A Newton–Faraday approach to electromagnetic energy and angular momentum storage in an electromechanical system

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The dynamical behavior of a simple rotating mechanical system that carries a charge on its surface and is accelerated by a falling mass is considered. It is shown that a fraction of the total kinetic energy is missing and that exactly this fraction of energy has been stored in the magnetic field distribution. The conservation of electromagnetic angular momentum is also discussed on the same basis. The concept of an electromagnetic moment of inertia is introduced to establish a close parallel with the concept of mechanical moment of inertia in classical dynamics. It is suggested that the present mechanical system can be used as a teaching tool at the early-to-intermediate level of an undergraduate physics program to ease the transition from the dynamics of rigid bodies to that of more abstract fields.
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I. INTRODUCTION

Most introductory presentations of the concept of magnetic energy storage rest on the assumed current–voltage relation for a linear self-inductance element,[1–3]

$$V = L\frac{di}{dt}. \tag{1}$$

The instantaneous power P supplied to the inductor is then shown to equal the rate of change of the stored energy, $\frac{1}{2}Li^2$:

$$P = iV = \frac{d}{dt}\left(\frac{1}{2}Li^2\right). \tag{2}$$

In these expressions, i is the current, V is the instantaneous voltage drop, and L is the self-inductance parameter.

Another important topic, that of electromagnetic angular momentum, is not covered[1–3] at all at the introductory level, in spite of the great physical significance of overall angular momentum conservation. The subject is mentioned at the more advanced undergraduate stage however,[4–6] but the number of applications that are discussed is usually limited, presumably due to the mathematical complexity of the subject.[7–9]

Simple presentations of magnetic energy and of electromagnetic angular momentum are done mostly on the basis of qualitative arguments and usually not from the perspective of the work done by the individual forces that are involved in a specific case. A student may often not appreciate until the more advanced undergraduate level that a system's magnetic energy and electromagnetic angular momentum are often stored at the expense of their mechanical counterparts.

Introductory-to-intermediate level discussions that approach the overall conservation of electromechanical energy from a mechanical perspective do not appear to be available in textbooks or in the pedagogical literature, although Ref. 10 does provide a simple example using a parallel-plate inductor arrangement. As mentioned, there also is a similar lack of early coverage of electromechanical angular momentum, and these observations partly motivated the present article.

The article considers an electromechanical system whose motion can be understood readily by students who have applied Newton's laws to the motion of rigid bodies and who have solved some elementary cases of Faraday's law. The model was originally developed by the author for explaining the role of Faraday's law of induction in terms of the basic conservation laws of physics. It will be explicitly shown that energy and angular momentum have to be ascribed to the distribution of the electromagnetic fields in order to restore overall energy and angular momentum conservation.

The model system will be described in Sec. II and its dynamical behavior will then be discussed. Section II will focus on the stored kinetic and magnetic energy aspects while Sec. III will consider the important issue of overall angular momentum conservation.

II. GRAVITATIONAL WORK BECOMES MECHANICAL KINETIC ENERGY AND MAGNETIC ENERGY

Consider a long thin uniform electrically insulating and magnetically nonpermeable cylindrical shell of length l, radius R, and moment of inertia I, that can rotate freely about its horizontal symmetry axis; a net uniformly distributed charge Q is on the shell. A massless string is wound around the shell surface and a vertically hanging mass m is attached to its free end and released from rest at time $t=0$. The problem is to determine the angular acceleration, the mechanical kinetic energy and mechanical angular momentum of the assembly after the hanging mass has fallen a distance H (see Fig. 1). A cylindrical coordinate system (ρ,ϕ,z) with unit vectors $(\hat{\rho},\hat{\phi},\hat{z})$ will be used.

Quasistationary acceleration conditions of motion are assumed to hold throughout the process. In this approximation, radiation losses due to acceleration effects are not significant, and Maxwell's displacement current term may be neglected when doing the field calculations.

