Uncertainty principle applied to the deuteron

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Many undergraduate texts[1,2] apply the uncertainty principle to the problem of the existence of atoms. The problem, as is well known, is to account for the stability of the ground state of atoms, which contrary to prequantum expectations do not radiate energy. By using the uncertainty principle as a constraint and minimizing the total energy, one obtains an excellent estimate of the ground-state energy, the radius of the electron distribution, and the momentum of the electrons in hydrogenic atoms. In this note it is shown that similar arguments can be used to estimate the size and energy structure of the deuteron, as well as the strength of the strong force.

In the quantum mechanics course at this university the Yukawa potential and the relationship between the range of the potential and the mase of the pion μ,

$$r' = \hbar/\mu c, \tag{1}$$

are obtained following the lines of the discussion given by Eisberg and Resnick.[2] The total energy of the deuteron is then written

$$E = p^2/2M - g^2(e^{-r/r'}/r), \tag{2}$$

where g^2 gives the strength of the interaction and M is the reduced mass of the deuteron. Minimizing Eq. (2) subject to the constraint implied by the uncertainty principle

$$pr \approx \hbar \tag{3}$$

gives

$$\frac{\hbar^2}{Mr} = \left(\frac{r' + r}{r'}\right) g^2 e^{-r/r'}, \tag{3}$$

and eliminating g^2, the energy is then

$$E = \frac{\hbar^2}{2Mr^2}\left(\frac{r - r'}{r + r'}\right). \tag{4}$$

The ground-state binding energy of -2.2 MeV is readily obtained from the mass defect of the deuteron and Eqs. (1) and (4) then give 1 fm for the radius of the deuteron. Equations (1) and (2) give 28 MeV for the kinetic energy and, with the aid of Eq. (3), -30 MeV for the potential energy.[3] Finally, Eq. (3) and the value obtained for the radius of the deuteron give the ratio of the strength of the strong force to the electromagnetic force expressed as $g^2 4\pi\epsilon_0/e^2$ a value of 60. The virial theorem[4] may also be demonstrated since it is easy to show using Eq. (3) that

$$2\langle T\rangle = \langle \mathbf{r}\cdot\nabla V\rangle. \tag{5}$$

Using very simple arguments, it is therefore possible to derive estimates of the size, kinetic energy, and potential energy of the deuteron, as well as the strength of the strong force. The only data required are the masses of the neutron, proton, pion, and deuteron. This analysis also demonstrates the central role played by the uncertainty principle in quantum mechanics. Students can thus be introduced to a number of important results without recourse to complicated mathematical formalisms.

[1]R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1964).
[2]R. Eisberg and R. Resnick, *Quantum Physics of Atoms, Molecules, Solids, Nuclei and Particles* (Wiley, New York, 1974), pp. 269, 690.
[3]L. R. B. Elton, *Introductory Nuclear Theory* (Pitman, London, 1959), pp. 61, 75, 241.
[4]L. I. Schiff, *Quantum Mechanics* (McGraw-Hill ISE, Kogakusha, Tokyo, 1968), p. 180.

Dipole radiation: A film

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This note reports the production of a new animated film, in super-8 cassette form, dealing with electromagnetic radiation.

The film is divided into three parts. In the first a charge accelerates uniformly from rest, then coasts at constant velocity $v = 0.5c$ (see Fig. 1). In the second part a charge undergoes simple harmonic motion with maximum velocity $v = 0.5c$ (see Fig. 2). In the third part a dipole radiates (see Fig. 3).

The equations and procedures involved in the first two parts are described by Hamilton and Schwartz.[1] Differences between their films and the one discussed here include: (i) Because of the generally lower velocity in this film, field lines are less distorted by the acceleration; (ii) This film shows a radiating dipole.

The equations and procedures involved in calculating the field lines of the radiating dipole will now be described. An electric dipole will be assumed; the magnetic field lines of a radiating magnetic dipole are identical.

