PROCEEDINGS OF

THE ROYAL SOCIETY.

Section A.—Mathematical and Physical Sciences.

A Note on the Theory of Directive Antennæ or Unsymmetrical Hertzian Oscillators.

By J. A. Fleming, M.A., D.Sc., F.R.S.

(Received March 15,—Read March 22,—Received in revised form May 24, 1906.)

The employment in wireless telegraphy of antennæ or radiating wires which have part of their length vertical and part horizontal, and possess the power of radiating unequally in various horizontal directions, draws attention to the particular qualities of bent electric oscillators when used as radiators or transmitters.* The object of the following note is to show that the properties of such radiators can be deduced from accepted principles, and that experimental results so far obtained are in general accordance with theory.

The nature of the field of electric and magnetic force round a short straight electric oscillator or doublet has been expressed analytically and delineated graphically in well-known memoirs by Hertz,† Pearson and Lee,‡ and others, and that around a not very short rod or linear oscillator has been

* See G. Marconi, LL.D., D.Sc., "On Methods whereby the Radiation of Electric Waves may be mainly confined to certain Directions, and whereby the Receptivity of a Receiver may be restricted to Electric Waves emanating from certain Directions," 'Roy. Soc. Proc.,' vol. 77, p. 413, 1906.

† H. Hertz, 'Wied. Ann.,' vol. 36, p. 1, 1889, or 'Electric Waves,' English translation, p. 137.

‡ Pearson and Lee, 'Phil. Trans.,' A, vol. 193, p. 159, 1900.

investigated by M. Abraham* and depicted by F. Hack.† In these cases we have perfect symmetry of radiation round the axis of the oscillator.

This equality is, however, destroyed by bending the oscillator, and it then radiates unequally in different directions taken in the equatorial or symmetrical plane of the oscillator through its centre, being somewhat greater or stronger on the convex side of the oscillator.

The analytical treatment of the subject presents, however, enormous difficulties unless we limit consideration to the case in which the current in the oscillator is assumed to have the same value at all points at the same time, and also that the dimensions of the oscillator are small compared with the distance from it of the points at which the field is considered.

One form of bent oscillator of the above kind may be considered to be made up by the superposition of three Hertzian electric doublets placed at right angles to each other, the poles being so arranged that at the two corners poles of opposite sign are superimposed, the oscillations in all being synchronous and similarly directed (see fig. 1). Hence, to obtain the field of the bent oscillator, we need merely to calculate those of the components and add them together.

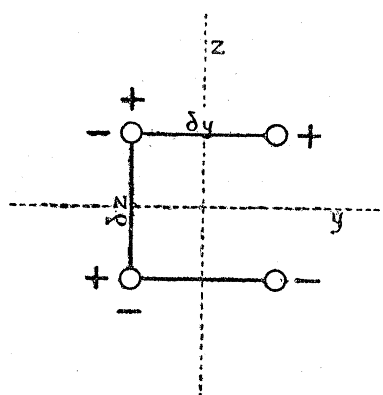

FIG. 1.

Let a single electric oscillator or doublet be placed with centre at the origin and axis coinciding with the z axis. Let oscillations exist in it of period $2\pi/n$, and radiation be emitted of wave-length $2\pi/m$. Suppose the length of the oscillator to be denoted by δz, the electric charge at either pole at any instant by q, the uniform current in the axis by i, whilst Q and I are the maximum values of q and i which vary so that $q = \mathrm{Q} \sin nt$ and $i = \mathrm{I} \cos nt$. Also let $\phi = \mathrm{Q}\delta z$ be the maximum electric moment of the

* M. Abraham, 'Ann. der Physik.,' vol. 66, p. 435, 1898.

† F. Hack, 'Ann. der Physik.,' vol. 14, p. 539, 1904.

oscillator. We have, therefore, $I = Qn$ or $\phi n = I\delta z$, and $n/m = v$, the velocity of propagation of the radiation through space. The scalar potential V at any point P whose distance from the origin is r is given by

$$V = -\frac{\phi}{k}\frac{d}{dz}\left(\frac{\sin(mr-nt)}{r}\right), \tag{1}$$

where k is the dielectric constant of the medium, and $r^2 = x^2+y^2+z^2$.