If there were no charge on the shell, the angular acceleration, assumed to be positive, would be

$$\alpha = \frac{mgR}{I+mR^2}, \tag{3}$$

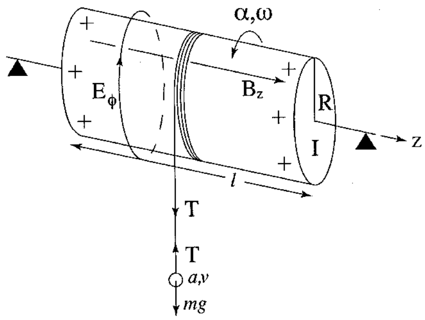

Fig. 1. Schematic diagram of the system: B_z=induced magnetic field; E_ϕ=induced electric field, T=tension in the string; mg=weight of suspended mass; a=linear acceleration; v=linear velocity; H=distance of fall; α=angular acceleration; ω=angular velocity; R=radius of the shell; l=length of the shell; + signs denote distributed charge.

where g is the acceleration due to gravity. Consequently, the total kinetic energy after the hanging mass has dropped a distance H would be

$$K=\tfrac{1}{2}I\omega^2+\tfrac{1}{2}mv^2=I\alpha\Delta\phi+mR^2\alpha\Delta\phi$$

$$=(I+mR^2)\frac{mgR}{(I+mR^2)}\frac{H}{R}=mgH, \quad (4)$$

where v is the linear velocity of the falling weight, $\Delta\phi=H/R$ is the angle turned by the cylindrical shell, and $\omega=v/R=\sqrt{2\alpha\Delta\phi}$ is its angular velocity. These results follow from an elementary exercise in dynamics and the details are omitted.

When there is charge on the shell, however, the situation is different because the accelerating charge creates a time-dependent magnetic field inside the shell, thereby giving rise to a back emf, in accordance with Faraday's law of induction. Because of the assumed cylindrical symmetry of the charge distribution, one can use Faraday's law in integral form[1–3] to obtain the induced electric field E_ϕ that acts on the deposited charge Q,

$$\oint_C \vec{E}\cdot d\vec{r}=-\frac{d}{dt}\int\int_S \vec{B}\cdot d\vec{s}, \quad (5a)$$

that is,

$$2\pi RE_\phi=-\frac{d}{dt}(\pi R^2 B_z), \quad (5b)$$

where B_z is the axially directed uniform magnetic field produced inside the shell by the moving charge (see Fig. 1). Because quasistationary conditions of motion are assumed to hold, the magnetic field outside the cylindrical shell remains very small at all times, and the interior field is the same as that of a very long solenoid of length l carrying a total current i:[1–3]

$$B_z=\frac{\mu_0 i}{l}, \quad (6a)$$

$$i\equiv\frac{dQ}{dt}=\frac{Q\omega}{2\pi}. \quad (6b)$$

The reaction field is then obtained by inserting Eq. (6) into Eq. (5b), and the result is

$$E_\phi=-\left(\frac{\mu_0 QR}{4\pi l}\right)\alpha, \quad (7)$$

where $a=dw/dt$. This field produces a retarding torque on the rotation of the shell, and this torque is proportional to the angular acceleration α. We now examine the rotational–translational equations of motion for the shell-weight assembly.

We first express the electromagnetic contribution to the net torque, τ_{em}, as an integral involving the circular force $E_\phi(\rho)dQ$, which acts on an element of charge dQ inside the shell, times the lever arm ρ of that force with respect to the spinning axis. The result is

$$\tau_{\mathrm{em}}\equiv\int dQ\,E_\phi(\rho)\rho=QE_\phi(R)R. \quad (8)$$

To arrive at the second line of Eq. (8), we note that the layer in which the electric charge is uniformly distributed is assumed to have a finite thickness, although it is very thin, and $E_\phi(\rho)$ is the circular component of the net electric field within that layer of charge. Now, Maxwell's theory predicts that the tangential component of the net electric field must be continuous across an interface. Hence the circular component of the net electric field must be continuous across the shell thickness. Consequently every charge element dQ in the shell will experience the same circular torque.

Under the above conditions, the equations of motion for the assembly are

$$I\alpha=TR+QE_\phi R \quad (9a)$$

and

$$-T+mg=m\alpha R, \quad (9b)$$

with T the tension in the string; also, $E_\phi\equiv E_\phi(R)$ is the induced circular component of the electric field on the shell and is given by Eq. (7).

