $\mathbf{f}$ is given by

$$\mathbf{v}\cdot\mathbf{f} = \frac{2}{3}Q^2\frac{da}{dT} - \mathbb{R} \tag{4.28}$$

In (4.25) $\mathbf{E}$ is the total field, $\mathbf{E} = \mathbf{E}_{ext} + \mathbf{E}'$, where $\mathbf{E}'$ is the field of the charge itself. At the position of the charge $Q\mathbf{E}' = \mathbf{f}$, and hence in (4.25) we have $Q\mathbf{v}\cdot\mathbf{E} = Q\mathbf{v}\cdot\mathbf{E}_{ext} + \mathbf{v}\cdot\mathbf{f}$. We are thus, as one should expect, lead from (4.25) and (4.26) to the conservation law

$$\frac{d(E_{em}+E_K)}{dT} = -\int \mathbf{S}\cdot d\boldsymbol{\sigma} \tag{4.29}$$

The change of field energy plus the change of mechanical energy inside a volume is equal to the flux of energy through the boundary of the volume.

In the case of hyperbolic motion, with vanishing radiation friction force $\mathbf{f} = 0$, Poynting's theorem (4.25) reduces to

$$\frac{dE_{em}}{dT} = -Q\mathbf{v}\cdot\mathbf{E}_{ext} - \int \mathbf{S}\cdot d\boldsymbol{\sigma} \tag{4.30}$$

and the time component of the equation of motion (3.2) takes the form

$$\frac{dE_K}{dT} = Q\mathbf{v} \cdot \mathbf{E}_{ext} \tag{4.31}$$

Ginzburg then writes:

> The vanishing of the radiation force during the uniformly accelerated motion is in no way paradoxical, in spite of the presence of radiation. Indeed, the non-vanishing total energy flux through a surface surrounding the charge, while the radiation force equals zero, is exactly equal to the decrease in the field energy in the volume enclosed by that surface. In the general case, however, all three quantities dE_{em}/dT, $\mathbf{v} \cdot \mathbf{f}$, and $\int \mathbf{S} \cdot d\boldsymbol{\sigma}$ are different from zero. There are no grounds for expecting that the work done by the radiation force, $\mathbf{v} \cdot \mathbf{f}$, and the energy flux $\int \mathbf{S} \cdot d\boldsymbol{\sigma}$ or the radiation flux $\mathbb{R}$ are necessarily equal, especially as the force is applied to the charge, while the flux is calculated through a spherical surface around the charge.

J. Cohn [42] has performed an interesting investigation of "hyperbolic motion and radiation," with the intention to clarify the connection between the Poynting vector and the rate of emission of energy by an accelerated charge. He formulates the problem as follows:

> Consider a charge moving eternally with constant intrinsic acceleration. The particle comes from infinity, momentarily comes to rest and goes back out to infinity with velocity reversed. At the moment the charge changes direction, the magnetic field, and therefore the Poynting vector, is everywhere zero. The *interpretation* of this conclusion is central to the problem. On the one hand, as the charge approaches its turning point it certainly seems that it must radiate according to the Larmor formula. But on the other hand, the fact that $\mathbf{B} = 0$ at this moment implies that the Poynting vector vanishes everywhere, which seems to impy that there is no radiation at this moment.

Cohn then consider straight motion of a charged particle and arrives at some useful results:

> The Poynting vector vanishes everywhere (in the observing frame) on the "light sphere" whose radius r is given by $r = -v/\gamma^2 a$. We shall call such a light sphere a "null" light sphere. Note that when the particle's velocity is zero and its acceleration is not zero, the null light sphere is of radius zero.
>
> It is possible to transform the Poynting vector to zero over the entire surface of any light sphere by a suitable Lorentz transformation. Hyperbolic motion is the unique straight-line motion for which the set of such spheres occupies all of space at the moment the charge is at the turning point.

He goes on and calculates the sign of the Poynting flux

$$F = \int \mathbf{S} \cdot \mathbf{n} \, d\sigma \tag{4.32}$$

over an *arbitrary* light sphere in the given reference frame. He finds that $F \geq 0$. The *total* energy flux is never negative. Then he considers the contributions from the Coulomb and radiation fields, i.e., the fields I and II of Teitelboim [22], and finds that $F_I \geq 0$, $F_{II} \geq 0$, and $F_{I,II} \leq 0$, where $F_{I,II}$ is the contribution to the flux coming from mixed terms involving both radiation and Coulomb fields,

$$\mathbf{S}_{I,II} \propto (\mathbf{E}_I \times \mathbf{B}_{II} + \mathbf{E}_{II} \times \mathbf{B}_I).$$

Cohn further comments:

> *By assumption* we consider the radiant energy flux to be *always* given by F_{II} (indeed, this is the only contribution to F that cannot be transformed to zero by some Lorentz transformation). Only when mixed and Coulomb contributions are negligible can this be identified with F. Thus *radiant* energy flux is not generally given by F. When the light sphere under consideration happens to be much larger (or later) than the null light sphere, we can identify F_{II} as just F, and when it is not much larger (or later) than the null light sphere we must only use F_{II} to evaluate the radiant flux. This yields the customary Larmor result.

Finally Cohn applies the above results to the case of a hyperbolically moving charge and comments on Pauli's conclusion that a hyperbolically moving charge does not radiate:

The mistake made by Pauli was to use F instead of F_{II} to indicate radiant energy flux. Only when the light sphere is much larger than the null light sphere is this permissible. In the case of eternal hyperbolic motion such light spheres only exist some time after the turning point. On such a sphere $F \approx F_{II}$ is then determined by the retarded kinematical properties of the charge at the emission time in the remote past, yielding the customary Larmor result.

5. THE RELATIONSHIP OF THE FIELD EQUATIONS AND THE EQUATION OF MOTION

Dirac [4] deduced an equation of motion of a charged point particle from Maxwell's equations and the principle of conservation of energy and momentum,

$$T^{\mu v}_{;v} = 0, \tag{5.1}$$

where $T^{\mu v}$ is the energy-momentum tensor of the electromagnetic field. The deduction has been reviewed by Rohrlich [21] and developed further by himself [43], Teitelboim [22], and co-workers [44]. We shall only note the main points here.

Dirac considers a tube surrounding the world-line of the charge. The tube has an invariant radius ϵ, and a surface-element of the tube is $d^3\sigma_v$. For any two points 1 and 2 on the world line, the flow of four-momentum out of the surface between these points is

$$\Delta P^{\mu} = \int_1^2 T^{\mu v} d^3\sigma_v. \tag{5.2}$$

Dirac showed (by a rather long calculation) that this can be expressed (in our notation) as

$$\Delta P^{\mu} = \int_1^2 \left(\frac{Q^2}{2\epsilon} A^{\mu} - Q U_v F^{\mu v}_{ext} - \Gamma^{\mu} \right) d\tau, \tag{5.3}$$

where Γ^{μ} is the Abraham four-vector given in Eq. (3.2), and $F^{\mu v}_{ext}$ is the external electromagnetic field. Using the conservation law (5.1) Dirac further proved that the integral (5.3) depends only on the end points 1 and 2, so that the integral has to be a perfect differential of some vector B^{μ}, i.e., equal to $\dot{B}^{\mu}\, d\tau$. Like the integrand $\dot{B}^{\mu}$ must be orthogonal to U^{μ}. The simplest choice, but not the only possible one, is to put $B^{\mu} = kU^{\mu}$, where k is a constant.

Dirac further puts $k = Q^2/2\epsilon - m$, where m is another constant, and gets the following equation

$$\frac{Q^2}{2\epsilon} A^{\mu} - Q U_v F^{\mu v}_{ext} - \Gamma^{\mu} = \left(\frac{Q^2}{2\epsilon} - m \right) A^{\mu} \tag{5.4}$$

or

$$m A^{\mu} = Q U_v F^{\mu v}_{ext} + \Gamma^{\mu} \tag{5.5}$$

which is interpreted as the equation of motion of a charged particle in an external electromagnetic field. This is the Lorentz–Dirac equation (3.1).

Dirac has interpreted the time component of Eq. (3.1), i.e., the equation of energy conservation, in the following way:

The rate at which work is done on the charge, is equated to the sum of three terms; $m\dot{U}^0$, $-(2/3)Q^2\ddot{U}^0$ and $(2/3)Q^2a^2U^0$. The first two of these are perfect differentials and the things they are differentials of, namely mU^0

> and $-(2/3)Q^2\dot{U}^0$, may be considered as intrinsic energies of the charge. The former is just the usual expression for the kinetic energy of a particle of rest mass m, while the latter is what is called the "acceleration energy" [20]. Changes in the acceleration energy correspond to a reversible form of emission or absorption of field energy, which never gets very far from the charge. The third term $(2/3)Q^2a^2U^0$ corresponds to irreversible emission of radiation and gives the effect of radiation damping on the motion of the charge.

Note that Rohrlich's definition of radiation reaction, Eq. (4.6), is in agreement with Dirac's interpretation.

A particularly interesting feature about Dirac's deduction is that it establishes a connection between Maxwell's equations and the equation of motion for a charged particle. It shows that the presence of the Abraham four-vector in the equation of motion comes from conservation of energy and momentum of the electromagnetic field. It must be present in the equation of motion of a charged particle in order for this to be consistent with energy-momentum conservation for a closed system consisting of a charge and its field.

We have seen in the preceding paragraph that the source of radiation energy in the case of a hyperbolically accelerated charge was still not fully understood in the sixties. A significant advance was made by Teitelboim [22]. He made a Lorentz invariant separation of the electromagnetic field tensor into two parts

$$F^{\mu\nu} = F_I^{\mu\nu} + F_{II}^{\mu\nu}, \tag{5.6}$$

where $F_I^{\mu\nu}$ is the velocity field, and $F_{II}^{\mu\nu}$ is the acceleration field. Inserting these fields into the energy-momentum tensor of the electromagnetic field, Teitelboim finds that the energy-momentum tensor contains terms of three types: a part $T_{I,I}^{\mu\nu}$ independent of the acceleration, a part $T_{I,II}^{\mu\nu}$ depending linearly upon the acceleration, and a part $T_{II,II}^{\mu\nu}$ depending linearly upon the square of the acceleration of the charge producing the fields.

It is a consequence of Maxwell's equations that the total energy-momentum tensor is covariantly divergence free outside the world line of the charge,

$$T^{\mu\nu}_{;\nu} = 0. \tag{5.7}$$

This expresses the conservation of energy and momentum of the electromagnetic field. Teitelboim shows that

$$T^{\mu\nu}_{II,II;\nu} = 0. \tag{5.8}$$

Then we have

$$T^{\mu\nu} = T_I^{\mu\nu} + T_{II}^{\mu\nu} \tag{5.9}$$

with

$$T^{\mu\nu}_{I;\nu} = 0, \qquad T^{\mu\nu}_{II;\nu} = 0, \tag{5.10}$$

where

$$T_I^{\mu\nu} \equiv T_{I,I}^{\mu\nu} + T_{I,II}^{\mu\nu}, \qquad T_{II}^{\mu\nu} \equiv T_{II,II}^{\mu\nu}, \tag{5.11}$$

where the separate conservation equations in (5.10) are valid off the world line of the charge. The contribution of the interference between the fields I and II has been included in $T_I^{\mu\nu}$, whereas the tensor $T_{II}^{\mu\nu}$ is related only to the part of the field depending upon the square of the acceleration.

Teitelboim further shows that the energy-momentum associated with the field $F_{II}^{\mu v}$ is travelling with the speed of light. The field fronts are spheres centered at the corresponding emission points. The four-momentum associated with $T_I^{\mu v}$ remains bound to the charge.

Teitelboim goes on and calculates these four-momenta in terms of the properties of the charge. The radiated four-momentum present at the proper time τ is defined by

$$P_{II}^{\mu}(\tau) \equiv \int_{\sigma(\tau)} T_{II}^{\mu v} U_v(\tau)\, d^3\sigma, \tag{5.12}$$

where σ is an arbitrary spacelike surface that intercepts the world line of the charge at the point of time τ. The result of Teitelboim's calculation is

$$P_{II}^{\mu}(\tau) = \int_{-\infty}^{\tau} \frac{2}{3} Q^2 a^2(\tau) U^{\mu}(\tau)\, d\tau. \tag{5.13}$$

Differentiating this equation, he finds that when a charge is being accelerated, four-momentum is being radiated at the instant τ in accordance with the relativistic Larmor formula (3.20).

The bound four-momentum present at the proper time τ is defined by

$$P_I^{\mu}(\tau) = \int_{\sigma(\tau)} T_I^{\mu v} U_v(\tau)\, d^3\sigma, \tag{5.14}$$

where $\sigma(\tau)$ is the three space at time T as viewed from the rest system of the charge at the proper time τ. Using the asymptotic condition $\lim_{\tau\to-\infty}(\text{motion}) = \text{uniform motion}$, his calculation leads to

$$P_I^{\mu} = \frac{Q^2}{2\epsilon} U^{\mu} - \frac{2}{3} Q^2 A^{\mu}. \tag{5.15}$$

This expression shows that P_I^{μ} is a state function of the charge; i.e., it depends only upon the world line of the particle at its position. This is a confirmation of the bound character of P_I^{μ}. Differentiation gives the rate of change of bound field four-momentum

$$\frac{dP_I^{\mu}}{d\tau} = \frac{Q^2}{2\epsilon} A^{\mu} - \frac{2}{3} Q^2 \dot{A}^{\mu}. \tag{5.16}$$

Teitelboim summarizes the above result with the words:

> The "bound" electromagnetic four-momentum contains, besides the generally accepted "Coulomb mass" × four-velocity term, the extra term $-(2/3)Q^2 A^{\mu}$, whose time derivative is precisely the negative of the yet-to-be-explained Schott term.

Teitelboim then derives the Lorentz–Dirac equation in a manner similar to that employed by Dirac and Rohrlich, but obtains an additional insight as to the significance of the Schott term by utilizing the results above.

> Since the charged particle cannot be separated from its bound electromagnetic four-momentum, the four-momentum of the particle is the sum of the mechanical or "bare" momentum and the electromagnetic one; that is to say,
>
> $$P^{\mu} = P_{(bare)}^{\mu} + P_I^{\mu} \tag{5.17}$$

If we assume the bare four-momentum to have the usual form for an uncharged particle, we obtain

$$P^{\mu} = \left(m_{(bare)} + \frac{Q^2}{2\epsilon} \right) U^{\mu} - \frac{2}{3} Q^2 A^{\mu} \tag{5.18}$$

To handle the divergence (when $\epsilon \to 0$), we make the usual identification

$$m = m_{(bare)} + \frac{Q^2}{2\epsilon} \tag{5.19}$$

Thus, for the four-momentum of a point charge in arbitrary motion we have

$$P^{\mu} = mU^{\mu} - \frac{2}{3} Q^2 \mu \tag{5.20}$$

The equation of motion for a charged particle which is not under the action of any external force follows readily from the conservation of momentum for the closed system particle plus radiation; that is to say,

$$mA^{\mu} - \frac{2}{3} Q^2 \dot{A}^{\mu} = -\frac{2}{3} Q^2 a^2 U^{\mu} \tag{5.21}$$

When the particle is acted upon by an external four-force, Eq. (5.21) must, of course, be replaced by the Lorentz-Dirac Equation (3.1).

Defining the *Schott four-momentum*

$$P_S^{\mu} = -\frac{2}{3} Q^2 A^{\mu} \tag{5.22}$$

with the Schott energy (3.24) in the time component, the four-momentum Eq. (5.20) corresponding to the energy Eq. (4.16) takes the form

$$P^{\mu} = mU^{\mu} + P_S^{\mu}. \tag{5.23}$$

A further comment is given in [44]:

The magnitude of P^{μ} is not conserved. However mU^{μ} does have a conserved magnitude. The difference between mU^{μ} and P^{μ} is the Schott term, which vanishes if the particle is free and negligible in weak external fields. Therefore, as long as physical measurements are effected when the particle is free or nearly free, the empirical evidence cannot distinguish between the conservation of $P_{\mu} P^{\mu}$ and that of $m^2 U_{\mu} U^{\mu} = -m^2$.