Also, if F, G, and H are the components of vector potential at P, we have in this case $F = G = 0$, and

$$H = -I\delta z\frac{\cos(mr-nt)}{r}. \tag{2}$$

If we employ the symbol Π to stand for $\sin(mr-nt)/r$, we can write the above expressions (1) and (2) in the form

$$V = -\frac{\phi}{k}\frac{d\Pi}{dz}, \qquad H = \phi\frac{d\Pi}{dt}. \tag{3}$$

If we suppose this doublet to be moved parallel to itself in the negative direction so that its centre is displaced by a distance $-\frac{1}{2}\delta y$, the scalar and vector potentials at P become—

$$V = \left(-\frac{\phi}{k}\frac{d\Pi}{dz}\right)+\frac{1}{2}\frac{d}{dy}\left(\frac{\phi}{k}\frac{d\Pi}{dz}\right)\delta y, \tag{4}$$

$$H = \phi\frac{d\Pi}{dt}-\frac{1}{2}\frac{d}{dy}\left(\phi\frac{d\Pi}{dt}\right)\delta y. \tag{5}$$

Consider, then, two other similar doublets of length δy, and maximum moment ϕ', placed with poles pointing in opposite directions and axes parallel to the axis of y, the doublets having centres at distances $+\frac{1}{2}\delta z$ and $-\frac{1}{2}\delta z$ from the origin and poles arranged as in fig. 1. The scalar and vector potentials at the point P of these last two doublets constituting together a double-doublet are obviously given by—

$$V = -\frac{\phi'}{k}\frac{d^2\Pi}{dz\,dy}\delta z, \tag{6}$$

$$G = \phi'\frac{d^2\Pi}{dz\,dt}\delta z. \tag{7}$$

Hence, if three such short, straight oscillators, having equal currents and charges, are placed round the origin so as to create a doubly-bent oscillator, the scalar and vector potentials of this oscillator at a point P (see fig. 2), the distance of which from the origin is large compared with the linear dimensions of the oscillator, are given by—

$$V = -\frac{\phi}{k}\frac{d\Pi}{dz}+\frac{1}{2}\frac{\phi}{k}\frac{d^2\Pi}{dy\,dz}\delta y-\frac{\phi'}{k}\frac{d^2\Pi}{dz\,dy}\delta z. \tag{8}$$

$$\left.\begin{aligned} F &= 0, \\ G' &= \phi' \frac{d^2\Pi}{dz\,dt}\delta z, \\ H &= \phi\frac{d\Pi}{dt} - \tfrac{1}{2}\phi\frac{d^2\Pi}{dy\,dt}\delta y, \end{aligned}\right\} \tag{9}$$

where $\phi'\delta z = \phi\delta y$.

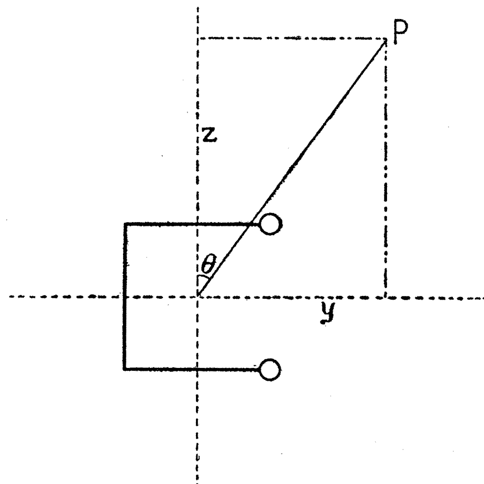

Fig. 2.