Equation (9) is readily solved for the angular acceleration, α, and hence for the angular velocity of the spinning shell, ω. The results are

$$\alpha=\frac{mgR}{I+mR^2+\dfrac{\mu_0 Q^2R^2}{4\pi l}} \quad (10a)$$

and

$$\omega=\sqrt{\frac{2\alpha H}{R}}=\sqrt{\frac{2mgH}{I+mR^2+\dfrac{\mu_0Q^2R^2}{4\pi l}}}, \quad (10b)$$

respectively. Consequently, the total mechanical kinetic energy of the spinning shell-weight system is

$$\mathrm{K}=\frac{1}{2}I\omega^2+\frac{1}{2}mv^2=mgH\frac{(I+mR^2)}{\left(I+mR^2+\dfrac{\mu_0Q^2R^2}{4\pi l}\right)}. \quad (11)$$

The net acceleration of the spinning shell is indeed smaller than that predicted in Eq. (3), due to the charge-dependent increase, $\mu_0Q^2R^2/4\pi l$, in the inertial factor $I+mR^2$ in the denominator of Eq. (10a). The total kinetic energy after a drop H is thus reduced from mgH, as seen in the second line of Eq. (11). The missing kinetic energy is seen to be

$$\Delta K = -mgH \frac{\dfrac{\mu_0 Q^2 R^2}{4\pi l}}{\left(I + mR^2 + \dfrac{\mu_0 Q^2 R^2}{4\pi l}\right)}. \tag{12}$$

We now transform Eq. (12) by rewriting the denominator using Eq. (10a), and then using Eq. (6) to replace the factor Q^2 in the numerator. The result is that

$$\Delta K = -\left(\frac{\alpha H}{R}\right)\left(\frac{\mu_0 R^2}{4\pi l}\right)\left(\frac{4\pi^2 l^2 B_z^2}{\mu_0^2 \omega^2}\right). \tag{13}$$

Equation (13) can be further simplified because by Eq. (10b), $\omega^2 = 2\alpha H/R$. The final expression for the missing kinetic energy ΔK is then

$$\Delta K = -\frac{B_z^2}{2\mu_0}\pi R^2 l, \tag{14}$$

and this value is equal to minus the stored magnetic energy, as predicted by Maxwell's theory:[4–6]

$$U_m = \int\int\int_{\text{space}} \frac{B_z^2}{2\mu_0} d\tau = \frac{B_z^2}{2\mu_0}\pi R^2 l. \tag{15}$$

The statement of overall energy conservation for this electromechanical system then reads

$$K + U_m = mgH. \tag{16}$$

The missing kinetic energy has in effect been stored in the magnetic field distribution.

Finally, to make contact with the current-voltage approach mentioned in Eqs. (1) and (2), we use Eq. (6) to eliminate B_z^2 in Eq. (15). The resulting expression then gives the stored magnetic energy in the more traditional form in terms of the square of the net current carried by the arrangement:

$$U_m = \frac{1}{2}\left(\frac{\pi\mu_0 R^2}{l}\right) i^2. \tag{17}$$

In this expression, $\pi\mu_0 R^2/l$ represents the self-inductance parameter.

III. TRANSFORMATION OF EXTERNAL GRAVITATIONAL TORQUE TO MECHANICAL AND ELECTROMAGNETIC ANGULAR MOMENTUM

We now determine where the angular momentum is distributed in the system. In the denominator of Eq. (12), the quantity

$$I_{\text{em}} \equiv \frac{\mu_0 Q^2 R^2}{4\pi l} \tag{18}$$

clearly plays the role of a moment of inertia, but is associated with the charge itself, not with the mass. For this reason we will call it the electromagnetic moment of inertia of the assembly, to distinguish it from the more familiar mechanical moment of inertia. The physical significance of this novel type of moment of inertia will be emphasized at the end of Sec. III where we show that it appears in the expressions for the stored electromagnetic angular momentum and the stored magnetic energy.

Let

$$\vec{\Gamma} \equiv \int_0^t dt\, \vec{\tau}_{\text{ext}} = \hat{z}\int_0^t dt(mgR) \tag{19}$$

be the net angular impulse of the external gravitational torque $\vec{\tau}_{\text{ext}} = z(mgR)$ which is provided by the falling mass m. This impulse has taken the spinning shell from its original state of rest at time $t=0$, and brought it to its final angular velocity $\vec{\omega}$ at time t, as the hanging mass fell by the distance H. The assumed quasistationary conditions again mean that radiation losses that are due to the acceleration of the spinning shell can be neglected.

The angular impulse of the external torque is readily evaluated, and we find that

$$\begin{aligned}\vec{\Gamma} &= \hat{z}[(mgR)t] = \hat{z}\left[mgR\frac{\omega}{\alpha}\right]\\ &= \hat{z}\left[\sqrt{2mgH(I + mR^2 + I_{\text{em}})}\right].\end{aligned} \tag{20}$$

To arrive at the second line in Eq. (20), it was noted that $\omega = \alpha t$ because the motion is characterized by a constant acceleration. Finally, Eq. (10) was used to deduce the last line of Eq. (20), along with the definition of the electromagnetic moment of inertia that is given in Eq. (18).