However, from our point of view, it would be conceptually mistaken to identify mU^{μ} as the true four-momentum of the particle, as this interpretation would be tenable only if the term ("Abraham four-vector") Γ^{μ} in the Lorentz-Dirac equation could be considered as the negative of the instantaneous rate of emission of electromagnetic radiation. Such an interpretation cannot possibly be correct, since Γ^{μ} is spacelike and hence the sign of its time component can be changed by a Lorentz transformation, which is not consistent with the irreversible character of electromagnetic radiation.

Finally, Teitelboim [22] mentions briefly the hyperbolically accelerated charge and its field.

Hyperbolic motion is a special case in which all radiated energy comes from the bound electromagnetic energy of the particle. In the general case there is a conversion of both mechanical and bound electromagnetic four-momentum into radiation.

The conversion of four-momentum of type I into momentum of type II is forbidden in the whole spacetime off the world line of the particle, since in this region the tensors corresponding to both parts conserve separately. The change of status of the four-momentum occurs only at the singularity of the fields, where both tensors have their sources.

Therefore a charge in hyperbolic motion can be pictured as being only a source of radiated four-momentum and a sink of bound four-momentum.

The results of Rohrlich and Teitelboim have been summarized by P. Pearle [45] who writes:

The term Γ^{μ} in the Lorentz-Dirac equation, as given in Eq. (3.2), is called the Abraham force. Its first term, $(2/3)Q^2\dot{A}^{\mu}$, is called the Schott term, and its second, $-(2/3)Q^2 A^{\upsilon} A_{\upsilon} U^{\mu}$, the radiation reaction term.

The zeroth component of the radiation reaction term is to be interpreted as the radiation rate. Indeed, the scalar product of this term with U_μ is the relativistic version of the Larmor formula. The spatial component of this term, proportional to $-\mathbf{v}$ like a viscous drag force, may similarly be interpreted as the radiation reaction force on the electron.

The physical meaning of the Schott term has been puzzled over for a long time. Its zeroth component represents a power which adds "Schott acceleration energy" to the electron and its associated electromagnetic field. The work done by an external force not only goes into electromagnetic radiation and into increasing the electron's kinetic energy, but it causes an increase in the "Schott acceleration energy" as well. This change can be ascribed to a change in the "bound" electromagnetic energy in the electron's induction field, just as the last term in Eq. (3.22) can be ascribed to a change in the "free" electromagnetic energy in the electron's radiation field.

What meaning should be given to the Schott term? Teitelboim [22] has argued convincingly that when an electron accelerates, its near field is modified so that a correct integration of the electromagnetic four-momentum of the electron includes not only the Coulomb four-momentum $(Q^2/2\epsilon)U^\mu$, but an extra four-momentum $-(2/3)Q^2A^\mu$ of the bound electromagnetic field. This suggests that the Lorentz-Dirac equation be written in the form (5.21). The Schott term is the negative rate of change of Teitelboim's four-momentum.

6. IS THE SCHOTT ENERGY LOCALIZED AT THE PARTICLE?

Rowe [23] has modified Teitelboim's separation of the energy momentum tensor of an electromagnetic field and introduced a separation into three divergence free parts. We shall examine what this separation reveals about the localization of the Schott energy.

Teitelboim's separation is given in Eqs. (5.6), (5.9), and (5.11). Calculating the expressions for the separate parts of the energy momentum tensor, one arrives at

$$T^{\mu v}_{I,I} = \frac{Q^2}{4\pi}\left[\frac{1}{2}\eta^{\mu v} + \frac{U^\mu R^v + U^v R^\mu}{s} - \frac{R^\mu R^v}{s^2}\right]\frac{1}{s^4} \tag{6.1}$$

$$T^{\mu v}_{I,II} = \frac{Q^2}{4\pi}\left[A^\mu R^v + A^v R^\mu + \frac{(R_\beta A^\beta)(U^\mu R^v + U^v R^\mu)}{s} - \frac{2(R_\beta A^\beta)R^\mu R^v}{s^2}\right]\frac{1}{s^4} \tag{6.2}$$

$$T^{\mu v}_{II,II} = \frac{Q^2}{4\pi}\left[A_\beta A^\beta - \frac{(R_\beta A^\beta)^2}{s^2}\right]\frac{R^\mu R^v}{s^4}, \tag{6.3}$$

where the null-vector $R^v = (T - T_Q, \mathbf{X} - \mathbf{X}_Q)$ is the distance four-vector between the observation event, i.e., the field point, $(T, \mathbf{X})$, and the emission event, i.e., the source point, $(T_Q, \mathbf{X}_Q)$, and $s = -U_v R^v = \kappa\gamma R$, where $R = T - T_Q$, $\kappa = 1 - \mathbf{n}\cdot\mathbf{v}$, and $\mathbf{n}$ is a unit vector directed from the emission point to the field point in space.

Teitelboim writes the energy momentum tensor $T^{\mu v}$ as the sum of a tensor $T^{\mu v}_{II} = T^{\mu v}_{II,II}$ for the radiation field, II, and a tensor $T^{\mu v}_{I} \equiv T^{\mu v}_{I,I} + T^{\mu v}_{I,II}$ for the Coulomb/velocity field, I, including the cross terms with the radiation field. He shows that the tensors $T^{\mu v}_I$ and $T^{\mu v}_{II}$ are both divergence free. Hence, there is no exchange of energy nor momentum between T^μ_I and T^μ_{II}, except possibly at the point $s = 0$ where the tensors are not defined. The four-momentum of the radiation field depends upon the whole prehistory of the particle. According to Teitelboim the four-momentum of type I, the so-called bound momentum, is given by the *instantaneous* values of the velocity and acceleration and consists of the Schott energy-momentum, $-(2/3)Q^2A^\mu$, and of the Coulomb energy-momentum, $(Q^2/2\epsilon)U^\mu$.

Rowe's separation of the energy momentum tensor is

$$T^{\mu v} = T^{\mu v}_1 + T^{\mu v}_2 + T^{\mu v}_3, \tag{6.4}$$

where

$$T_1^{\mu v} = T_{I,I}^{\mu v} + \frac{Q^2}{2\pi} \frac{R_\beta A^\beta R^\mu R^v}{s^6} \tag{6.5a}$$

$$T_2^{\mu v} = T_{I,II} - \frac{Q^2}{2\pi} \frac{R_\beta A^\beta R^\mu R^v}{s^6} \tag{6.5b}$$

$$T_3^{\mu v} = T_{II,II}^{\mu v}. \tag{6.5c}$$

This is a separation of the energy momentum tensor into three symmetrical tensors. Computation (differentiation of retarded quantities) shows that each of the tensors are divergence free.

We shall first study the energy and momentum of type 2, at a point of time T, of the electromagnetic field from a charged particle, which we consider as a point charge. The field produced at a point of time T_Q is found at the time T at an eikonal (a light front) shaped as a spherical surface K with radius $T - T_Q$. The field produced by the particle during an infinitesimal time interval from T_Q to $T_Q + dT_Q$ is found between two non-concentric spherical surfaces with distance κdT_Q in the direction **n**. The energy-momentum at time T of type 2 in this region is given by the space angle integral

$$dP^v = (T - T_Q)^2\, dT_Q \oint \kappa T_2^{0v}\, d\Omega. \tag{6.6}$$

Calculation shows that all integrals $\oint \kappa T_2^{\mu v} d\Omega$ vanish. In other words, integrated over all directions the particle does not emit any energy-momentum of type 2. Hence, the total amount of energy and momentum of type 2 in the space V′ outside an arbitrary eikonal K is zero,

$$\int_{V'} T^{0v}\, d^3X = 0. \tag{6.7}$$

We shall now find the integrals $\int_V T_2^{0v}\, dV$ over the volume V outside an ellipsoid (corresponding to a spherical surface in the rest frame of the charge) with half axes ϵ and ϵ/γ enclosing the particle, and inside an eikonal K which is just outside the ellipsoid. We want to express the result in terms of the instantaneous values of the particle's velocity and acceleration. The calculation of the integral follows the same procedure as the calculations in Section 3 of Ref. [46] and is not given in detail here. We find the following expressions for the total energy and momentum of type 2 in V in the limit $\epsilon \to 0$,

$$\int_V T_2^{00}\, dV = \frac{2}{3} Q^2 \gamma^4 \mathbf{a} \cdot \mathbf{v} = \frac{2}{3} Q^2 A^0 \tag{6.8a}$$

$$\int_V T_2^{0i}\, dV = \frac{2}{3} Q^2 \gamma^4 (\mathbf{a} \cdot \mathbf{v})\mathbf{v} = \frac{2}{3} Q^2 A^0 \mathbf{v}. \tag{6.8b}$$

Outside the eikonal the corresponding energy and momentum are zero. Hence the energy and momentum of type 2 are given by Eq. (6.8) and are localized inside an eikonal K of arbitrary size. This energy and momentum do not form a four-vector, but may be interpreted as the energy and momentum of a system with variable mass (energy) moving with a velocity **v**. It may be noted that in order to arrive at the results (6.8) for the energy and momentum of type 2, we omitted a Lorentz contracted sphere of radius ϵ around the particle and took the limit $\epsilon \to 0$. No divergent integrals appeared in this limit. This means that as far as the energy and momentum of type 2 are concerned, there are no divergences for the total energy and momentum anywhere in the space, not even arbitrarily close to the point charge.

Considering the field from a point particle Rowe finds that the expressions (6.5a), (6.5b) should be completed by δ-function expressions at the position of the particle. He finds, using distribution theory, that in spite of the fact that $T_1^{\mu v}$ and $T_2^{\mu v}$ are symmetrical for $s \neq 0$, this is not the case at $s = 0$. He symmetrizes the $s = 0$-term and finds a symmetrical extension of the original tensors. This leads to the following modified tensor of type 2 at laboratory time T,

$$T_{2new}^{\mu v} = T_2^{\mu v} - \frac{2}{3} Q^2 \gamma^{-1} (A^\mu U^v + A^v U^\mu) \delta(\mathbf{X} - \mathbf{X}_Q), \tag{6.9}$$

where A^μ, U^v, $\mathbf{X}_Q$ refer to the particle at the point of time T. For $T_1^{\mu v}$ there is a corresponding term with opposite sign, so there is no δ-function contribution to the sum $T_{1new}^{\mu v} + T_{2new}^{\mu v}$. Integrating the last term in Eq. (6.9) over all of space one finds that the contribution of the δ-function term to the energy is

$$\Delta P_2^0 = -\frac{4}{3} Q^2 A^0 \tag{6.10a}$$

and to the momentum

$$\Delta \mathbf{P}_2 = -\frac{2}{3} Q^2 (A^0 \mathbf{v} + \mathbf{A}). \tag{6.10b}$$

Adding this to the expressions (6.8) we find that the total energy-momentum of type 2 inside K is given by the Schott four-momentum $P_S^v = -(2/3) Q^2 A^v$.

According to Rowe's assumptions and our calculations we arrive at the following physical picture. At an arbitrary point of time T space may be thought of as filled by spherical eikonals. Each spherical surface has center at a retarded position of the charge, so that $R = T - T_Q$, where T_Q is the retarded time. The total energy and momentum of type 2 and "2new" summarized over all directions vanish outside an eikonal of arbitrarily small radius, but inside the eikonal there is a Schott four-momentum,

$$P_S^v = -(2/3) Q^2 A^v \tag{6.11}$$

which is partly concentrated at the particle as a δ-function distribution. Note, however, that totally there is no energy or momentum situated at the particle since the δ-functions from "1new" and "2new" cancel each other.

The energy and momentum of type "1new" are found from the relation

$$T_I^{\mu v} = T_{1new}^{\mu v} + T_{2new}^{\mu v} = T_1^{\mu v} + T_2^{\mu v}. \tag{6.12}$$

We find from Eqs. (3.8) and (3.12) in Ref. [46] that the energy and momentum of type I, which were produced from $T_{Q1} = -\infty$ to T_{Q2}, are at time T localized in the space outside an eikonal K with radius $T - T_{Q2}$, and are given by

$$U_I(-\infty, T_{Q2}, T) = \frac{Q^2}{2(T - T_{Q2})} \gamma^2(T_{Q2}) \left(1 + \frac{1}{3} v^2(T_{Q2})\right) \tag{6.13a}$$

$$\mathbf{P}_I(-\infty, T_{Q2}, T) = \frac{2}{3} \frac{Q^2}{T - T_{Q2}} \gamma^2(T_{Q2}) \mathbf{v}(T_{Q2}). \tag{6.13b}$$

TABLE I

Type of tensor	At the particle	Inside K (outside the ellipsoid)	Outside K	In all space
$T_1^{\mu v}$	0	$U_C + 2E_S - U_I$	U_I	$U_{Coul} + 2E_S$
$T_2^{\mu v}$	0	$-E_S$	0	$-E_S$
$T_{1new}^{\mu v}$	$-2E_S$	$U_C + 2E_S - U_I$	U_I	U_{Coul}
$T_{2new}^{\mu v}$	$2E_S$	$-E_S$	0	E_S
$T_I^{\mu v}$	0	$U_C + E_S - U_I$	U_I	$U_{Coul} + E_S$

Here we have assumed that $\gamma^2(T_{Q1})/(T - T_{Q1}) \rightarrow 0$ when $T_{Q1} \rightarrow -\infty$.

Similarly we find from Eqs. (3.8), (3.12), and (3.13) in [46] that the energy and momentum of type I in the space outside the ellipsoid and inside an eikonal K of arbitrary size are

$$\int_V u_I \, d^3X = \frac{Q^2}{2\epsilon}\gamma\left(1 + \frac{1}{3}v^2\right) - \frac{2}{3}Q^2 A^0 - U_I(-\infty, T_{Q2}, T) \tag{6.14a}$$

$$\int_V \mathbf{P}_I \, d^3X = \frac{2}{3}\frac{Q^2}{\epsilon}\gamma\mathbf{v} - \frac{2}{3}Q^2\mathbf{A} - \mathbf{P}_I(-\infty, T_{Q2}, T), \tag{6.14b}$$

where the symbols without argument refer to the point of time T.

By means of Eqs. (6.11)–(6.14) we may summarize the results of Rowe's and Teitelboim's separations in a table showing the distribution of energy inside and outside an eikonal K of arbitrary size. In Table I, $E_S = -(2/3)Q^2 A^0 = -(2/3)Q^2\gamma^4\mathbf{a}\cdot\mathbf{v}$ is the Schott energy, $U_C = (Q^2\gamma/2\epsilon)(1 + v^2/3)$, and U_I is given by Eq. (6.13a).

Table I shows the distribution of field energy for the tensor $T_I^{\mu v}$ and its separations $T_I^{\mu v} = T_1^{\mu v} + T_2^{\mu v}$ and $T_I^{\mu v} = T_{1new}^{\mu v} + T_{2new}^{\mu v}$.

The Schott energy—interpreted as a field energy—is localized inside the eikonal K. We shall introduce the concept of *interaction energy*, i.e., the field energy due to the interaction between the fields $F_I^{\mu v}$ and $F_{II}^{\mu v}$, and examine whether this energy is related to the Schott energy. For this purpose we shall make use of the bound energy-momentum tensor $T_I^{\mu v}$, being the sum of Coulomb terms and interaction terms.

Consider a particle moving along the X-axis. We assume that the motion was uniform in the far past, but at a certain time some force started acting upon the particle, and since then it has had an arbitrary motion. At the point of time T_{Q2} the particle has a velocity v_2. The field which the particle has emitted up to T_{Q2} has at time $T > T_{Q2}$ an energy of type I given by U_I, which is dependent only on the velocity v_2 at this point of time. This is the same energy as if the particle had moved with constant velocity v_2 up to T_{Q2}.