The electric and magnetic forces at the point P, of which the axial components are P, Q, R, and a, b, c, can be obtained from equations (8) and (9) at once by the aid of the relations—

$$\left.\begin{aligned} P &= -\frac{dF}{dt} - \frac{dV}{dx}, & a &= \frac{dH}{dy} - \frac{dG}{dz}, \\ Q &= -\frac{dG}{dt} - \frac{dV}{dy}, & b &= \frac{dF}{dz} - \frac{dH}{dx}, \\ R &= -\frac{dH}{dt} - \frac{dV}{dz}, & c &= \frac{dG}{dx} - \frac{dF}{dy}, \end{aligned}\right\} \tag{10}$$

It is obvious that if the electric circuit is completed by placing a pair of double-doublets of moments ϕ and ϕ' at right angles to each other with poles directed in like sense all round the origin, the free charges cancel each other pair and pair, and we are left with a closed electric circuit of area $\delta y \,.\, \delta z$, traversed by a maximum current I. This quadruple doublet creates potentials such that

$$\left.\begin{aligned} V &= 0, & F &= 0, \\ G &= \phi'\frac{d^2\Pi}{dz\,dt}\delta z, & H &= \phi\frac{d^2\Pi}{dy\,dt}\delta y, \end{aligned}\right\} \tag{11}$$

where $\phi'\delta z = \phi\delta y$.

Such a completely closed circuit or *magnetic doublet* creates, therefore, periodic electric and magnetic forces in its field when current oscillations are set up in the circuit.*

For the present purposes we require only the electric and magnetic forces perpendicular to the radius vector r, taken at its extremity, when that radius is taken in the plane xy, which is normal to the plane yz in which the oscillator is situated. Hence we need only calculate the value of R, a, and b for the case in question.

If we write M for $\mathrm{I}\,\delta y\,\delta z$ and call this the magnetic moment of the bent oscillator, so that $\phi\delta y = \phi'\delta z = \mathrm{M}/n$, we have the following equations for the potentials and forces in the field at points not very near the oscillator—

$$\begin{aligned}
\mathrm{V} &= -\frac{\phi}{k}\frac{d\Pi}{dz}+\frac{\phi}{2k}\frac{d^2\Pi}{dy\,dz}\delta y-\frac{\phi'}{k}\frac{d^2\Pi}{dz\,dy}\delta z = -\frac{\phi}{k}\frac{d\Pi}{dz}-\frac{\mathrm{M}}{2kn}\frac{d\Pi}{dy\,dz},\\
\mathrm{G} &= \phi'\frac{d^2\Pi}{dz\,dt}\delta z = \frac{\mathrm{M}}{n}\frac{d^2\Pi}{dy\,dt},\\
\mathrm{H} &= \phi\frac{d\Pi}{dt}-\frac{\phi}{2}\frac{d^2\Pi}{dy\,dt^2}\delta y = \phi\frac{d\Pi}{dt}-\frac{\mathrm{M}}{2n}\frac{d^2\Pi}{dy\,dt},\\
\mathrm{R} &= -\phi\frac{d^2\Pi}{dt^2}+\frac{\mathrm{M}}{2n}\frac{d^3\Pi}{dy\,dt^2}+\frac{\phi}{k}\frac{d^2\Pi}{dz^2}+\frac{\mathrm{M}}{2kn}\frac{d^3\Pi}{dy\,dz^2},\\
a &= \phi\frac{d^2\Pi}{dy\,dt}-\frac{\mathrm{M}}{2n}\frac{d^3\Pi}{dy^2\,dt}-\frac{\mathrm{M}}{n}\frac{d^3\Pi}{dz^2\,dt},\\
b &= -\phi\frac{d^2\Pi}{dx\,dt}+\frac{\mathrm{M}}{2n}\frac{d^3\Pi}{dx\,dy\,dt}.
\end{aligned}\qquad (12)$$

Performing the necessary differentiations on the function $\Pi = \sin(mr-nt)/r$ and collecting terms in $\sin(mr-nt)$ and $\cos(mr-nt)$, which for shortness will be written $\sin \mathrm{X}$ and $\cos \mathrm{X}$, also putting v for n/m or $(\mu k)^{-\frac{1}{2}}$, where μ and k are the permeability and dielectric constant of the medium, we have the following expressions for R, a, and b.

