Structure effects on the radiation emitted from an electron

G. W. Ford*
Institut Laue-Langevin, 38042 Grenoble CEDEX, France

R. F. O'Connell
Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803-4001
(Received 10 June 1991)

Nonrelativistic radiation reaction theory points out the need for incorporating structure in the theory of the electron (since the assumption of a point electron leads to problems with causality). Thus the question arises as to whether other well-known results need to be modified to incorporate structure effects. Here, we show that the Larmor formula is one such result requiring modification.

PACS number(s): 05.30.−d, 03.50.De, 12.20.Ds

It is now well established that the theory of nonrelativistic quantum and classical electrodynamics demands that the point electron theory must be abandoned (see, for example, Ref. [1] and references therein). Thus the question arises as to whether other well-known results need to be modified to incorporate the inclusion of electron structure. Using a derivation that, under the assumption of a point electron, leads to the Larmor formula [2] here we show that, with the inclusion of structure effects, we obtain a modification of the Larmor result.

Our starting point is Maxwell's equations in Fourier transform language [2]:

$$\mathbf{B}_{\mathbf{k},\omega}=\frac{c}{\omega}\mathbf{k}\times\mathbf{E}_{\mathbf{k},\omega} \qquad (1)$$

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E}_{\mathbf{k},\omega})+\frac{\omega^2}{c^2}\mathbf{E}_{\mathbf{k},\omega}=-i\frac{4\pi\omega}{c^2}\mathbf{j}_{\mathbf{k},\omega}\ , \qquad (2)$$

so that

$$\mathbf{E}_{\mathbf{k},\omega}=\frac{4\pi}{i\omega}(\hat{\mathbf{k}}\cdot\mathbf{j}_{\mathbf{k},\omega})\hat{\mathbf{k}} -i\frac{4\pi\omega}{\omega^2-c^2k^2}[\mathbf{j}_{\mathbf{k},\omega}-(\hat{\mathbf{k}}\cdot\mathbf{j}_{\mathbf{k},\omega})\hat{\mathbf{k}}]\ , \qquad (3)$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, respectively, and $\mathbf{j}$ is the current produced by the external force.

The negative of the total work done on the current by the radiated electric field is equal to the radiated energy W_R. Hence

$$\begin{aligned}W_R&=-\int_{-\infty}^{\infty}dt\int d\mathbf{r}\,\mathbf{j}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t)\\&=-\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}d\omega\int d\mathbf{k}(\mathbf{j}^*_{\mathbf{k},\omega}\cdot\mathbf{E}_{\mathbf{k},\omega})\\&=-\frac{1}{4\pi^3}\int_{-\infty}^{\infty}d\omega\int d\mathbf{k}\,\mathrm{Re}\left[\frac{1}{i\omega}|\hat{\mathbf{k}}\cdot\mathbf{j}_{\mathbf{k},\omega}|^2-\frac{i\omega}{\omega^2-c^2k^2}|\mathbf{j}_{\mathbf{k},\omega}-(\hat{\mathbf{k}}\cdot\mathbf{j}_{\mathbf{k},\omega})\hat{\mathbf{k}}|^2\right]\\&=\frac{1}{8\pi^2}\int_{-\infty}^{\infty}d\omega\int d\mathbf{k}|\mathbf{j}_{\mathbf{k},\omega}-(\hat{\mathbf{k}}\cdot\mathbf{j}_{\mathbf{k},\omega})\hat{\mathbf{k}}|^2[\delta(\omega-ck)+\delta(\omega+ck)]\ .\end{aligned} \qquad (4)$$

Here, to obtain the last line in Eq. (4) we have used first of all the fact that the first term in the third line, while divergent, vanishes on account of the continuity equation. In the second term on the third line we have used the fact that ω is understood to have a small positive imaginary part. If follows from the reality condition obeyed by the current that

$$W_R=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}d\omega\int d\mathbf{k}\,|\mathbf{j}_{\mathbf{k},\omega}-(\hat{\mathbf{k}}\cdot\mathbf{j}_{\mathbf{k},\omega})\hat{\mathbf{k}}|^2\delta(\omega-ck)\ . \qquad (5)$$