We may choose T_{Q2} gradually closer to T until the eikonal touches the ellipsoid from outside. Then we have a situation as shown in Fig. 1, where $T - T_{Q2} \simeq \gamma\epsilon(1 + v)$ and $X(T) - X(T_{Q2}) \simeq v(T - T_{Q2})$.

In the region V there is an energy energy $U_C + E_S - U_I$. This is a region close to the particle which vanishes in the limit $\epsilon \rightarrow \infty$.

The Schott energy E_S is a quantity of first order in the acceleration. One may wonder whether E_S is a measure of the interaction energy.

Let us consider the density $u_{I,II}$ of the interaction energy in the future light cone from the particle as given by Eq. (6.2). In three dimensional notation the expression takes the form

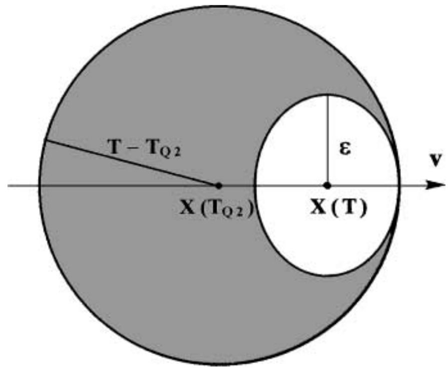

FIG. 1. The Schott energy is localized in the shaded region between the eikonal and the ellipsoid surrounding the particle. Here $v = 0.6$.

$$u_{I,II} = \frac{Q^2}{2\pi R^3 \gamma^2 \kappa^6}[\kappa \mathbf{a} \cdot \mathbf{v} + (\kappa - \gamma^{-2})\mathbf{a} \cdot \mathbf{v}], \tag{6.15}$$

where $\mathbf{v}$, γ, $\mathbf{a}$ refer to the retarded time T_Q, $R = T - T_Q$, $\kappa = 1 - \mathbf{n} \cdot \mathbf{v}$, and $\mathbf{n} = \mathbf{R}/R$.

The field produced by the particle in the infinitesimal time interval from T_Q to $T_Q + dT_Q$ is at time T situated in the region between two eikonals with radii $T - T_Q$ and $T - T_Q - dT_Q$, respectively. The interaction energy in this region is

$$dU_{I,II} = (T - T_Q)^2\, dT_Q \int \kappa u_{I,II}\, d\Omega = \frac{4}{3} Q^2 \gamma^4 \frac{\mathbf{a} \cdot \mathbf{v}}{T - T_Q}\, dT_Q = -\frac{2}{T - T_Q} E_S(T_Q)\, dT_Q. \tag{6.16}$$

Thus, the interaction energy produced from $T_{QI} = -\infty$ to T_{Q2} is at time T given by (assuming constant velocity in the infinite past)

$$U_{I,II}(-\infty, T_{Q2}, T) = -2 \int_{-\infty}^{T_{Q2}} \frac{E_S(T_Q)}{T - T_Q}\, dT_Q. \tag{6.17}$$

We shall now show by considering a particular situation that in general there is no connection between the Schott energy and the amount of interaction energy. Consider a particle being in uniform motion in the infinite past, and having arbitrary motion during a finite period until it comes to rest at time the T_{Q2}. When the particle is at rest for $T > T_{Q2}$ the space close to the particle is filled by a Coulomb field. Hence, at a point of time T after T_{Q2} the total interaction field energy is correctly given by Eq. (6.17) and spreads over all space outside the eikonal from T_{Q2}. The integral (6.17) depends upon the prehistory of the particle and may have any value. On the other hand the Schott energy at the time T is zero.

The general expression for the interaction energy at time T in the region V between the ellipsoid and the eikonal just outside it may be found by the same procedure as used in the calculations in Section 3 of Ref. [46]. The result is

$$\int_V u_{I,II}\, d^3X = \left(2 \ln \frac{\epsilon}{T - T_{Q2}} + f(v)\right) E_S(T), \tag{6.18}$$

where

$$f(v) = -\frac{17}{12} + \frac{1}{4v^2} + \frac{3v^4 + 6v^2 - 1}{4v^3} \operatorname{artanh} v = \frac{6}{5}v^2 + \frac{3}{14}v^4 + \cdots \tag{6.19}$$

and $v = v(T)$. Adding the expression (6.17) and omitting terms which vanish in the limit $\epsilon = 0$, we find that the total amount of interaction energy in all of space is

$$\int u_{I,II}\, d^3X = (2\ln\epsilon + f(v))E_S(T) - 2\int_{-\infty}^{T} E_S'(T_Q)\ln(T - T_Q)\, dT_Q. \tag{6.20}$$

Due to the last term we see that the interaction energy is not a state function of the particle.

As an example consider a particle moving in the negative X-direction from $X = \infty$ at $T = -\infty$, the motion being uniform until the point of time T_1. Then the motion becomes hyperbolic with proper acceleration g and with turning point at $T = 0$. We shall find the interaction energy in the field at $T = 0$. We put $T = 0$ and $v = 0$ in Eq. (6.20) and get

$$\int u_{I,II}\, d^3X = -2\int_{-\infty}^{0} E_S'(T_Q)\ln(-T_Q)\, dT_Q. \tag{6.21}$$

Here we introduce $E_S(T_Q) = 0$ for $T_Q < T_1$ and $E_S(T_Q) = -(2/3)Q^2g^2T_Q$ for $T_Q > T_1$. Taking into account that $E'(T_Q)$ has a δ-function contribution,$-(2/3)Q^2g^2T_1\delta(T_Q - T_1)$, we find that the interaction energy when the particle is at the turning point is not equal to the Schott energy $E_S(0) = 0$, but

$$\int u_{I,II}\, dX = -2E_S(T_1), \tag{6.22}$$

where $E_S(T_1)$ is the Schott energy just after the particle has entered the hyperbolic motion.

7. THE FIELD REACTION FORCE

A calculation of the self force upon an accelerated charge was originally made by Lorentz [1] and has been reviewed among others by Jackson [38] and by Panowski and Phillips [47], who performed the calculation in the instantaneous rest frame of the charge. It is not obvious that a more general calculation, with arbitrary velocity, shall give the same result as the calculation in the rest frame of the charge, Lorentz transformed to an arbitrary reference frame, because the calculations involve an integration over the volume of the charge. This volume is defined as a simultaneity space *in the laboratory frame*, which is different from the simultaneity space in the rest frame of the charge. For this reason we have made a general calculation, which is found in Appendix B (see also Eqs. (A.15) and (A.24) in Appendix A). The result of the calculation is as follows.

Consider a non-rotating particle moving along a straight line in the laboratory. The particle is assumed to be Born rigid and spherically symmetric in its rest frame. Then the motion of each element of the charge is determined by specifying the motion of one element, for example, the center of the charge.

The electromagnetic force between two elements of the charge will in general not satisfy Newton's 3.law; the force from an element dQ_1 on an element dQ_2 is different from the force from dQ_2 on dQ_1.

The self force is the vector-sum obtained by adding internal electromagnetic forces by simultaneity in the laboratory frame.

The calculation is simplified due to the following circumstance. The force acting upon an element dQ_1 from all the other elements of the charge is partly due to a magnetic field and partly an electrical field with one component normal to the direction of motion of the charge and one component E^X along this direction. Due to the rotational symmetry of the charge about the direction of motion, only the component E^X contributes to the self force. Moreover, E^X is invariant against a transformation in the X-direction so the force on each element of the charge may be calculated in the rest frame of the element. But the integration that defines the resultant force, F, on the whole charge must be performed by simultaneity in the laboratory frame.

We find that the self force in the laboratory system is equal to the self force in the rest system. Also we have performed a separation in a Coulomb/velocity part and a radiation/acceleration part, following Teitelboim, and found that the field of type I does not contribute to the self force, when the charge-distribution which is spherically symmetric in its rest frame, has vanishing extension. Hence

$$F = F_I + F_{II}, \qquad F_I = 0, \qquad F_{II} = -\frac{4}{3}V_0 g + \frac{2}{3}Q^2\frac{dg}{d\tau}, \tag{7.1}$$

where τ is the proper time of the charge, V_0 is its electrostatic energy, and $g = d(\gamma v)/dT$ its acceleration in the instantaneous rest frame. The last term is the Lorentz–Dirac field reaction force for rectilinear motion.

Newton's 2.law as applied to a charged particle acted upon by an external force F_{ext} then takes the form

$$M_0 g = F_{ext} + \frac{2}{3}Q^2\gamma\frac{dg}{dT} - \frac{4}{3}V_0 g. \tag{7.2}$$

Here M_0 is formally a mechanical rest mass. The last term expresses "a resistance against being put into motion," and acts as an addition to the rest mass. We normalize to a physical rest mass

$$m_0 = M_0 + \frac{4}{3}V_0. \tag{7.3}$$

Equation (3.1) then takes the form

$$F_{ext} = m_0 g - \frac{2}{3}Q^2\gamma\frac{dg}{dT}. \tag{7.4}$$

The field reaction force may be written

$$\frac{2}{3}Q^2\gamma\frac{dg}{dT} = \frac{2}{3}Q^2\frac{d(\gamma v g)}{v\,dT} - \frac{2}{3}Q^2\frac{g^2}{v}. \tag{7.5}$$

Using this in Eq. (7.4) we find the following expression for the work performed by the external force

$$W_{ext} = \int_{T'}^{T''} F_{ext} v\,dT = \Delta E_K + \Delta E_S + \frac{2}{3}Q^2\int_{T'}^{T''} g^2\,dT, \tag{7.6}$$

where E_K is the kinetic energy of the particle and E_S its Schott energy, defined in Eq. (3.24). The last term in Eq. (7.6) is the work W_R which must be performed to overcome a sort of "resistance" (different from the radiation reaction force) which always acts against the accelerated motion of a charged particle. From the field point of view this work contributes the energy which is emitted by the particle from T' to T''.

The Schott energy is sometimes thought of as a quantity belonging to the particle, but can also be perceived as a field quantity. These different ways of interpreting the Schott energy are not contradictory, but have a complementary character. In connection with the equation of motion of the charge and the associated energy budget, the particle aspect is significant. But when it comes to localizing the Schott energy, the field aspect is the central one.

In this section we have seen how a charged particle acts upon itself because of internal retarded forces that summarize to a field reaction force $(2/3)Q^2\gamma dg/dT$, according to Eq. (7.4). In the deduction we have assumed a spherically symmetric distribution of the charge in the rest frame of the particle, and we have considered the case of rectilinear motion.

We shall now present an alternative point of view, referring to Section 3 of Ref. [46], leading to the same result. There we studied the field outside the particle produced by the particle from an infinitely remote point of time up to an arbitrary point of time. We considered a particle with arbitrary (curvilinear) motion and calculated the retarded field from the charge. The region outside the charge was defined as the space outside a spherical boundary in the rest frame of the particle with the charge concentrated as a point charge in the center.

The Coulomb energy, which is included in the mass of the particle, diverges in the limit when the radius of the charge approaches zero, but there are no other divergences in this limit.

The four-momentum P^μ for the system of particle and field is given in Eq. (3.20) of Ref. [46]. If the particle does not interact with the world outside, P^μ will be constant. However, if an external force F^μ_{ext} acts upon the particle, then $F^\mu_{ext} = \dot{P}^\mu$, and we get

$$F^\mu_{ext} = m_0 A^\mu - \frac{2}{3}Q^2 \dot{A}^\mu + \dot{P}^\mu_R, \tag{7.7}$$

where

$$\dot{P}^\mu_R = \frac{2}{3}Q^2 g^2 U^\mu \tag{7.8}$$

in accordance with Larmor's formula. We find the field reaction formula

$$\Gamma^\mu = \frac{2}{3}Q^2(\dot{A}^\mu - g^2 U^\mu) \tag{7.9}$$

in agreement with Eq. (3.2). There are two contributions to the Abraham-vector Γ^μ: the term $(2/3)Q^2\dot{A}^\mu$, which is the Schott vector representing the acceleration reaction, is due to the bound four-momentum of the field (Teitelbaum's type I); while the Rohrlich-vector $-(2/3)Q^2g^2U^\mu$ representing the radiation reaction, is due to the radiation field (i.e., field of type II). The latter term has recently been discussed by Hartemann and Luhman [48]. They deduced it by integrating the radiation pressure due to the radiation emitted by the particle, over an eikonal, and taking the limit when the radius tends to zero.

We should also like to point out the somewhat astonishing circumstance that the term g^2U^μ, which appears in the expression for Γ^μ_R, also appears in the expression for $\dot{A}^\mu$. By linear motion

$$\dot{A}^\mu = g^2 U^\mu + \gamma(\mathbf{v}\cdot\mathbf{g}, \mathbf{g}) \tag{7.10}$$

and Eq. (3.2) for the Abraham vector reduces to

$$\Gamma^\mu = \frac{2}{3}Q^2\gamma(\mathbf{v}\cdot\dot{\mathbf{g}}, \dot{\mathbf{g}}), \tag{7.11}$$

i.e.,

$$\boldsymbol{\Gamma} = \frac{2}{3} Q^2 \dot{\mathbf{g}} \tag{7.12}$$

in accordance with Eqs. (3.17) and (7.4).

Summarizing the contents of this subsection, we have considered two methods of finding the field reaction force $\boldsymbol{\Gamma}$:

(i) The interpretation of $\boldsymbol{\Gamma}$ obtained by summing up the internal electromagnetic forces in the particle is that the field reaction force is entirely due to the acceleration (radiation) field $F_{II}^{\mu v}$.

(ii) By taking the time rate of the momentum in all of space contained in the field produced by the particle from $T = -\infty$ and up to the present time, we get $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_A + \boldsymbol{\Gamma}_R$ where $\boldsymbol{\Gamma}_A$ is due to the bound momentum (Teitelboim's tensor $T_I^{\mu v}$), and Γ_R^{μ} is due to the radiation field. In an instantaneous rest frame of the charge $\boldsymbol{\Gamma}_R = 0$ and $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_A$, so referring to this frame the field reaction force $\boldsymbol{\Gamma}$ depends only upon the bound momentum.

We find it rather remarkable that $\boldsymbol{\Gamma}$ may be calculated by two methods being so different, and that the parts I and II of the field seem to have complementary roles in connection with the two ways of calculating this single quantity.

8. ENERGY CONSERVATION FOR A CHARGE ENTERING A REGION OF HYPERBOLIC MOTION

As mentioned in Section 3 even if a charge radiates as observed from the laboratory system during hyperbolic motion, the field reaction vanishes. All of the work performed by the force causing the hyperbolic motion is used to change the kinetic energy of the particle. Nothing of it contributes to the energy radiated by the charge. So where does the radiated energy come from?

This question may be investigated by considering, not the eternal hyperbolic motion, but a situation where a charge initially moves with constant velocity, and then, at a position X_1 and point of time T_1, enters a region, H, where it is acted upon by a constant force $F_{ext} = m_0 g$. In this connection phenomena such as runaway motion, pre-acceleration, and acceleration in the opposite direction of that given by the Lorentz equation of motion turn up in certain solutions of the LAD-equation. The non-causal character of these phenomena has lead to several investigations [49–58] trying to find out how they can be resolved either by imposing asymptotic conditions, or by modifying the equation. C. C. Yan [53] has recently introduced a new definition of momentum which removes preacceleration and runaway solution. M. A. Oliver [54], on the other hand, has claimed that the LAD-equation contains an error of sign, and that when this is corrected one obtains an equation of motion with only physically well behaved solutions. This claim is, however, due to an unmotivated change of sign in Eq. (10) in Oliver's deduction of the equation of motion.