* In the discussion on Mr. Marconi's paper (*loc. cit.*), read at the Royal Society on March 22, 1906, Professor J. Larmor, Sec. R.S., pointed out that the bent antenna employed by Mr. Marconi was equivalent in effect at distant points to the combination of a magnetic doublet or bi-pole magnetic oscillator of the kind investigated, in advance of Hertz's discovery of electric radiation, by Professor G. F. Fitzgerald (see 'The Scientific Writings of the late Professor G. F. Fitzgerald,' edited by Professor J. Larmor, p. 128), and a straight Hertzian doublet or bi-pole electric oscillator. It is obvious that if a straight electric oscillator is placed in contiguity to one side of a closed rectangular circuit or magnetic oscillator, the currents being of the same value and directed in an opposite sense in the open and adjacent side of the closed circuit, the resultant electromagnetic effect must be that of a doubly bent oscillator of the type considered in the text. The equations (8), (9), (4), (5), and (11) are consistent with this mode of viewing the facts.

$$R=\left\{\phi\,(m^2r^2-1)-\phi\,(m^2r^2-3)\left(\frac{z}{r}\right)^2+\frac{M}{2v}\frac{3}{mr}\left(\frac{y}{r}\right)+\frac{M}{2v}\frac{(8m^2r^2-15)}{mr}\left(\frac{y}{r}\right)\left(\frac{z}{r}\right)^2\right\}\frac{\sin X}{kr^3}$$
$$+\left\{\phi\,mr-\phi\,3mr\left(\frac{z}{r}\right)^2-\frac{M}{2v}(m^2r^2+3)\left(\frac{y}{r}\right)+\frac{M}{2v}(m^2r^2+18)\left(\frac{y}{r}\right)\left(\frac{z}{r}\right)^2\right\}\frac{\cos X}{kr^3}. \quad (13)$$

$$a=\left\{\phi vm^2r^2\left(\frac{y}{r}\right)+\frac{M}{2}3mr\left(\frac{y}{r}\right)^2-\frac{M}{2}3mr+3Mmr\left(\frac{z}{r}\right)^2\right\}\frac{\sin X}{r^3}$$
$$+\left\{\phi vmr\left(\frac{y}{r}\right)-\frac{M}{2}(m^2r^2-3)\left(\frac{y}{r}\right)^2-\frac{3M}{2}-M\,(m^2r^2-3)\left(\frac{z}{r}\right)^2\right\}\frac{\cos X}{r^3}. \quad (14)$$

$$b=-\left\{\phi vm^2r^2\left(\frac{x}{r}\right)+\frac{M}{2}3mr\left(\frac{x}{r}\right)\left(\frac{y}{r}\right)\right\}\frac{\sin X}{r^3}$$
$$-\left\{\phi vmr\left(\frac{x}{r}\right)-\frac{M}{2}(m^2r^2-3)\left(\frac{x}{r}\right)\left(\frac{y}{r}\right)\right\}\frac{\cos X}{r^3}. \quad (15)$$

Suppose we limit attention to the value of the electric force e and the magnetic force d at right angles to the extremity of the radius vector r, the former being parallel to the z-axis and the latter being drawn in the plane of xy. The magnetic force d in this direction is equal to $ay/r-bx/r$. Hence we obtain its value by multiplication of the values of a and b by y/r and x/r and subtraction. Then putting $z=0$ in the above equations and writing $\cos\theta$ for y/r we have,

$$e=\left\{\phi\,(m^2r^2-1)+\frac{M}{2v}\frac{3}{mr}\cos\theta\right\}\frac{\sin X}{kr^3}+\left\{\phi mr-\frac{M}{2v}(m^2r^2+3)\cos\theta\right\}\frac{\cos X}{kr^3}, \quad (16)$$

$$d=\{\phi vm^2r^2\}\frac{\sin X}{r^3}+\left\{\phi vmr-\frac{M}{2}m^2r^2\cos\theta\right\}\frac{\cos X}{r^3}. \quad (17)$$

If we denote the amplitudes of e and d by E and H, we have finally

$$E=\frac{1}{kr^3}\left[\left\{\phi\,(m^2r^2-1)+\frac{3}{2}\frac{M}{v}\frac{\cos\theta}{mr}\right\}^2+\left\{\phi mr-\frac{M}{2v}(m^2r^2+3)\cos\theta\right\}^2\right]^{\frac{1}{2}}, \quad (18)$$

$$H=\frac{1}{r^3}\left[(\phi vm^2r^2)^2+\left(\phi vmr-\frac{M}{2}m^2r^2\cos\theta\right)^2\right]^{\frac{1}{2}}, \quad (19)$$

where E is the amplitude of the electric force perpendicular to the radius vector and to the equatorial plane, and H is the amplitude of the magnetic force perpendicular to the radius vector and in the equatorial plane.