According to Newton's law for rotational motion, the angular impulse due to the external torque must equal the change of the angular momentum of the system, that is, $\vec{\Gamma} = \Delta\vec{L}_{\text{syst}}$. As a result, the final angular momentum of the system is

$$\vec{L}_{\text{syst}} = \hat{z}L_{\text{syst}} = \hat{z}\left[\sqrt{2mgH(I + mR^2 + I_{\text{em}})}\right], \tag{21}$$

because the initial angular momentum equals zero. We now examine how the total angular momentum is stored by the system.

The mechanical contribution to the angular momentum arises exclusively from the mechanical inertia of the rotating shell and from that of the falling mass, and is

$$\vec{L}_{\text{mech}} = (I + mR^2)\vec{\omega} = \hat{z}(I + mR^2)\sqrt{\frac{2mgH}{(I + mR^2 + I_{\text{em}})}}. \tag{22}$$

By comparing this result with Eq. (21), one can see that $\vec{L}_{\text{mech}}$ is smaller than $\vec{L}_{\text{syst}}$. Therefore some of the angular momentum imparted by the angular impulse of the external torque is missing from the mechanical contribution, and this missing part is to be found in the electromagnetic angular momentum. Indeed, we can simply add the quantity $\vec{L}_{\text{em}} = I_{\text{em}}\vec{\omega}$ to both sides of Eq. (22) to obtain

$$\begin{aligned}\vec{L}_{\text{mech}} + \vec{L}_{\text{em}} &= (I + mR^2 + I_{\text{em}})\vec{\omega}\\ &= \hat{z}(I + mR^2 + I_{\text{em}})\sqrt{\frac{2mgH}{(I + mR^2 + I_{\text{em}})}}\\ &= \hat{z}\sqrt{2mgH(I + mR^2 + I_{\text{em}})},\end{aligned} \tag{23}$$

where I_{em} is given in Eq. (18). In other words, the missing angular momentum has been stored in the electromagnetic fields because the last line in Eq. (23) is equal to the full angular impulse of the external torque by Eq. (21). We conclude that

$$\vec{L}_{\text{mech}} + \vec{L}_{\text{em}} = \vec{L}_{\text{syst}}, \tag{24a}$$

$$\vec{L}_{\text{mech}} = I\vec{\omega}, \tag{24b}$$

$$\vec{L}_{\text{em}} = \frac{\mu_0 Q^2 R^2}{4\pi l}\vec{\omega} \equiv I_{\text{em}}\vec{\omega}. \tag{24c}$$

According to Newton's law for rotational motion, we then have that

$$\frac{d}{dt}[\vec{L}_{\text{mech}} + \vec{L}_{\text{em}}] = \vec{\tau}_{\text{ext}}, \tag{25}$$

or, equivalently,

$$(I + mR^2 + I_{\text{em}})\vec{\alpha} = z(mgR), \tag{26}$$

where $\vec{\tau}_{\text{ext}}$ is the gravitational torque. That is, the time derivative of Eq. (21) equals the torque in Eq. (19): the rate of change of the total angular momentum of the system is equal to the external torque applied to that system. If the external torque vanishes, the total angular momentum, mechanical plus electromagnetic, remains constant in time, and the system continues to spin without losses, in accordance with the law of inertia.

We now show that the concept of an electromagnetic moment of inertia also helps to simplify the energy discussion of Eqs. (11)–(16). Indeed, using Eq. (6), we readily find that the magnetic energy of Eq. (15) may be written in the form of a mechanical-like kinetic energy term

$$U_m = \tfrac{1}{2} I_{\text{em}} \omega^2, \tag{27}$$

where I_{em} is as defined in Eq. (18). The electromagnetic moments of inertia that are associated with the electromagnetic angular momentum and with the magnetic energy are the same, just as in nonrelativistic mechanics. Equation (16) can thus be rewritten as

$$\tfrac{1}{2}(I + mR^2 + I_{\text{em}})\omega^2 = mgH. \tag{28}$$

The statement of overall energy conservation can now be formulated as follows: the total change in the energy of the system, mechanical plus magnetic, equals the work done by the external force. In the absence of external work, the total energy of the system is conserved. Note that the electromagnetic angular momentum for the present system can be calculated, as usual, by invoking the field integral of the Poynting vector.[4–7] Such a rigorous calculation has been performed for the system under consideration here and the results support the findings given in Eq. (24),[11] a simpler approach is now sketched.