Consider a point charge initially moving with constant velocity at a finite distance from a bounded region H with a uniform electric field. The equation of motion, Eq. (4.20), is a first order linear differential equation in $\dot{\alpha}$ with general solution

$$\dot{\alpha} = \left(C - \frac{Q}{m_0 \tau_0} \int \mathrm{E}(\tau) e^{-\tau/\tau_0}\, d\tau \right) e^{\tau/\tau_0}, \qquad \tau_0 = \frac{2}{3} \frac{Q^2}{m_0}, \tag{8.1}$$

where C is a constant of integration. Inserting the electric field (4.17), and letting $g = Q\mathrm{E}_0/m_0$, Eq. (4.20) leads to

$$\ddot{\alpha} = \frac{A_1}{\tau_0} e^{\tau/\tau_0}, \qquad \dot{\alpha} = A_1 e^{\tau/\tau_0}, \qquad \alpha = A_1 \tau_0 e^{\tau/\tau_0} + A_2 \tag{8.2}$$

outside the field, and

$$\ddot{\alpha} = (B_1/\tau_0) e^{\tau/\tau_0}, \qquad \dot{\alpha} = g + B_1 e^{\tau/\tau_0}, \qquad \alpha = g\tau + B_1 \tau_0 e^{\tau/\tau_0} + B_2 \tag{8.3}$$

inside the field. Here A_1, A_2, B_1, B_2 are constants of integration. In the case that the charge enters the field and later leaves it again, the constants A_1, A_2 may have different values in the different field free parts of the motion. It is demanded that the acceleration vanishes in the asymptotic region far from the field, i.e., that the solution contains no so-called runaway term. This requires that $A_{1after} = 0$.

The solutions (8.2) and (8.3) have been discussed by C. J. Eliezer [49], who asks,

> From where does the electron obtain the energy to increase its kinetic energy and also at the same time to continue to lose energy by radiation?

Considering the free motion (8.2) he writes,

> For non zero A_1 the motion is non physical with the velocity steadily increasing and tending to the velocity of light. In this motion there is a loss of energy by radiation. This outflow of energy can take place at the same time as the electron is increasing its mechanical energy, mU^0, because the field energy in the immediate neighbourhood of the electron can become increasingly more and more negative, thus releasing energy for radiation. The energy tensor corresponding to the equation of motion (3.1) is such that the field energy near the electron is not necessarily positive definite. It is the acceleration energy, $(2/3)Q^2A^0$, which contributes towards the increase of mechanical energy, mU^0, and also towards loss of energy by radiation at a rate $(2/3)Q^2A_\mu A^\mu$. The energy of the electron and the field near it is $mU^0 - (2/3)Q^2A^0$, which is negative after sufficient lapse of time. It is however desirable to regard the energy portion $-(2/3)Q^2A^0$ not as field energy but as energy possessed by the electron in virtue of its acceleration. As the electron acquires this negative energy it releases an equal amount of positive energy which supplies the increase of kinetic energy and also the loss of radiation.

Considering the motion (8.3) in a uniform electrical field, Eliezer writes,

> B_1 equal to zero gives the physical motion with the velocity gradually increasing in the direction of the field. For this motion $\dot{A}^\mu - A^\nu A_\nu U^\mu = 0$. The entire energy released by the electron, as it requires its negative acceleration energy, is being lost by radiation. The electric field contributes towards the increase of the kinetic energy of the electron.

We shall now consider in more detail two different types of solutions of Eq. (3.1).

(A) The conventional solution with pre-acceleration.

(B) Motion with reaction forces neutralized.

8A. Motion with Pre-acceleration

It may be noted that the relativistic LAD-equation (4.20) and the nonrelativistic equation (2.1) are mathematically identical in the case of rectilinear motion. The latter may be written in terms of the velocity v and the external field E as

$$(2/3)Q^2 v''(T) + QE = m_0 v'(T) \tag{8.4}$$

which is exactly the same equation as Eq. (4.20) if we replace v by the rapidity α and T by the proper time τ. Thus there is a one to one correspondence between these equations as applied to rectilinear motion.

According to Eq. (2.2) the relativistic general solution may be written

$$\dot{\alpha}(\tau) = e^{\tau/\tau_0}\left(\dot{\alpha}(0) - \frac{1}{m_0\tau_0}\int_0^{\tau} e^{-\tau'/\tau_0} QE(\tau')\,d\tau'\right). \tag{8.5}$$

In order to supress runaway motion (getting pre-acceleration instead) we have in the non-relativistic case the condition (2.3) with the solution (2.4). Translated to the relativistic case we get the condition

$$m_0\tau_0\dot{\alpha}(0) = \int_0^{\infty} e^{-\tau'/\tau_0} QE(\tau')\,d\tau' \tag{8.6}$$

with the solution

$$m_0\dot{\alpha}(\tau) = \int_0^{\infty} e^{-s} QE(\tau + \tau_0 s)\,ds. \tag{8.7}$$

From Eq. (4.19) we have that $\dot{\alpha} = (A^{\mu}A_{\mu})^{1/2} = g$, where $g = \gamma^3 a$ is the acceleration in the rest frame of the charge. Equation (8.7) thus demonstrates that the particle's acceleration depends on the future values of the external force.

Solutions without runaway motion have been considered by Haag [33], Plass [34], and Bradbury [35] for the following case. A charged free particle comes from an infinitely remote distance with rapidity α_0. At a point of time $\tau = \tau_1$ it enters H, where it performs approximately (except for the pre-acceleration) hyperbolic motion. The electric field in H is given by Eq. (4.17). The particle moves out of H at a point of time τ_2. Demanding that $\dot{\alpha}$ and α be continuous and writing $\alpha(-\infty) = \alpha_0 = \tanh v_0$, the solution of the LAD-equation (4.20) for the present case becomes

$$\begin{aligned}
&\ddot{\alpha} = \frac{\dot{\alpha}}{\tau_0}, \quad \dot{\alpha} = g\left(e^{-\frac{\tau_1}{\tau_0}} - e^{-\frac{\tau_2}{\tau_0}}\right)e^{\frac{\tau}{\tau_0}}, \quad \alpha = \alpha_0 + \dot{\alpha}\tau_0, \quad \tau < \tau_1 \\
&\ddot{\alpha} = -\frac{g}{\tau_0}e^{-\frac{\tau_2-\tau}{\tau_0}}, \quad \dot{\alpha} = g\left(1 - e^{-\frac{\tau_2-\tau}{\tau_0}}\right), \quad \alpha = \alpha_0 + g(\tau - \tau_1) + \dot{\alpha}\tau_0, \quad \tau_1 < \tau < \tau_2 \\
&\ddot{\alpha} = 0, \quad \dot{\alpha} = 0, \quad \alpha = \alpha_0 + g(\tau_2 - \tau_1), \quad \tau > \tau_2.
\end{aligned} \tag{8.8}$$

The graphs of these functions are shown in Fig. 2.

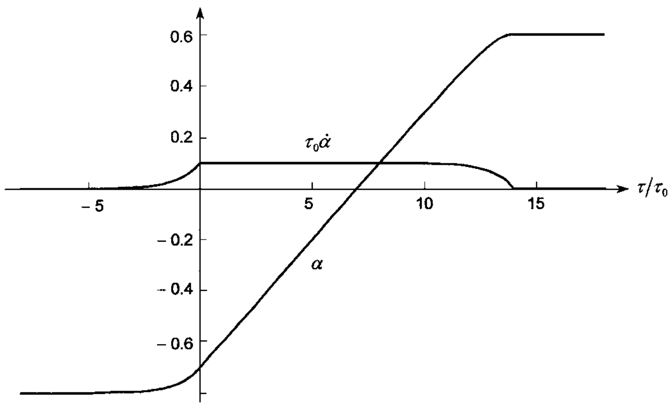

FIG. 2. The rapidity and its rate of change times τ_0 as functions of proper time. Here $\alpha_0 = \alpha(-\infty) = -0.8$ and $g\tau_0 = 0.1$. The particle enters H at $\tau = 0$ and leaves H at $\tau = 14\tau_0$.

The particle gets a pre-acceleration just before it enters and just before it leaves H.

We shall first investigate the motion and energy of the particle during the period $\tau < \tau_1$ before the charge enters H. Assuming that $\tau_0 \ll \tau_2 - \tau_1$, i.e., that the time τ_0 is much less than the time the charge is inside the region H, the solution given in the first of Eqs. (8.8) reduces to

$$\dot{\alpha} = g e^{\frac{\tau-\tau_1}{\tau_0}}, \qquad \alpha = \alpha_0 + g\tau_0 e^{\frac{\tau-\tau_1}{\tau_0}}, \quad \tau < \tau_1 \tag{8.9}$$

$$\dot{\alpha} = g, \qquad \alpha = \alpha_0 + g\tau_0 \equiv \alpha_1, \quad \tau = \tau_1, \tag{8.10}$$

where $\alpha_0 < 0$ is the limiting initial value of the rapidity for $\tau \to -\infty$.

The values of the kinetic energy of the particle and the Schott energy, respectively, for $\tau = -\infty$, are

$$E_K = m_0(\cosh\alpha_0 - 1), \qquad E_S = 0. \tag{8.11}$$

At the moment $\tau = \tau_1$ when the particle enters H, the values of these energies are

$$E_K = m_0(\cosh\alpha_1 - 1), \qquad E_S = -\frac{2}{3}Q^2 g \sinh\alpha_1. \tag{8.12}$$

To 2.order in $g\tau_0$, which we assume is much less than the velocity of light, $c = 1$, the changes in the kinetic energy and the Schott energy from $\tau = -\infty$ to $\tau = \tau_1$ are

$$\Delta E_K = \gamma_0 m_0 v_0 g\tau_0 + \frac{1}{2}\gamma_0 m_0 g^2\tau_0^2 \tag{8.13}$$

$$\Delta E_S = -\gamma_0 m_0 v_0 g\tau_0 - \gamma_0 m_0 g^2\tau_0^2. \tag{8.14}$$

From the energy equation (3.30), which for $\tau < \tau_1$ takes the form $\Delta E_R + \Delta E_K + \Delta E_S = 0$, it follows that the radiated energy during this period is

$$\Delta E_R = \frac{1}{2}\gamma_0 m_0 g^2\tau_0^2. \tag{8.15}$$

From Eqs. (8.13)–(8.15) it follows that the particle during the pre-acceleration gets an increase of the Schott energy which is nearly equal to the loss of kinetic energy of the particle. Only a minor part of the particle's loss in kinetic energy (2.order in $g\tau_0$) is radiated away.

The change of velocity of the particle, $\tanh\alpha_1 - \tanh\alpha_0$, during the pre-acceleration may be expressed as

$$v_1 - v_0 = \frac{\sinh g\tau_0}{\gamma_1\gamma_0} \tag{8.16}$$

which shows that $v_1 - v_0 \to 0$ when $\gamma_0 \to \infty$.

We shall now consider the energy budget during the time when the charge is within H, i.e., for $\tau_1 < \tau < \tau_2$. From Eqs. (3.28) and (8.8) we get an energy equation $\dot{W}_{ext} = \dot{E}_K + E_S + \dot{E}_R$ with the following rates of change per unit proper time

$$\dot{W}_{ext} = m_0 g \sinh\alpha \tag{8.17}$$

$$\dot{E}_K = m_0\dot{\alpha}\sinh\alpha \tag{8.18}$$

$$\dot{E}_S = m_0(g - \dot{\alpha})\sinh\alpha - \frac{2}{3}Q^2\dot{\alpha}^2\cosh\alpha \tag{8.19}$$

$$\dot{E}_R = \frac{2}{3}Q^2\dot{\alpha}^2\cosh\alpha. \tag{8.20}$$

The equations shall now be interpreted for the case that $\dot{\alpha} \approx g$, i.e., when the motion of the particle is approximately hyperbolic. According to Eq. (8.8) this is the case when $\tau \ll \tau_2 - \tau_0$, i.e., when the particle is not too near its exit from H. In this region $\dot{W}_{ext} = \dot{E}_K$ so that there is no other effect of the external force than a change in the kinetic energy. Thus the sum of the kinetic energy of the charge and its potential energy in the field of force accelerating it in H is approximately constant, and the radiated energy is taken from the Schott energy, which according to Eq. (8.19) decreases uniformly with time,

$$\frac{dE_S}{dT} = \frac{\dot{E}_S}{\gamma} \approx -\frac{2}{3}Q^2g^2. \tag{8.21}$$

When the particle arrives at the point where it turns back, there is no Schott energy left, and at this moment the energy that has been radiated by the particle is equal to its loss of kinetic energy during the pre-acceleration. The energy has been radiated as a pulse with energy ΔE_R, given in Eq. (8.15), during the pre-acceleration, and then with a constant effect $(2/3)Q^2g^2$ during the hyperbolic motion.

The situation changes when the particle approaches the position where it leaves H, i.e., when $\tau \approx \tau_2 - \tau_0$. Then the particle experiences a new non-negligible pre-acceleration, which reduces the acceleration from $\approx g$ to 0, and the emitted power is reduced from $\approx(2/3)Q^2g^2$ to 0. The velocity still increases during this period, but less than in the case of hyperbolic motion. The Schott energy, which until now (in H) has decreased at a constant rate, increases from the negative value $-(2/3)Q^2g\gamma v$ to zero. All the energies E_R, E_K, E_S increase during this pre-acceleration. The energy is provided by the work of the external force $F_{ext} = m_0 g$, or in other words from the loss of potential energy of the particle in the field of this force.

In the region where the motion can be considered as hyperbolic, $\dot{\alpha} = g =$ constant, and the reaction force $m_0\tau_0\ddot{\alpha}$ vanishes. Here $F_{ext} = m_0 g$ is the only force acting upon the particle, and $E_K + E_P =$ constant. This is no longer the case when the particle approaches the exit of H, where the pre-acceleration makes $\ddot{\alpha} \neq 0$.

In order to make a complete energy budget in the region H, we must know the proper time τ_2 when the particle leaves H. The position $X(\tau)$ of the particle at a point of time τ is given by

$$X(\tau) - X_1 = \int_{\tau_1}^{\tau} \gamma v\, d\tau = \int_{\tau_1}^{\tau} \sinh\alpha\, d\tau, \tag{8.22a}$$

where α is given by Eq. (8.8). The point of time τ_2 when the particle leaves H is found from the equation $X(\tau_2) = X_1$. Solving the integral (to second order in $g\tau_0$) the equation reads

$$g\int_{\tau_1}^{\tau_2} \sinh\alpha\, d\tau = \left(1 - g^2\tau_0^2\right)\cosh(\alpha_1 + g(\tau_2 - \tau_1)) - \cosh\alpha_1, \tag{8.22b}$$

where we have utilized that $\tau_0 \ll \tau_2 - \tau_1$. We get the following solution to second order in $g\tau_0$

$$\tau_2 - \tau_1 = -\frac{1}{g}\left(2\alpha_1 + g^2\tau_0^2\coth\alpha_1\right), \tag{8.23}$$

where $\alpha_1 = \alpha_0 + g\tau_0$ is the rapidity of the particle at the moment it enters H. The term $-2\alpha_1/g$, which is dominating, is the proper time that the particle would have spent inside H, if the motion had been hyperbolic. Then $\dot{\alpha} = g$, so the travelling proper time would be $\Delta\tau = \Delta\alpha/g$, where $\Delta\alpha = -2\alpha_1$ is the increase of α during the motion in H. Equation (8.23) tells that $\tau_2 - \tau_1$ is a little larger than this value.