At this point, we deviate from the usual derivation [2] by including a form factor and writing

$$\mathbf{j}(\mathbf{r},t)=ef(\mathbf{r}-\mathbf{R})\mathbf{V}\ , \qquad (6)$$

where $f(\mathbf{r}-\mathbf{R})$ is the form factor of the electron, $\mathbf{R}(t)$ is the center of charge of the electron, and $\mathbf{V}=d\mathbf{R}/dt$. Hence

$$\mathbf{j}_{\mathbf{k},\omega}=ef_k\int_{-\infty}^{\infty}dt\,e^{-i[\mathbf{k}\cdot\mathbf{R}(t)-\omega t]}\mathbf{V}(t)\simeq ef_k\mathbf{V}_\omega\ , \qquad (7)$$

where the last equality implies use of the electric dipole approximation (an excellent approximation in the present context). If follows that

$$W_R=\frac{e^2}{6\pi^2}\int_{-\infty}^{\infty}d\omega\int d\mathbf{k}|f_k|^2|\mathbf{V}_\omega|^2\delta(\omega-ck)$$
$$=\frac{2e^2}{3\pi}\int_{-\infty}^{\infty}d\omega\int_0^{\infty}dk\,k^2|f_k|^2|\mathbf{V}_\omega|^2\delta(\omega-ck)$$
$$=\frac{e^2}{3\pi c^3}\int_{-\infty}^{\infty}d\omega\,\omega^2|f_k|^2|\mathbf{V}_\omega|^2\,. \quad (8)$$

We recall [3,4] that

$$|f_k|^2=\frac{1}{1+\omega^2\tau_e^2}\,, \quad (9)$$

where $\tau_e=2e^2/3Mc^3$, M being the mass of the electron.

Without loss of generality, we can confine our attention to one dimension (x) and thus we can write [replacing $\mathbf{V}_\omega$ by $\tilde{v}(\omega)$, and so on, the superposed tilde always denoting the Fourier transform]

$$|\tilde{v}(\omega)|^2=\omega^2|\tilde{x}(\omega)|^2\,. \quad (10)$$

In addition, we have

$$\tilde{x}(\omega)=\alpha(\omega)\tilde{f}(\omega)\,, \quad (11)$$

where $\alpha(\omega)$ is the generalized susceptibility, given by [3,4]

$$\alpha(\omega)=\frac{1-i\omega\tau_e}{-M\omega^2}\,,\quad \mathrm{Im}\omega>0\,, \quad (12)$$

and where $\tilde{f}(\omega)$ is the Fourier transform of the external force $f(t)$. It follows, using (11) and (12) in (10), that

$$|\tilde{v}(\omega)|^2=\omega^2|\alpha(\omega)|^2|\tilde{f}(\omega)|^2$$
$$=\frac{1}{\omega^2M^2}|(1-i\omega\tau_e)|^2|\tilde{f}(\omega)|^2\,. \quad (13)$$

Substituting (9) and (13) in (8) leads to

$$W_R=\frac{e^2}{3\pi c^3M^2}\int_{-\infty}^{\infty}d\omega|\tilde{f}(\omega)|^2$$
$$=\frac{2e^2}{3c^3}\int_{-\infty}^{\infty}dt[f(t)/M]^2\,, \quad (14)$$

where the last equality follows from the use of Parseval's relation.

This is our generalization of the Larmor formula. It arises from the inclusion of structure effects in the analysis of the electron interaction with the electromagnetic field. Our result is not the same as the Larmor result by virtue of the fact that it contains $f(t)/M$, which differs from the usual $\ddot{x}(t)$ term because of radiation reaction effects occurring in the equation of motion. In addition, we note that, whereas the Larmor formula is compatible with the Abraham-Lorentz equation (in the sense that, in conjunction with energy conservation considerations, the former leads to the latter), it may also be shown that (14) is compatible with our generalization [1] of the Abraham-Lorentz equation.

This research was partially supported by the U.S. Office of Naval Research under Contract No. N00014-90-J-1124 and by the National Science Foundation Grant No. INT-8902519.

---

*Permanent address: Department of Physics, University of Michigan, Ann Arbor, MI 48109-1120.

[1] G. W. Ford and R. F. O'Connell, Phys. Lett. A **157**, 217 (1991).
[2] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), Chap. 17.
[3] G. W. Ford, J. T. Lewis, and R. F. O'Connell, Phys. Rev. A **36**, 1466 (1987).
[4] G. W. Ford, J. T. Lewis, and R. F. O'Connell, Phys. Rev. A **37**, 4419 (1988).