Hence, since m^2r^2 is always much greater than $1/mr$, it is clear that when θ is 180° the values of E and H are both greater than when $\theta=0°$.

If we put $M = 0$ in the above equations they reduce to the values given by Hertz for the electric and magnetic forces of the short straight oscillator or doublet taken in the equatorial plane, the electric force being parallel to the axis and magnetic force at right angles. When mr is large compared with unity we have $kE^2 = \mu H^2$, showing that the energies of the magnetic and electric components of the wave then become equal. Also there is a minimum value of H and E corresponding to a value of θ, such that

$$\cos\theta = \frac{2\phi v}{M}\frac{1}{mr}, \qquad \text{or} \qquad \cos\theta = \frac{2\phi v}{M}\frac{mr}{m^2r^2+3}. \tag{20}$$

The above expressions are numerically small when r is large compared with the wave-length of the radiation. Hence a minimum value of the forces at the extremity of the radius vector is found, corresponding to some azimuthal angle θ rather less than 90° reckoned from the direction in which the free ends of the bent oscillator point.

A reference to the records of observations made by Mr. Marconi* on the radiation from a bent antenna shows that the above deductions from theory agree with his observed facts. Hence we conclude that whereas a straight vertical oscillator earthed at the lower end radiates equally in all horizontal directions or azimuths, the result of bending the antenna over to one side, so that a portion of it is horizontal, is to cause it to radiate less vigorously in the direction in which the free end points than in the opposite direction, and to create a minimum radiation in two other directions equally inclined to the direction of maximum radiation.

The degree of this fore-and-aft inequality in the plane of the oscillator will depend upon the ratio of the magnitude of the quantities $\frac{1}{2}Mmr$ and ϕv, or upon the ratio of δy to $2/m^2r$, that is upon the ratio of

$$\frac{\delta y}{\frac{1}{2}\delta z+\delta y} \quad \text{to} \quad \frac{4}{m^2rl}, \quad \textit{i.e.}, \text{ to} \quad \frac{\lambda^2}{\pi^2rl},$$

where $2l$ is the total length of the bent oscillator. The greatest inequality between the fore and aft radiation in the plane of the oscillator will exist when π^2 times the ratio of the sum of the lengths of the two horizontal parts of the oscillator to its total length is as nearly as possible equal to the product of the ratios of λ^2/l^2 and l/r. The ratio λ/l is fixed by the geometrical form of the oscillator, hence the inequality in radiative power in the fore and aft directions for a given oscillator essentially depends upon the ratio of wave-length to the distance of the point at which observations are made, and at large distances will only be sensible when long wave-

* See 'Roy. Soc. Proc.,' A, vol. 77, p. 415, 1906.

lengths are employed. This result also agrees with the observations of Mr. Marconi, who says* :—

"I have observed that, in order that the effects should be well marked, it is necessary that the length of the horizontal conductors should be great in proportion to their height above the ground, and that the wave-lengths employed should be considerable, a condition which makes it difficult to carry out such experiments within the walls of a laboratory.

"I have found the results to be well marked for wave-lengths of 150 metres and over, but have not been able to obtain as well-defined results when employing much shorter waves, the effects following some law which I have not yet had time to investigate."

The above theoretical examination of this operation of a bent oscillator shows clearly that its unsymmetrical radiation in the equatorial plane depends not upon absolute wave-length, but upon the ratio of wave-length to the distance of the receiving point and upon the proportion between the length of the vertical and of the horizontal portions of the oscillator.

The theory is thus supported by the observed facts. If necessary, it would be possible from the equations given above to delineate the lines of electric force in the field of the bent oscillator for various epochs. Thus if R and Q are the components of the electric force in the plane yz, the differential equation to the lines of electric force in that plane is $R dy - Q dz = 0$. The substitution of the values of R and Q and integration of this equation would furnish the equation to the lines of electric force in the yz plane from which they might be delineated, but its complexity does not make the task of actually delineating the lines of force for the bent oscillator an inviting one.

* *Loc. cit.*, p. 420.