Inserting τ_2 from Eq. (8.23) into the second of Eq. (8.8) we get

$$\alpha(\tau_2) = -\alpha_1 - g\tau_0 - g^2\tau_0^2 \coth\alpha_1 \tag{8.24}$$

which gives

$$\cosh\alpha(\tau_2) = \left(1 + \frac{3}{2}g^2\tau_0^2\right)\cosh\alpha_1 + g\tau_0 \sinh\alpha_1. \tag{8.25}$$

From this we find the following (negative) changes of the kinetic energy of the charge and its Schott energy during the period, $\tau_1 < \tau < \tau_2$, when the charge moves in H,

$$\Delta E_K = m_0 g\tau_0\gamma_1 v_1 + \frac{3}{2}m_0 g^2\tau_0^2\gamma_1 \tag{8.26}$$

$$\Delta E_S = m_0 g\tau_0\gamma_1 v_1. \tag{8.27}$$

Since the total work performed by the external force F_{ext} upon the charge during its motion in H vanishes, the energy equation (3.30) gives for the energy radiated by the charge during this motion

$$\Delta E_R = -\Delta E_K - \Delta E_S = -2m_0 g\tau_0\gamma_1 v_1 - \frac{3}{2}m_0 g^2\tau_0^2\gamma_1. \tag{8.28}$$

The dominating term, $-2m_0 g\tau_0\gamma_1 v_1$, may be interpreted as the energy radiated by a particle with exact hyperbolic motion. This is seen as follows. Using that

$$g = \dot{\alpha} = \gamma\frac{d\alpha}{dT} = \cosh\alpha\frac{d\alpha}{dT} = \frac{d(\sinh\alpha)}{dT} \tag{8.29}$$

for hyperbolic motion, leads to

$$\Delta(\sinh\alpha) = g\Delta T. \tag{8.30}$$

Thus, the time that the charge stays inside H is

$$\Delta T = -2\sinh\alpha_1/g. \tag{8.31}$$

The dominating term in Eq. (8.28) may be written

$$m_0 g^2\tau_0\frac{-2\sinh\alpha_1}{g} = m_0 g^2\tau_0\Delta T = \frac{2}{3}Q^2 g^2\Delta T \tag{8.32}$$

in agreement with Larmor's formula.

According to Eqs. (8.26) and (8.27) the Schott energy and the kinetic energy decrease by about the same amount, which means that the Schott energy and the kinetic energy give approximately the same contribution to the radiated energy.

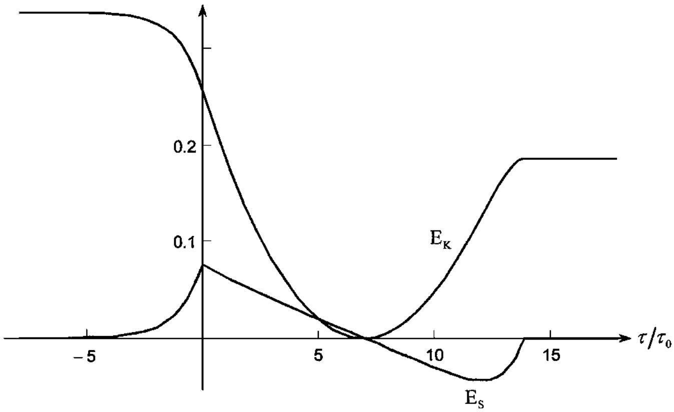

FIG. 3. Kinetic energy and Schott energy in units of m_0 as functions of proper time for the motion shown in Fig. 2.

To get the complete energy budget from $\tau = -\infty$ to $\tau = \infty$ we utilize that $W_{ext} = 0$ and that $E_S(-\infty) = E_S(\infty) = 0$. Then, according to Eq. (3.30), $\Delta E_R + \Delta E_K = 0$, where ΔE_R is the sum of the expressions (8.15) and (8.28). Expressing the relationships in terms of the velocity v_0 (which is negative) at $\tau = -\infty$, we find that the radiated energy is

$$\Delta E_R = \Delta E_K = E_K(-\infty) - E_K(\infty) = -2m_0 g\tau_0\gamma_0 v_0 - 3m_0 g^2\tau_0^2\gamma_0. \tag{8.33}$$

Diagrams with E_K, E_S, E_R are shown in Figs. 3 and 4.

Let us summarize what happens to the particle and its energy from $\tau = -\infty$ to $\tau = \infty$. The charge comes from an infinitely far region with constant velocity. It moves towards a region H with, say, a constant electrical field anti-parallel to its direction of motion. Approaching H it gets an increasing pre-acceleration, which causes the kinetic energy of the particle to decrease. A Schott energy of about the same magnitude appears. Also a small amout of energy is radiated away by the particle.

In the region H the particle moves approximately hyperbolically until it experiences a new pre-acceleration before it leaves H. During the hyperbolic part of the motion the external work performed by the field force upon the particle is used only to change the kinetic energy of the particle. The

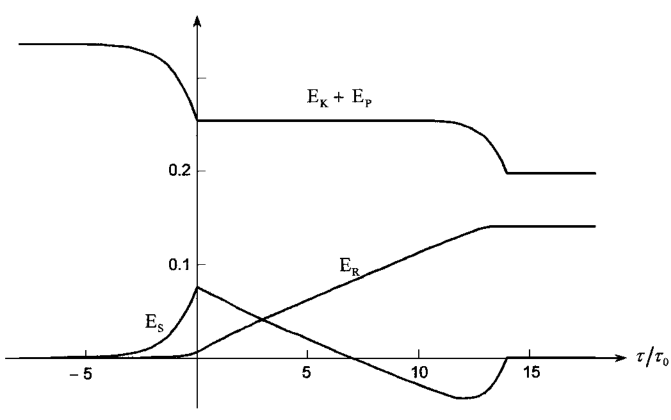

FIG. 4. Mechanical energy, i.e., $E_K + E_P$, Schott energy, and radiated energy as functions of proper time. Here E_P is the potential energy in the force field $m_0 g$ with $E_P = 0$ for $X > X_1$, and $E_P = m_0 g(X_1 - X)$. Note that $E_K + E_P + E_S + E_R =$ constant. All energies are in units of m_0.

particle radiates at a constant rate, and the radiated energy comes from the Schott energy, which decreases steadily during this part of the motion. Before the particle leaves H the pre-acceleration decreases the acceleration towards zero. The particle still radiates although the Schott energy now increases.

What happens all together while the particle is in H is that the kinetic energy and the Schott energy decrease by about the same amount, giving about the same contribution to the radiated energy.

When the particle has left H and disappears towards an infinite remote region, the Schott energy has vanished again. The particle has lost kinetic energy, and this loss of energy is equal to the energy that the particle has radiated.

8B. Motion with Reaction Forces Neutralized

We now let the particle, in addition to the given force F_{ext}, be acted upon by a force f_{ext} which is opposite to the field reaction force. That is, we consider a modified problem of motion by introducing a force which at every point neutralizes the field reaction force. Then the particle moves as if there were no field reaction force, and consequently no pre-acceleration or runaway solution.

We consider a particle moving in an external field F^{μ}_{ext}. If we neglect the field reaction force, the equation of motion is

$$F^{\mu}_{ext} = m_0 A^{\mu}. \tag{8.34}$$

Let $x^{\mu}(\tau)$ be a solution of this equation. Then the field reaction force Γ^{μ}, which we have neglected, is

$$\Gamma^{\mu} = \frac{2}{3}Q^2 \dot{A}^{\mu} - \frac{2}{3}Q^2 g^2 U^{\mu}, \qquad A^{\mu} = \dot{U}^{\mu}, \qquad U^{\mu} = \dot{X}^{\mu}. \tag{8.35}$$

Now we introduce a compensating force f^{μ}_{ext},

$$f^{\mu}_{ext} = -\Gamma^{\mu}. \tag{8.36}$$

Then Eq. (3.1) can be written

$$F^{\mu}_{ext} + f^{\mu}_{ext} + \Gamma^{\mu} = m_0 A^{\mu} \tag{8.37}$$

which is the LAD-equation with an extra force f^{μ}_{ext}. Thus by solving Eq. (8.34) for a given force F^{μ}_{ext} and introducing a force f^{μ}_{ext} as calculated by inserting the found solution, $x^{\mu}(\tau)$, into Eqs. (8.35), (8.36), we have a solution of the Lorentz–Dirac equation (8.37).

The rate of work performed per unit proper time by the force f^{μ}_{ext} is

$$f^{0}_{ext} = -\frac{2}{3}Q^2 \dot{A}^0 + \frac{2}{3}Q^2 \gamma g^2 = \dot{E}_S + \dot{E}_R, \tag{8.38}$$

where E_S is the Schott energy and E_R is the radiation energy. The rate of change of momentum per unit proper time due to the force $\mathbf{f}_{ext}$ is

$$\gamma \mathbf{f}_{ext} = -\frac{2}{3}Q^2 \dot{\mathbf{A}} + \frac{2}{3}Q^2 g^2 \gamma \mathbf{v} = \dot{\mathbf{P}}_S + \dot{\mathbf{P}}_R, \tag{8.39}$$

where $\mathbf{f}_{ext}$ is a 3-force, $\mathbf{P}_S$ is the Schott momentum, and

$$(\dot{E}_R, \dot{\mathbf{P}}_R) = \frac{2}{3}Q^2 g^2 \gamma(1, \mathbf{v}) \tag{8.40}$$

is the 4-momentum radiated per unit proper time.

Assume that the charged particle experiences a force F^{μ}_{ext}, given by a step function, making the 4-acceleration A^{μ} increase by ΔA^{μ} during a vanishingly small time interval. The velocity is then (approximately) unchanged, since the acceleration is finite and the time interval is arbitrarily small. In this case the change of 4-momentum due to the force f^{μ}_{ext} is

$$\Delta P^{\mu} = \int f^{\mu}_{ext}\, d\tau = \int -\frac{2}{3}Q^2 \dot{A}^{\mu}\, d\tau = -\frac{2}{3}Q^2 \Delta A^{\mu}. \tag{8.41}$$

Such a blow thus leads to a change $-(2/3)Q^2\gamma\mathbf{v}\cdot\Delta\mathbf{g}$ of Schott energy and a change $-(2/3)Q^2\Delta\mathbf{A}$ of Schott momentum. In the instantaneous rest frame of the particle $\Delta P^0 = 0$, which means that in this frame there is only a change of momentum and not of energy.

We now specialize to linear motion in which the charge moves towards the region H and enters it. Then $\mathbf{g} = g\mathbf{e}_x$, $\mathbf{f}_{ext} = f_{ext}\mathbf{e}_x$, and

$$f_{ext} = -\frac{2}{3}Q^2\dot{g}, \tag{8.42}$$

i.e.,

$$f_{ext} = -m_0\tau_0\ddot{\alpha}, \tag{8.43}$$

where $\tau_0 = (2/3)Q^2/m_0$ and $g = \dot{\alpha}$.

In the case considered here the solution is that the particle comes from $X = \infty$ with constant rapidity α_1 (velocity v_1). At proper time $\tau = \tau_1 = \alpha_1/g$ (laboratory time T_1 and position X_1) the proper acceleration $\dot{\alpha}$ is changed discontinuously from $\dot{\alpha} = 0$ to $\dot{\alpha} = g$. During the time that the particle is in H it has a constant proper acceleration $\dot{\alpha} = g$, i.e., $\alpha = g\tau$. The particle performs a hyperbolic motion from $T = T_1$ to $T = -T_1$. When the particle leaves H at the point of time $\tau = \tau_2 = -\alpha_1/g = -\tau_1$, the proper acceleration $\dot{\alpha}$ changes discontinuously from $\dot{\alpha} = g$ to $\dot{\alpha} = 0$. Thus, the solution is

$$\begin{aligned} &\dot{\alpha} = 0, && \alpha = \alpha_1, \quad \tau < \tau_1 \\ &\dot{\alpha} = g, && \alpha = g\tau, \quad \tau_1 < \tau < \tau_2 \\ &\dot{\alpha} = 0, && \alpha = -\alpha_1, \quad \tau > \tau_2 \\ &\ddot{\alpha} = g[\delta(\tau - \tau_1) - \delta(\tau - \tau_2)]. \end{aligned} \tag{8.44}$$

This is a special case of the general solution (8.1) with $A_1 = B_1 = 0$ and $B_2 = A_{2before} = -A_{2after} = \alpha_1$.

In the present case the extra force f_{ext} vanishes everywhere except at the transition into and out of H,

$$f_{ext} = -m_0 g\tau_0\delta(\tau - \tau_1) + m_0 g\tau_0\delta(\tau - \tau_2) = -\frac{2}{3}Qg\gamma\delta(T - T_1) + \frac{2}{3}Qg\gamma\delta(T - T_2). \tag{8.45}$$

This means that the particle must be given an impulse in the direction of motion when it enters H, and also when it leaves H, in order to compensate for the field reaction force.

Due to the acausal behaviour of the pre-acceleration that takes place in the ordinary case, the force f_{ext} that prevents this pre-acceleration in the present case has the strange property of acting just at the end of the periods of pre-acceleration. Thus, the effect of the force f_{ext} comes before the force is acting.

The energy provided by the force f_{ext} when the particle enters H is

$$\begin{aligned} w_{ext} &= \int f_{ext} v\, dT = -m_0 g \tau_0 \int \delta(\tau - \tau_1) v \gamma\, d\tau \\ &= -m_0 g \tau_0 v_1 \gamma_1 = -\frac{2}{3} Q^2 g v_1 \gamma_1 = -\frac{2}{3} Q^2 g^2 T_1. \end{aligned} \tag{8.46}$$

The equation shows that the energy which must be given to the particle in order to bring it from a state of motion with vanishing proper acceleration to a state with the same velocity but a proper acceleration $\dot{\alpha} = g$ is given by the increase of the Schott energy. The same amount of energy must be conveyed to the particle in order that it can leave H with unchanged velocity and a sudden change of proper acceleration from $\dot{\alpha} = g$ to $\dot{\alpha} = 0$.

We shall write down the energy expressions for a particle entering and leaving a hyperbolic region H. The equation of motion is Eq. (8.34). The particle comes from the infinite far ($X = \infty$) with a constant velocity v_1 which is negative. At a point of time $T_1 < 0$ the particle enters the region H, the velocity being continuous. In H the proper acceleration g is constant. The particle stops at $T = 0$, then reverses its motion and leaves the region at $-T_1$, and returns to $X = 0$ with constant velocity $-v_1$.

During the hyperbolic motion

$$v = T/X, \qquad \gamma = X/L = gX, \qquad X = \sqrt{L^2 + T^2}. \tag{8.47}$$

The energy equation for the system of the particle and the field is

$$W_{ext} + w_{ext} = \Delta E_K + \Delta E_S + \Delta E_R, \tag{8.48}$$

where W_{ext} is the work done by the force $m_0 g$, and w_{ext} is the work done by the impact force f_{ext} in Eq. (8.45). The energy expressions are for

$$T < T_1 \qquad E_K = \gamma_1 m_0 - m_0 = (gX_1 - 1) m_0, \quad E_S = E_R = 0 \tag{8.49a–c}$$

$$T_1 < T < -T_1 \qquad \begin{aligned} E_K &= (gX - 1) m_0 \\ E_S &= -(2/3) Q^2 \gamma v g = -(2/3) Q^2 g^2 T \\ E_R &= (2/3) \int_{T_1}^{T} g^2\, dT = (2/3) Q^2 g^2 (T - T_1) \end{aligned} \tag{8.50a–c}$$

$$T > -T_1 \qquad E_K = (gX_1 - 1) m_0, \quad E_S = 0, \quad E_R = -(4/3) Q^2 g^2 T_1. \tag{8.51a–c}$$

Note that when the particle is in H, $E_S + E_R = \text{constant} = -(2/3) Q^2 g^2 T_1$ and $w_{ext} = 0$. It follows that $\Delta E_R = -\Delta E_S$ and that $W_{ext} = \Delta E_K$.

Let E_P be the potential energy of the particle in the given field,

$$E_P = -m_0 g(X - X_1) \quad \text{for } X < X_1, \qquad \text{and} \qquad E_P = 0 \quad \text{for } X > X_1 \tag{8.52}$$

Then the mechanical energy of the particle is

$$E_K + E_P = \text{constant} = (gX_1 - 1)m_0 \tag{8.53}$$

which is equal to the kinetic energy of the particle before it enters the region H. Equation (8.53) is valid in the whole domain of X.