Let

$$\vec{g} = \varepsilon_0 \vec{E} \times \vec{B} = \varepsilon_0 (E_\rho \rho) \times (B_z z) = -(\varepsilon_0 E_\rho B_z)\phi \tag{29}$$

be the electromagnetic momentum density, where by Gauss's law and symmetry,

$$E_\rho = 0, \quad 0 \leqslant \rho < R, \qquad E_\rho = \frac{Q}{2\pi\rho l}, \quad R < \rho, \tag{30a}$$

and

$$B_z = B_z(\rho) \quad (0 \leqslant \rho). \tag{30b}$$

We have assumed that the shell, which extends from $z_1 = -l/2$ to $z_2 = -l/2$, is very long, $l \to \infty$. If we take $d\tau$ to be the spatial volume element in cylindrical coordinates, the electromagnetic angular momentum is

$$\begin{aligned}
\vec{L}_{\text{em}} &= \int\int\int_{\text{space}} d\tau\, \vec{r} \times \vec{g} \\
&= \int_R^\infty d\rho\, \rho \int_0^{2\pi} d\phi \int_{-l/2}^{+l/2} dz \left(-\frac{Q}{2\pi\rho l} B_z\right) \\
&\quad \times [(\rho\hat{\rho} + z\hat{z}) \times \hat{\phi}] \\
&= \int_R^\infty d\rho\, \rho \int_0^{2\pi} d\phi \int_{-l/2}^{+l/2} dz \left(-\rho \frac{Q}{2\pi\rho l}\hat{z} + z\frac{Q}{2\pi\rho l}\hat{\rho}\right) B_z.
\end{aligned} \tag{31}$$

Equation (31) reduces to

$$\vec{L}_{\text{em}} = -\frac{Q}{2\pi}\left(\int_R^\infty d\rho\, \rho \int_0^{2\pi} d\phi B_z\right) z. \tag{32}$$

To arrive at Eq. (32), the term involving the unit radial vector $\hat{\rho}$ was integrated over the azimuthal angle ϕ, and the result was found to vanish because of the assumed axial symmetry. Note that the factor in parentheses on the right-hand side of Eq. (32) cannot equal zero, in spite of the fact that the magnetic field of the spinning shell is extremely small at all exterior points. Indeed, the integral on the right-hand side gives the return magnetic flux linking the entire region outside the cylindrical shell and, according to Maxwell's theory, this return magnetic flux must be equal and opposite to the magnetic flux in the interior of the shell, $\pi R^2(\mu_0 Q\omega/2\pi l)$. As a result, we find that

$$\vec{L}_{\text{em}} = +\frac{Q}{2\pi}\left[\pi R^2 \frac{\mu_0 Q}{2\pi l}\omega\right] z, \tag{33}$$

which agrees with Eq. (24).

IV. DISCUSSION

We have solved a simple model problem where the mechanical and electromagnetic behaviors are coupled to one another by the appropriate Newton–Faraday equations of motion. We have shown that part of the kinetic energy was used to establish the magnetic field distribution created by the motion of the assembly. When energy is ascribed to the magnetic field distribution, we recover overall energy conservation.

The model was also used to establish that we must similarly ascribe angular momentum to the distribution of the electromagnetic fields of the system. The concept of an electromagnetic moment of inertia was introduced to facilitate the general description of the problem. It was then established that the same electromagnetic moment of inertia determines both the electromagnetic angular momentum and the magnetic energy, in a striking parallel with the more familiar mechanical situation. A more general presentation of some of the ideas that are discussed in the present article is currently being considered for publication.[11]

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PHYSICS FOR POETS

He wandered the hallways of the department like a ghost and was condemned, for his quantum heresy, to teach the course they called ''Physics for Poets.'' But he had taken to his sentence with an unbecoming gladness. It was the first work since the hidden variables that he had loved, although he was far more taken with the poets than the physics. He was teaching Physics for Poets to the baffled undergraduates who had wanted only to fulfill their science requirement without being dragged through the mental anguish that they called mathematics. The tag was meant to convey only what was mercifully missing, but he had taken fiercely to the notion of the noetic poetic.

Rebecca Goldstein, *Properties of Light: A Novel of Love, Betrayal, and Quantum Physics* (Houghton Mifflin Company, New York, NY, 2000), p. 17.

Submitted by Alan DeWeerd.