The different types of energy as functions of the time are shown in Fig. 5.

At the transition into H the particle (or its field) is "loaded" with Schott energy, so that immediately after the particle has entered H, it has a positive Schott energy $-(2/3)Q^2\gamma_1 v_1 g$. During the motion inside H the field reaction force upon the particle vanishes, and the work performed by the external force upon the particle is used only to change its kinetic energy. This means that the energy radiated by the particle must be provided by the Schott energy.

From the moment when the particle enters H until it stops at $X = L$, the Schott energy diminishes by $-(2/3)Q^2\gamma_1 v_1 g$, i.e., by $-(2/3)Q^2 g^2 T_1$. This is just the energy radiated by the particle, according to Larmor's formula, from the time T_1 to $T = 0$. During the motion back to the boundary of H, the Schott energy is further diminished by the same amount of energy, which is radiated away by the particle. When the particle arrives at X_1 again, and is about to leave H, its Schott energy is negative. In order to change its proper acceleration from g to zero, it must be acted upon by a force, which performs a positive work upon the particle. This work provides the energy necessary to make the Schott energy equal to zero at the transition.

We have found that the Schott energy during the hyperbolic motion is $-(2/3)Q^2 g^2 T$ at the point of time T. This expression shows that the Schott energy has the limit ∞ as $T \to -\infty$. Thus, in the case of eternal hyperbolic motion the Schott energy diminishes from ∞ at an infinitely far past time to $-\infty$ in the indefinite future.

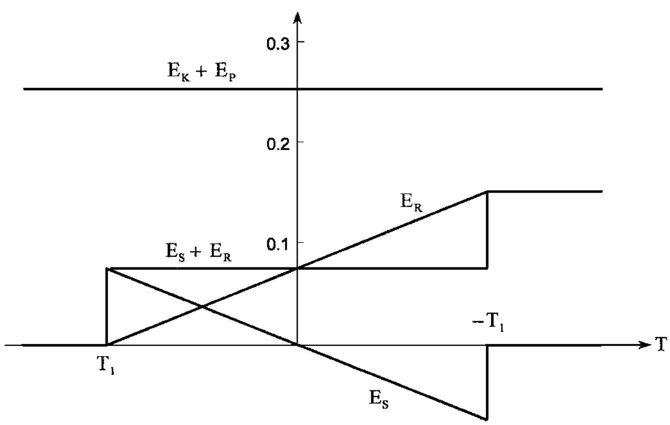

FIG. 5. The particle comes from $X = \infty$ with constant velocity v_1. An external impact force performing a work $-(2/3)Q^2 g\gamma_1 v_1$ is acting upon the particle when it enters the hyperbolic region at $T = T_1$, and when it leaves it at $T = -T_1$. All of this work is used to increase the Schott energy E_S. These steps are found in the graphs of E_S and of the sum $E_S + E_R$. The work performed by the force $m_0 g$ in H is equal to the change of the kinetic energy E_K of the particle. Hence the mechanical energy $E_K + E_P$ is constant. In the figure we have put $g\tau_0 = 0, 1$, $T_1 = -7, 6\tau_0$, and $v_1 = -\tanh 0, 7 = -0, 6$. All energies are in units of m_0. $T_1 = -7.6\tau_0$.

9. FIELD ENERGIES IN THE CASE WITH NEUTRALIZED REACTION FORCES

We shall consider the field energies in the case where an extra force acts upon the charge so that there is no pre-acceleration. The total field energy is in general the sum of the radiated energy E_R and the bound energy U_I. The energy E_R is given by the prehistory of the particle, according to Larmor's formula,

$$E_R(T) = \frac{2}{3}Q^2 \int_{-\infty}^{T} g^2(T_Q)\,dT_Q, \tag{9.1}$$

where $g(T_Q)$ is the proper acceleration. The bound energy $U_I = U_{I,I} + U_{I,II}$ is the field energy due to the zeroth and first order terms in the particle's acceleration, i.e., the sum of the energy of the Coulomb/velocity field and the interaction energy between the fields $F_I^{\mu v}$ and $F_{II}^{\mu v}$ (the latter energy is not necessarily positive). The energies $U_{I,I}$ and $U_{I,II}$ are not state functions of the particle, but their sum is. From Eq. (3.18A) in Ref. [46],

$$U_I = E_S + U_C, \tag{9.2a}$$

where

$$E_S = -\frac{2}{3}Q^2\gamma v g \tag{9.2b}$$

and

$$U_C = \frac{Q^2\gamma}{2\epsilon}\left(1 + \frac{1}{3}v^2\right). \tag{9.2c}$$

These equations show that the energy has a remarkable property. When the particle has come into a state of constant velocity after it has left the region H, the bound energy is given by this velocity, just as if it had moved with this contant velocity all the time.

Like the interaction energy, the Schott energy E_S is of first order in the acceleration, but as shown in Section 6 it cannot be identified with the interaction energy in the field. Note also that U_C is not equal to the energy of the Coulomb field produced by the charge. For instance, when the particle in hyperbolic motion stops at $T = 0$, then $E_S = 0$ and $U_C = Q^2/2\epsilon$, but as shown in Eq. (6.22), the interaction energy in the field is $U_{I,II} = (4/3)Q^2\gamma_1 v_1 g = (4/3)Q^2 g^2 T_1$, where we have used Eq. (8.47). Hence $U_{I,I} = Q^2/2\epsilon - (4/3)Q^2 g^2 T_1$.

When the particle comes from $X = \infty$ with constant velocity v_1 outside H, the field is a (Lorentz contracted) Coulomb field symmetric about a plane through the particle normal to the direction of motion, and with energy given by Eq. (9.2c),

$$U_C = \frac{Q^2\gamma_1}{2\epsilon}\left(1 + \frac{1}{3}v_1^2\right) = \frac{Q^2}{6\epsilon}\left(4gX_1 - \frac{1}{gX_1}\right) \simeq -\frac{2}{3}\frac{Q^2}{\epsilon}gT_1. \tag{9.3}$$

The last term is valid in the limit $X_1 \to \infty$, i.e., when $T_1 \to -\infty$. When the particle enters H, the field suddenly acquires an increase of energy since the proper acceleration then jumps from 0 to g. A radiation field $F_{II}^{\mu v}$ appears, and the total energy of the field increases instantaneously by

$E_S = -(2/3)Q^2\gamma_1 v_1 g$. This result is based upon our calculations in Ref. [46]. Hence these calculations provide an analytical proof of the discontinuity of the field energy in the present situation, which was found numerically by Ng [59].

When the particle is in H the radiated energy is provided by the bound energy of the field, such that the Schott energy E_S diminishes at the same rate as energy is radiated. The transition between the energies must take place at the position of the particle since the same amount of radiation energy passes per second through each spherical surface around the charge. Let K_1 be an eikonal with center at the point $X = X_1$ and with radius $T - T_1$ where $T_1 < T < -T_1$. The Coulomb field produced by the charge before the charge entered H is outside K_1. According to Eq. (3.8) in Ref. [46] the energy of this field at the time $T > T_1$ is

$$U_1(-\infty, T_1, T) = \frac{Q^2}{2(T - T_1)}\left(1 + \frac{4}{3}\gamma_1^2 v_1^2\right) = \frac{Q^2}{2(T - T_1)}\left(1 + \frac{4}{3}g^2 T_1^2\right) \simeq \frac{2}{3}Q^2 g^2(-T_1 - T), \tag{9.4}$$

where the last term is valid when $T_1 \to -\infty$ and T is finite. In the limit $T_1 \to -\infty$ this energy is concentrated at the plane following the virtual position of the particle, normal to the X-axis. This is the front-energy of the Bondi–Gold field [60, 61].

Inside K_1 is the Schott field with energy consisting of the bound energy U_I and the radiation energy E_R. From Eq. (3.18A) in Ref. [46],

$$\begin{aligned} U_I &= \frac{Q^2}{2\epsilon g X}\left(1 + \frac{4}{3}g^2 T^2\right) - \frac{2}{3}Q^2 g^2 T - \frac{Q^2}{2(T - T_1)}\left(1 + \frac{4}{3}g^2 T_1^2\right) \\ &\simeq \frac{Q^2}{2\epsilon g X}\left(1 + \frac{4}{3}g^2 T\right) + \frac{2}{3}Q^2 g^2 T_1 \end{aligned} \tag{9.5}$$

$$E_R = \frac{2}{3}Q^2 g^2 (T - T_1). \tag{9.6}$$

We see that infinite terms that appear in U_I and E_R in the limit $T_1 \to -\infty$ cancel each other, so that even in this limit the energy of the Schott field is finite, and given by

$$U = U_I + E_R \simeq \frac{Q^2}{2\epsilon g X}\left(1 + \frac{4}{3}g^2 T^2\right) + \frac{2}{3}Q^2 g^2 T, \tag{9.7}$$

where $X = \sqrt{L^2 + T^2}$.

After the particle has left H, the field has a bound energy equal to that before it entered H, since the magnitude of the velocity is the same. However, while the field was a pure Coulomb field before the particle entered H, this is not the case after it has left H. Let us introduce a second eikonal K_2 at $T > -T_1$ with center at $X = X_1$ and radius $T + T_1$. Inside K_2 and outside K_1 there are Coulomb fields produced by the particle with respective velocities $-v_1$ and v_1. Between the eikonals the field consists of a Coulomb field $F_I^{\mu\nu}$ and a radiation field $F_{II}^{\mu\nu}$. The energy of the radiation field is equal to the work performed by the neutralizing force $\mathbf{f}_{ext}$ acting when the particle enters and leaves H. This is again equal to the reduction of the Schott energy while the particle is in H. In addition to the radiation energy there is bound field energy consisting of Coulomb energy and interaction energy. The value of the bound energy between the eikonals is as if the velocity of the particle had been constant and equal to v_1 (or $-v_1$).

D. Villaroel [62] has recently considered the problem of causality violation in classical electrodynamics. In particular he has constructed analytic solutions without pre-acceleration to the Lorentz–Dirac equation, for situations with an electrostatic field which vanishes outside a region of limited extension. Villaroel pointed out that the existence of the non-causal solutions with pre-acceleration depends, among others, upon the requirement that the acceleration of the charge is continuous even at points where it enters or leaves an electric field with a discontinuous boundary. This represents, in itself, a non-physical behaviour. Thus, he permits discontinuities of the acceleration at points where the external force acting upon the charge is discontinuous.

The case considered in the present subsection represents a modification of Villaroel's method. We have introduced an external force counteracting the field reaction force. Thus, the rate of change of the rapidity with proper time is given by

$$QE = m_0\dot{\alpha}. \tag{9.8}$$

The equation was applied to a situation with a region H with a uniform electric field, $E = m_0 g$, and an exterior region with $E = 0$. In these regions $\ddot{\alpha} = 0$. This means that (8.44) is a solution of Eq. (4.20) in these regions. Furthermore the rapidity is continuous at the boundary between the regions. The solution is therefore in accordance with the solution Villaroel presents in order to prevent pre-acceleration.

However, our method is different from that of Villaroel in one essential respect. The particle must be supplied with energy and momentum as it enters and leaves H, since the acceleration shall change discontinuously. This is due to the fact that the particle, in addition to the kinetic four-momentum $m_0 U^\mu$, has a Schott four-momentum $P_S^\mu = -(2/3)Q^2 A^\mu$. This was not taken into account by Villaroel. The energy of the particle is

$$E_{part} = \gamma m_0 - \frac{2}{3}Q^2\gamma \mathbf{v}\cdot\mathbf{g} \tag{9.9}$$

whether the particle is pre-accelerated or not.

The Schott energy (or "acceleration energy") which must be supplied to the particle in order to suddenly change its acceleration as it enters H is just the energy that the particle radiates away in H before it stops instantaneously at the turning point.

10. SOME COMMENTS ON THE LORENTZ–DIRAC EQUATION AND DISCONTINUITIES IN THE DYNAMIC VARIABLES

We shall first discuss whether it is possible to have a discontinuity in the velocity when the charge enters H. Then we first consider a transition layer with finite thickness, and afterwards take the limit representing a sharp boundary for the region H. Assume that a charged particle changes the four velocity from U_1 at T_1 to $U_1 + \Delta U$ at $T_1 + \Delta T$. The acceleration in the instantaneous rest frame of the particle is easily shown to be equal to the derivative of U with respect to T, i.e., $U'(T) = (\gamma v)' = \gamma^3 a v^2 + \gamma a = \gamma^3 a = g$, so the radiated energy during the transition is

$$\Delta E_R = \frac{2}{3}Q^2 \int\limits_{T_1}^{T_1+\Delta T} U'(T)^2\, dT. \tag{10.1}$$

We shall find an equation for the minimum of this integral with the given boundary conditions. This is determined by a Hamiltonian variational principle, leading to the Euler–Lagrange equation

$$\left(\frac{d}{dT}\frac{\partial}{\partial U'} - \frac{\partial}{\partial U}\right)U'^2 = 0 \tag{10.2}$$

giving $U''(T) = 0$, i.e., $U'(T) = g_0 =$ constant. Thus, the radiation loss is minimal if the transition is performed as a hyperbolic motion. Then

$$U = g_0 T + \text{constant}, \tag{10.3}$$

where

$$g_0 = \frac{\Delta U}{\Delta T} \tag{10.4}$$

is the proper acceleration in the transition region. This leads to

$$\Delta E_R = \frac{2}{3} Q^2 \frac{(\Delta U)^2}{\Delta T}. \tag{10.5}$$

It follows that $\Delta E_R \to \infty$ when $\Delta T \to 0$ and $\Delta U \neq 0$. This infinite amount of radiation loss cannot be avoided by any type of transition with a discontinuity in the velocity, since it represents a minimum value.

The instantaneous work performed at T_1 is equal to the difference of Schott energy just before T_1 and just after T_1. If the acceleration is g_1 just before T_1 and g_0 just after T_1, the work at the point of time T_1 is

$$W_1 = \frac{2}{3} Q^2 (g_1 - g_0) U_1. \tag{10.6}$$

Just after $T_1 + \Delta T$, when the charge has moved through the transition region and entered H, it has hyperbolic motion with proper acceleration g. Thus, the work performed at $T_1 + \Delta T$ is

$$W_2 = \frac{2}{3} Q^2 (-g + g_0)(U_1 + \Delta U). \tag{10.7}$$

During the hyperbolic motion in the transition zone, from T_1 to $T_1 + \Delta T$, there acts a constant force $m_0 g_0$ performing the work

$$W_H = m_0 \Delta\gamma. \tag{10.8}$$

The total work performed from just before T_1 to just after $T_1 + \Delta T$ is

$$W = \frac{2}{3} Q^2 [g_1 U_1 - g(U_1 + \Delta U)] + \frac{2}{3} Q^2 g_0 \Delta U + m_0 \Delta\gamma \tag{10.9}$$

or

$$W = \Delta E_S + \Delta E_R + \Delta E_K, \tag{10.10}$$

where

$$\Delta E_S = \frac{2}{3} Q [g_1 U_1 - g(U_1 + \Delta U)] \tag{10.11}$$

is the increase of Schott energy from just before T_1 to just after $T_1 + \Delta T$,

$$\Delta E_R = \frac{2}{3} Q^2 g_0 \Delta U = \frac{2}{3} Q^2 \frac{(\Delta U)^2}{\Delta T} \tag{10.12}$$

is the radiated energy, and

$$\Delta E_K = m_0 \Delta \gamma. \tag{10.13}$$

is the increase of kinetic energy.

The infinitely large radiation energy in the limit $\Delta T \to 0$, $\Delta U \neq 0$ cannot be taken from the kinetic energy of the charge or its Schott energy, since the velocities and accelerations just before T_1 and after $T_1 + \Delta T$ are finite quantities. Thus the discontinuity of velocity must be due to a force performing an infinitely great work. Not even an infinitely large impulsive force, given by a δ-function, will let the particle enter H with a discontinuity in the velocity, since the work performed by such a force is finite.

Next we shall consider the case that the velocity is continuous, but the acceleration has a jump. At a transition of this type there is no radiated energy and there is no change of kinetic energy, but there is a discontinuity in the Schott energy. Such a transition requires that the particle is acted upon by a force performing a work equal to the change of the Schott energy, i.e., the Lorentz–Dirac equation requires an infinite force of a δ-function character if the acceleration makes a sudden jump. Thus, if the external forces acting on the particle are finite, the acceleration is continuous.

Consider a particle arriving from a region free of any forces infinitely far away, at $\tau = -\infty$, with asymptotic rapidity $\alpha_0 (<0)$. At $X = X_1$ the particle enters a region $X < X_1$ with a field of force (H in the case of a uniform electric field). The force per unit mass is $f = QE/m_0$, and points in the positive X-direction. The transition may be continuous or discontinuous in E, but we assume that E is finite at all points, and that the particle moves without being acted upon by any other external force. Then $\alpha(\tau)$ and $\dot{\alpha}(\tau)$ are continuous functions, but $\ddot{\alpha}(\tau)$ is discontinuous if f is so. In this case the Lorentz–Dirac equation may be written

$$\dot{\alpha} - \tau_0 \ddot{\alpha} = f. \tag{10.14}$$

Solving this equation, with the general solution given in Eq. (8.1), we find the following expressions for the rapidity, with pre-acceleration,

$$\alpha(\tau) = \alpha_0 + \int_{-\infty}^{\tau} f(\tau)\, d\tau + e^{\tau/\tau_0} \int_{\tau}^{\infty} e^{-\tau/\tau_0} f(\tau)\, d\tau, \tag{10.15}$$

and without pre-acceleration,

$$\alpha(\tau) = \alpha_0 + \int_{-\infty}^{\tau} f(\tau)\, d\tau - e^{\tau/\tau_0} \int_{-\infty}^{\tau} e^{-\tau/\tau_0} f(\tau)\, d\tau. \tag{10.16}$$

The two first terms at the right hand side are those expected without any field reaction. The effect of the field reaction is expressed by an integral over the future in the first case, and over the past in the latter case.

Let the particle enter the field at τ_1 and leave the field at τ_2. We assume that $f = g =$ constant for $\tau_1 < \tau < \tau_2$, and $f = 0$ for $\tau < \tau_1$ and $\tau > \tau_2$. Then Eq. (10.15) leads to Eq. (8.8), representing a motion where the particle moves away from the force field with constant velocity for $\tau > \tau_2$.

In the second case the motion is given by Eq. (10.16), which leads to

$$\begin{aligned} \alpha(\tau) &= \alpha_0, \qquad \tau < \tau_1 \\ \alpha(\tau) &= \alpha_0 + (\tau - \tau_1)g + \left(1 - e^{\frac{\tau-\tau_1}{\tau_0}}\right) g\tau_0, \qquad \tau > \tau_1. \end{aligned} \tag{10.17}$$

In this case a strange circumstance appears, that has previously been noted by C. J. Eliezer [55] and S. Parrot [52, 58], namely that the particle accelerates into the field in spite of the fact that the field force acts in the opposite direction. If the field had been oppositely directed, i.e., into the field, the particle would have behaved equally as strangely: moving into the field due to its velocity, being retarded, stopping, and accelerating back, moving out of the field.

In the case with pre-acceleration we find that the particle has an acceleration *before* it enters the field, and it has constant velocity after it has left the field. One may wonder: What about time reversal of this motion? Would not that mean a motion with constant velocity *before* the particle enters the field? The reason that this way of reasoning is not valid is the presence of the $\ddot{\alpha}$ term in the Lorentz–Dirac equation, which implies that the motions that it describes are not generally time reversible.

11. A MODIFIED LAD-EQUATION WITHOUT PRE-ACCELERATION

The LAD-equation is the preferred equation of motion of a radiating point charge. It reduces to Eq. (2.1) in the instantaneous rest frame of the charge. The runaway free solution given in Eq. (2.4) contains pre-acceleration. Yaghjian [19] has investigated the deduction of the LAD-equation in order to find the cause of pre-acceleration and to eliminate it.

The method of Yaghjian is to find the equation of motion of a uniformly charged shell with finite radius ϵ and retaining only terms that do not vanish in the limit $\epsilon \to 0$. He considers a charged shell which is not acted upon by any forces until a point of time $\tau = 0$ and notes that deduction involves a Taylor series expansion which converges only for $0 < \tau < 2\epsilon$. Hence the LAD-equation is not valid during this time interval. Correcting for this lack of convergence by renouncing to specify the motion of the shell during this time interval, he arrives at the following modified LAD-equation for a point charge

$$F^{\mu}_{ext} + \Theta(\tau)\Gamma^{\mu} = m_0 \dot{U}^{\mu}, \tag{11.1}$$

where Θ is the Heaviside step function. We shall call this equation the Lorentz–Abraham–Dirac–Yaghjian equation, or for short, the LADY-equation.

Applying this equation to the charge entering the region H there is no pre-acceleration in the passive case, and hence there is no need for an extra external force f_{ext} neutralizing the pre-acceleration as in Section 8B. However, this introduces a new problem. We saw that there is a discontinuity in the energy of the electromagnetic field of the charge when it enters H which is accounted for by the work performed by the force f_{ext}. This sudden increase in field energy cannot be accounted for if the equation of motion of the point charge is the LADY-equation. The explanation of the sudden energy increase is now to be found in the point particle limit. In the case of a charge distribution with finite extent there is a period from $T = 0$ to $T = 2\epsilon$ during which the Abraham vector in the LADY-equation is multiplied by a function $\eta(\tau)$ which increases from $\eta(0) = 0$ to $\eta(2\epsilon) = 1$. During this period the acceleration reaction force performs work which accounts for the increase of the field energy. In the limit of a point particle the time interval becomes infinitely short, acceleration reaction approaches a delta function, and the increase of field energy gets a step function behaviour.

12. CONCLUSION

The equation of motion of a radiating charge has some well known problems, i.e., the existence of runaway solutions and pre-acceleration. Also a full understanding of the so-called acceleration energy or Schott energy has been lacking. In the present work we have made an effort to clarify the significance of this energy in connection with energy-momentum conservation for a radiating charge and the field it produces.

The Lorentz–Abraham–Dirac equation (3.1) contains an additional term, called the Abraham four-vector, which is not present in the ordinary form of Newton's 2.law. We have followed Rohrlich and called the spatial component of this vector the field reaction force. This force is given in Eq. (3.11c). It is the sum of an acceleration reaction given in Eq. (3.a) and a radiation reaction given in Eq. (3.11b). This sum is equal to the rest frame acceleration reaction.

We have reviewed earlier works of the physics contained in the LAD-equation with special emphasis on a discussion presented by Rohrlich some years ago, where several points are made in a clear and illuminating way. We began our analysis of the Schott energy from a field point of view in Section 5, where Teitelboim's separation of the electromagnetic field into a velocity field and an acceleration field was introduced. Then we utilized Rowe's modification of this separation to argue that the Schott energy is localized close to the charge.

Also we have shown how the field reaction force can be found by two different considerations: Either by calculating the electromagnetic forces inside the charged particle, or by calculating the effect of its own external field upon itself.

A thorough understanding of the role of the Schott energy in the case of a radiating charge moving hyperbolically can be obtained by considering a charge entering and leaving a region of hyperbolic motion. We have analyzed two versions of such a situation, one in which the charge is not acted upon by any external force except the one in H. Then there is pre-acceleration. Second, we have considered a case with constant velocity outside H, which requires that the charge is acted upon by an extra force neutralizing the pre-acceleration. In both cases we have provided a detailed description of the evolution of energy-momentum of the electromagnetic field produced by the charge, and the role of the Schott energy has been clarified. Also continuity properties of the motion as the charge enters and leaves the region H have been discussed.

APPENDIX A

The Electron as an Extended Particle

We shall find the electromagnetic field of an accelerated spherical charged shell according to the approximation which is linear in the velocity and its derivatives. Also we shall calculate the electromagnetic force acting between two concentric charged shells, and letting the radius of the shells approach each other, the force of an accelerated charged shell upon itself is found.

A.1. *The Electromagnetic Field of an Accelerated Point Charge in the Linear Approximation*

The components of the retarded electromagnetic field of a point charge q_1 moving along the X-axis in the laboratory system are

$$E^X_{ret} = q_1\{(1+\dot{R})^{-3}R^{-3}[(s-Rv)(1-v^2)-\rho^2 a]\}_{ret} \tag{A.1a}$$

$$E^\rho_{ret} = q_1\{(1+\dot{R})^{-3}R^{-3}(1-v^2+sa)\rho\}_{ret} \tag{A.1b}$$

$$B^\varphi_{ret} = q_1\{(1+\dot{R})^{-3}R^{-3}[(1-v^2)v+Ra]\rho\}_{ret}. \tag{A.1c}$$

Here $R = |\mathbf{R}|$, where $\mathbf{R}$ is the vector from the point charge, $(X_1, \rho = 0)$, to the field point, (X_2, ρ_2), and the overdot denotes differentiation with respect to the laboratory time, $= d/dT$. Furthermore

$$s = X_2 - X_1, \qquad v = \dot{X}_1 = -\dot{s}, \qquad a = \dot{v} = -\ddot{s}, \qquad \dot{R} = -sv/R, \qquad \rho_{ret} = \rho. \tag{A.2}$$

In the linear approximation we neglect v^2 and approximate $(1 + \dot{R})^{-3}$ by

$$(1 + \dot{R})^{-3} \approx 1 - 3\dot{R} = 1 + 3sv/R. \tag{A.3}$$

In this approximation E^X_{ret} reduces to

$$E^X_{ret} \approx q_1\{3s^2 v R^{-4} + (s - a\rho^2)R^{-3} - vR^{-2}\}_{ret}. \tag{A.4}$$

For the calculation of retarded values we shall utilize a remarkable formula deduced by Page [64]. Let f be a quantity depending upon the position, velocity, and acceleration of the particle. Hence, f is a function of the time. The retarded value f_{ret} can be expressed by the instantaneous value f and the series of higher order derivatives by the formula

$$f_{ret} = f - R\dot{f} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} D^{n-1}(R^n \dot{f}), \tag{A.5}$$

where $D = d/dT$. R is the distance between the field point and the instantaneous position of the particle. We shall apply this formula to our linearized expressions. From Eq. (A.4), putting $f = E^X/q_1$ and neglecting non-linear terms,

$$\dot{f} = (3s^2 - R^2)vR^{-5} + (3s^2 - R^2)aR^{-4} - \rho^2 \dot{a} R^{-3}. \tag{A.6}$$

Here all the quantities have instantaneous values. When we insert this expression into Eq. (A.5), the operator D^{n-1} must act upon v, a, and $\dot{a}$ only in order that non-linear terms in these quantities shall not appear. Introducing

$$D^{n-1}v = -D^n s, \qquad D^{n-1}a = D^n v, \qquad D^{n-1}\dot{a} = D^n a \tag{A.7}$$

and utilizing a Taylor expansion, we obtain

$$(1/q_1)E^X_{ret} = f_{ret} = \frac{s}{R^3} + \frac{R^2 - 3s^2}{R^5}[s(T - R) - s] - \frac{R^2 - 3s^2}{R^4} v(T - R) - \frac{\rho^2}{R^3} a(T - R), \tag{A.8a}$$

where R and $s = R_X$ have instantaneous values. Similarly we find

$$(1/q_1)E^\rho_{ret} = \frac{\rho}{R^3} - \frac{3\rho s}{R^5}[s(T - R) - s] + \frac{3\rho s}{R^4} v(T - R) + \frac{\rho s}{R^3} a(T - R) \tag{A.8b}$$

$$(1/q_1)B^\varphi_{ret} = \frac{\rho}{R^3} v(T - R) + \frac{\rho}{R^2} a(T - R). \tag{A.8c}$$

A.2. *The Force between Two Concentric Shells*

We shall calculate the force between two rigid concentric spherical shells in the linear approximation. In the instantaneous inertial rest frame all points of the shell are momentarily at rest, but with different accelerations. A point with X-coordinate ΔX relative to the center of the shells has acceleration $g/(1+g\Delta X)\approx g-g^2\Delta X$, where g is the acceleration of the center. In the linear approximation the latter term is neglected such that all points of the shells have the same acceleration. Furthermore the acceleration a in the laboratory system is equal to the acceleration in the rest system, since (in the linear approximation) $a=g/\gamma^3\approx g$. The spherical shells have radii ϵ_1 and ϵ_2, $\epsilon_1<\epsilon_2$, with charges Q_1 and Q_2 uniformly distributed.

We shall first consider shell 1 as the active part; i.e., this shell produces a field acting upon shell 2 by a force called F_{12}. Let $\mathbf{R}$ be the vector from points on the shell 1 to points on the shell 2. It is a consequence of the symmetry that the net force is due to the component E^X, only. The force is found by integrating over two spherical surfaces. The term s/R^3 in Eq. (A.8a) will not contribute since s changes sign during the integration so that the contributions to the left and to the right of the center cancel each other. We now consider the next two terms, containing the factor R^2-3s^2 (times a function of R). Due to the spherical symmetry s^2 can be replaced by $R^2/3$ in the integrals, showing that these two terms do not contribute to the integral. Finally, in the last term we can replace ρ^2 by $2R^2/3$, so that we get

$$F_{12}=-\frac{2}{3}\frac{Q_1}{4\pi\epsilon_1^2}\frac{Q_2}{4\pi\epsilon_2^2}\int_{\sigma_1}\int_{\sigma_2}\frac{1}{R}a(T-R)\,d\sigma_1\,d\sigma_2, \tag{A.9}$$

where σ_1 and σ_2 signify the interior and exterior spherical surface, respectively. Note that the acceleration is taken at the time $T-R$ such that Eq. (A.9) is a relation between a present force and accelerations in the past.

Since both shells are assumed to have identical motions, the integral (A.9) can be interpreted to give the force, F_{21} from 2 upon 1, as well. Hence,

$$F_{12}=F_{21}. \tag{A.10}$$

This is opposite of what one would expect from Newton's 3.law.

Calculating the integrals, let us start by performing the integral over the internal surface. Then R is the distance from the point of integration to a point P on the external surface. We introduce polar coordinates (θ,φ) with the line from the center to the point P as axis and use $R=(\epsilon_1^2+\epsilon_2^2-2\epsilon_1\epsilon_2\cos\theta)^{1/2}$ as the integration variable with limits of integration $\epsilon_2-\epsilon_1$ and $\epsilon_2+\epsilon_1$. This gives

$$d\sigma=\epsilon_1^2\sin\theta\,d\theta\,d\varphi=(\epsilon_1/\epsilon_2)R\,dR\,d\varphi, \tag{A.11}$$

i.e.,

$$\int_{\sigma_1}\frac{1}{R}a(T-R)\,d\sigma_1=2\pi\frac{\epsilon_1}{\epsilon_2}\int_{\epsilon_2-\epsilon_1}^{\epsilon_2+\epsilon_1}a(T-R)\,dR=-2\pi\frac{\epsilon_1}{\epsilon_2}[v(T-\epsilon_2-\epsilon_1)-v(T-\epsilon_2+\epsilon_1)]. \tag{A.12}$$

Inserting this into Eqs. (A.9) and (A.10) leads to

$$F_{21}=F_{12}=\frac{Q_1Q_2}{3\epsilon_1\epsilon_2}[v(T-\epsilon_2-\epsilon_1)-v(T-\epsilon_2+\epsilon_1)]. \tag{A.13}$$

The result (A.13) contains some interesting special cases.

Letting $Q_2 = Q_1$ and $\epsilon_2 \to \epsilon_1$, we obtain the action of a spherical charged shell upon itself. Then

$$F_{11} = \frac{1}{3}\left(\frac{Q_1}{\epsilon_1}\right)^2 [v(T - 2\epsilon_1) - v(T)] \tag{A.14}$$

which is a well-known result [63]. Making a series expansion in powers of ϵ_1,

$$F_{11} = \frac{2}{3}Q_1^2\left[-\frac{a}{\epsilon_1} + \dot{a} - \frac{2}{3}\epsilon_1\ddot{a} + \cdots\right] \tag{A.15}$$

we recover the force of a point charge upon itself in the limit $\epsilon_1 \to 0$.

Another special case is obtained by letting $\epsilon_1 \to 0$ in Eq. (A.13). This gives the force between a charged spherical shell and a point charge at its center,

$$F_{12} = F_{21} = -\frac{2}{3}\frac{Q_1 Q_2}{\epsilon_2}a(T - \epsilon_2). \tag{A.16}$$

It follows that the electrical field strength at the center of the shell is

$$E^X = -\frac{2}{3}\frac{Q_2}{\epsilon_2}a(T - \epsilon_2) = Q_2\left[-\frac{2}{3}\frac{a}{\epsilon_2} + \frac{2}{3}\dot{a} - \frac{\epsilon_2}{3}\ddot{a} + \cdots\right]. \tag{A.17}$$

A.3. *The Field inside an Accelerated Charged Spherical Shell*

The shell has radius ϵ, and a uniformly distributed charge Q. It has an acceleration a and is instantaneouly at rest in the laboratory frame at the moment of observation. The expressions (A.8) shall be applied to find the elctromagnetic field inside the shell in the linear approximation.

Performing a series expansion in R we get

$$s(T - R) = s - \frac{1}{2}R^2 a + \frac{1}{6}R^3\dot{a} - \frac{1}{24}R^4\ddot{a} + \cdots \tag{A.18a}$$

$$v(T - R) = -Ra + \frac{1}{2}R^2\dot{a} - \frac{1}{6}R^3\ddot{a} + \cdots \tag{A.18b}$$

$$a(T - R) = a - R\dot{a} + \frac{1}{2}R^2\ddot{a} + \cdots, \tag{A.18c}$$

where $s, a, \dot{a}, \ddot{a}$ refer to instantaneous time T. We find the field at a point with position vector $\mathbf{r}$ relative to the center of the spherical shell. The vector $\mathbf{r}$ makes an angle θ with the positive X-axis the direction of the acceleration. Integration over the spherical shell gives the following components of the electromagnetic field for $r < \epsilon$ up to the second order derivative $\ddot{a}$

$$\frac{E^X}{Q} = -\frac{2}{3}\frac{a}{\epsilon} + \frac{2}{3}\dot{a} - \frac{\ddot{a}}{15\epsilon}(5\epsilon^2 + r^2 + r^2\sin^2\theta) \tag{A.19a}$$

$$\frac{E^\rho}{Q} = \frac{\ddot{a}}{30\epsilon}r^2\sin 2\theta \tag{A.19b}$$

$$\frac{B^\varphi}{Q} = -\frac{\dot{a}}{3\epsilon}r\sin\theta + \frac{\ddot{a}}{3}r\sin\theta. \tag{A.19c}$$

Putting $r = 0$ the result (A.17) for the field at the center of the spherical shell is confirmed.

A.4. *Self Force upon a Spherically Symmetric Spatial Charge Distribution*

The expression (A.9) for the force between charged spherical shells is easily generalized to the self force of a spherically symmetric charge distribution $\rho(r)$ in space.

Let $dq_1 = \rho(r_1)\, dV_1$ and $dq_2 = \rho(r_2)\, dV_2$ be infinitesimal elements of charge in the distribution. According to Eq. (A.9) the self force is given by

$$F = -\frac{2}{3} \int_{V_1} \int_{V_2} \frac{dq_1\, dq_2}{R} a(T - R), \tag{A.20}$$

where $R = |\mathbf{r}_2 - \mathbf{r}_1|$. Performing a series expansion we get

$$F = -\frac{2}{3} \int_{V_1} \int_{V_2} \frac{dq_1\, dq_2}{R} [a(T) - R\dot{a}(T) + \cdots]. \tag{A.21}$$

Here

$$\int_{V_1} \int_{V_2} \frac{dq_1\, dq_2}{R} = 2U \tag{A.22}$$

and

$$\int_{V_1} \int_{V_2} dq_1\, dq_2 = Q^2, \tag{A.23}$$

where U is the electrostatic self energy, and Q is the total charge of the distribution. Hence the force of the charge distribution upon itself is

$$F = -\frac{4}{3} U a + \frac{2}{3} Q^2 \dot{a} + \cdots \tag{A.24}$$

in accordance with Eq. (A.15) for a spherical shell.

A.5. *Equation of Motion of a Charged Shell*

We consider a spherical shell with radius ϵ and a uniformly distributed surface charge Q. From the deductions above we have two expressions for the self force, namely Eq. (A.20) where we let dq_1 and dq_1 denote infinitesimal surface charges,

$$F(T) = -\frac{2}{3} \int_{\sigma_1} \int_{\sigma_2} \frac{a(T - R)}{R} dq_1\, dq_2 \tag{A.25}$$

and Eq. (A.14)

$$F(T) = \frac{1}{3} \left(\frac{Q}{\epsilon} \right)^2 [v(T - 2\epsilon) - v(T)] \qquad (v^2 \ll 1). \tag{A.26}$$

Let m_0 be the mechanical (bare) mass, and f_{ext} an external force. Then we have the equation of motion

$$m_0 a(T) = f_{ext}(T) + \frac{1}{3}\left(\frac{Q}{\epsilon}\right)^2 [v(T - 2\epsilon) - v(T)]. \tag{A.27}$$

The dominating term in the self force is proportional to the acceleration (see Eq. (A.15)) and is given by

$$\frac{1}{3}\left(\frac{Q}{\epsilon}\right)^2 [-2\epsilon a(T)] = -\delta m a(T), \tag{A.28}$$

where $\delta m = (4/3)Q^2/2\epsilon$ is the electrodynamic contribution to the physical mass of the particle. We add $\delta m a(T)$ on both sides of Eq. (A.28) and obtain a mass renormalization due to the term δm. Then Eq. (A.27) becomes

$$m a(T) = f_{ext}(T) + \frac{1}{3}\left(\frac{Q}{\epsilon}\right)^2 [v(T - 2\epsilon) - v(T) + 2\epsilon a(T)], \tag{A.29}$$

where $m = m_0 + \delta m$ is the physical (observed) mass. Introducing $\tau_0 = (2/3)(Q^2/m)$ leads to $m = m_0 + (\tau_0/\epsilon)m$ or $m_0 = (1 - \tau_0/\epsilon)m$. Equation (A.29) can also be written

$$(1 - \tau_0/\epsilon)m a(T) = f_{ext}(T) + (\tau_0/2\epsilon^2)m[v(T - 2\epsilon) - v(T)]. \tag{A.30}$$

Note that the acceleration is multiplied by the mechanical mass m_0, not the physical mass m.

Making a series expansion for small ϵ we get

$$m a(T) = f_{ext}(T) + m\tau_0 \dot{a}(T) \tag{A.31}$$

which is the LAD-equation in the rest frame of the charge.

E. J. Moniz and D. H. Sharp [14] have investigated solutions of Eq. (A.30) for the cases of negative and positive mechanical mass. They found that there is no pre-acceleration or runaway solution for $m_0 > 0$. Furthermore they claim that the classical theory is valid for $m_0 > 0$ only. Hence there are no pathological solutions of the equation of motion of a charged shell within the region of applicability of the classical theory.

Taking the limit $\epsilon \to 0$ the mechanical mass, $m_0 = (1 - \tau/\epsilon)m$, becomes negative, which seems to imply that the LAD-equation, which is presumably valid for a point charge, is not valid at all according to Moniz and Sharp. However, the introduction of the mass $(1 - \tau_0/\epsilon)m$ at the left hand side of the equation may be somewhat misleading. Making a series expansion $v(T - 2\epsilon) = v(T) - 2\epsilon a(T) + 2\epsilon^2 \dot{a}(T) - \cdots$ the term $(\tau_0/\epsilon)ma$ is cancelled. This means that Eq. (A.29) and not Eq. (A.30) is the physically significant way of writing the equation of motion of the charged shell. In this equation only the physical mass of the charged shell appears, and there is no ambiguity of sign when the limit $\epsilon \to 0$ is taken.

APPENDIX B

The Self Force on a Charged Particle

Consider a charged particle moving along the X-axis in the laboratory frame, Σ. The self force on the charge shall here be found by summarizing the internal electromagnetic forces acting upon each element of the charge.

In the instantaneous inertial rest frame, Σ', of the particle it has a spherically symmetric distribution of charge. Let P_1 and P_2 be two arbitrary points in the charge distribution. Their distances from a plane orthogonal to the X-axis through the center of the particle are ξ_1 and ξ_2, respectively. The separation vector from P_1 to P_2 is $\mathbf{R}$. We define

$$R_{\parallel} = \xi_2 - \xi_1, \qquad R_{\perp} = \left[R^2 - R_{\parallel}^2\right]^{1/2}, \tag{B.1a,b}$$

where $R_{\parallel}$ and $R_{\perp}$ are the components of $\mathbf{R}$ parallel to and perpendicular to the X-direction, respectively.

All the points of the charge are at rest in Σ' at the point of time $t' = 0$. We shall first calculate the field strength dE^X at P_2 at $t' = 0$ due to the charge dq_1 at P_1. The state of motion of the element at P_1 at the time $t' = 0$ is given by

$$v_1' = 0, \qquad \left(\frac{dv_1'}{dt'}\right)_{t'=0} = a_1', \qquad \left(\frac{da_1'}{dt'}\right)_{t'=0} = b_1'. \tag{B.2}$$

The retarded values at P_1 are found in terms of a series expansion in R, where we have included the number of terms necessary to obtain the field strength dE^X correct to zero order in R. The Coulomb part (I) and the acceleration part (II) are calculated separately, using Eq. (2.1a), with the result

$$dE_I^X = \left\{R_{\parallel} + \frac{1}{2}\left(R^2 - 3R_{\parallel}^2\right)\left(a_1' - \frac{2}{3}Rb_1'\right) + \left(15R_{\parallel}^2 - 9R^2\right)R_{\parallel}a_1'^2/8\right\}R^{-3}\,dq_1 \tag{B.3a}$$

$$dE_{II}^X = R_{\perp}^2\left\{-a_1' + \frac{3}{2}R_{\parallel}a_1'^2 + Rb_1'\right\}R^{-3}\,dq_1. \tag{B.3b}$$

We shall first find the self force in Σ' for the case that the acceleration and its derivative are independent of the position in the charge. Hence, for arbitrary choice of P_1, $a_1' = g$ and $b_1' = dg/d\tau$, where g is the proper acceleration of the center. This leads to the following double integrals for the self force $F = F_I + F_{II}$:

$$F_I = \int\!\!\int dE_I^X\,dq_2 = 0 \tag{B.4a}$$

$$F_{II} = \int\!\!\int dE_{II}^X\,dq_2 = -\frac{4}{3}V_0 g + \frac{2}{3}Q^2\frac{dg}{d\tau}, \qquad V_0 = \frac{1}{2}\int\!\!\int \frac{dq_1 dq_2}{R}. \tag{B.4b}$$

Here V_0 is the electrostatic self energy, and Q the charge of the particle. Calculating the integrals we have utilized that the terms with $R_{\parallel}$ as a factor do not contribute to the integrals due to the spherical symmetry, and that $R_{\parallel}^2$ can be replaced by $R^2/3$ and $R_{\perp}^2$ by $2R^2/3$.

Next we consider the particle as a Born rigid body. Then the acceleration in the instantaneous rest frame of the particle depends upon the position in the charge distribution. At the point P_1,

$$a_1' = g/(1 + \xi_1 g) = g - \xi_1 g^2 + \cdots \tag{B.5a}$$

$$b_1' = \dot{g} - 2\xi_1 g\dot{g} + \cdots, \tag{B.5b}$$

where g is the proper acceleration of the center of the particle, and $\dot{g} = dg/d\tau$.

Inserting Eqs. (B.5) into Eqs. (B.3) we get to the necessary order the following field in the rest frame at the point P_2

$$dE_I^X = \left\{R_\parallel + \frac{1}{2}\left(R^2 - 3R_\parallel^2\right)\left(g - \xi_1 g^2 - \frac{2}{3}R\dot{g}\right) + \left(15R_\parallel^2 - 9R^2\right)\frac{R_\parallel g^2}{8}\right\}\frac{dq_1}{R^3} \tag{B.6a}$$

$$dE_{II}^X = R_\perp^2\left\{-g + \xi_1 g^2 + \frac{3}{2}R_\parallel g^2 + R\dot{g}\right\}\frac{dq_1}{R^3}. \tag{B.6b}$$

Putting these expressions into the integrals $\int dE_I^X\, dq_2$ and $\int dE_{II}^X\, dq_2$ we get the previous result Eq. (B.4). Thus the variation of the acceleration due to Born rigidity has no effect upon the self force in the rest frame of a spherically symmetric particle in the limit of vanishing extension.

We shall now calculate the self force in the laboratory frame Σ assuming that the charge performs a Born rigid motion. The electromagnetic self force is now found summarizing by simultaneity in Σ. We utilize that the field component E^X is the only one of significance for the self force, and that this component is invariant against a Lorentz transformation in the X-direction. Thus we may utilize Eqs. (B.6) that are valid in the rest frame, with all quantities referring to simultaneity in this frame, g being the proper acceleration of the center of the particle.

The electrical field shall be calculated at a point P_2 at time T in the laboratory frame Σ. Let Σ' be the instantaneous rest frame in P_2 at the point of time T. At simultaneity in the rest frame Σ', the time in the laboratory frame Σ at the center of the charge is T_0, and the velocity of the center is $v(T_0)$. The relationship between T and T_0 is found by making a Lorentz transformation between Σ and Σ', from which follows

$$T - T_0 = \gamma(T_0)v(T_0)\xi_2. \tag{B.7}$$

To first order in ξ_2,

$$T_0 = T - \gamma(T)v(T)\xi_2. \tag{B.8}$$

Hence the proper acceleration of the center at the point of time T_0 is to first order in ξ_2,

$$g(T_0) = g(T) - \frac{dg(T)}{dT}\gamma(T)v(T)\xi_2. \tag{B.9}$$

We now utilize Eq. (B.6), replacing g by

$$g(T_0) = g - \dot{g}v\xi_2, \tag{B.10}$$

where g, $\dot{g}$, v refer to the laboratory time T, and $\dot{g} = dg/d\tau$. Then, to the necessary order, the expressions for dE_I^X and dE_{II}^X must be supplied with the respective terms

$$-\frac{1}{2}\left(R^2 - 3R_\parallel^2\right)\dot{g}v\xi_2\frac{dq_1}{R^3} \tag{B.11a}$$

and

$$R_\perp^2\dot{g}v\xi_2\frac{dq_1}{R^3}. \tag{B.11b}$$

Due to the spherical symmetry these terms will not contribute to the self force, so we get the previous result, Eq. (B.4).

We have studied linear Born rigid motion of a charge distribution which is spherically symmetric in its rest frame, and which has vanishing extension. The result of the calculation is that the self force is the same in the laboratory frame as in the instantaneous rest frame of the particle.

Note also that from the present point of view the self force is a pure type II phenomenon. The Coulomb/velocity component does not contribute to the self force. The self force as calculated by summarizing the internal forces on the charge elements of the distribution is in accordance with the self force, Eq. (7.9), as calculated by considering the rate of change of the momentum of the electromagnetic field produced by the charge. But the interpretation of the self force is different in these two procedures. This is seen most clearly in the instantaneous rest frame of the charge. Here the self force is due to the bound momentum according to the field consideration and due to the radiation field according to the internal force consideration.

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