The differential equation governing the field lines is

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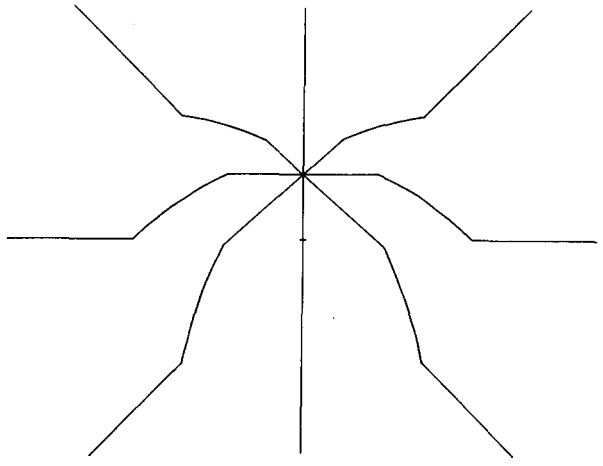

Fig. 1. Charge accelerates uniformly from rest, then coasts at constant velocity $v = 0.5c$.

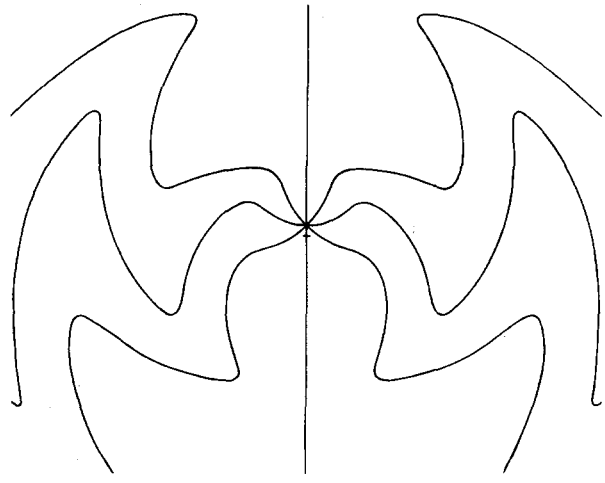

Fig. 2. Charge undergoes simple harmonic motion with maximum velocity $v = 0.5c$.

$$dr/r\,d\theta = E_r/E_\theta, \tag{1}$$

where r and θ are polar coordinates and E_r and E_θ are the corresponding electric field components. The field is to be found in standard texts[2]:

$$E_r = \frac{2p_0\cos\theta}{4\pi\epsilon_0 r^2 \lambda\!\!\!^{-}}\left[\frac{\lambda\!\!\!^{-}}{r}\cos\left(\frac{r}{\lambda\!\!\!^{-}}-\omega t\right)+\sin\left(\frac{r}{\lambda\!\!\!^{-}}-\omega t\right)\right]; \tag{2}$$

$$E_\theta = \frac{p_0\sin\theta}{4\pi\epsilon_0 r^2 \lambda\!\!\!^{-}}\left[\left(\frac{\lambda\!\!\!^{-}}{r}-\frac{r}{\lambda\!\!\!^{-}}\right)\cos\left(\frac{r}{\lambda\!\!\!^{-}}-\omega t\right)+\sin\left(\frac{r}{\lambda\!\!\!^{-}}-\omega t\right)\right], \tag{3}$$

where p_0 is the amplitude of the dipole moment, $\lambda\!\!\!^{-}$ is radian length, ω is circular frequency, and t is time. Field components from Eqs. (2) and (3) are now substituted into Eq. (1), resulting in

$$\frac{dr}{r\,d\theta} = \frac{2\cos\theta}{\sin\theta}\,\frac{\left(\frac{\lambda\!\!\!^{-}}{r}\right)\cos\left(\frac{r}{\lambda\!\!\!^{-}}-\omega t\right)+\sin\left(\frac{r}{\lambda\!\!\!^{-}}-\omega t\right)}{\left(\frac{\lambda\!\!\!^{-}}{r}-\frac{r}{\lambda\!\!\!^{-}}\right)\cos\left(\frac{r}{\lambda\!\!\!^{-}}-\omega t\right)+\sin\left(\frac{r}{\lambda\!\!\!^{-}}-\omega t\right)}. \tag{4}$$

Separating variables,

$$\frac{2\cos\theta}{\sin\theta}\,d\theta = \frac{\left(\frac{\lambda\!\!\!^{-2}}{r^2}-1\right)\cos\left(\frac{r}{\lambda\!\!\!^{-}}-\omega t\right)+\left(\frac{\lambda\!\!\!^{-}}{r}\right)\sin\left(\frac{r}{\lambda\!\!\!^{-}}-\omega t\right)}{\left(\frac{\lambda\!\!\!^{-}}{r}\right)\cos\left(\frac{r}{\lambda\!\!\!^{-}}-\omega t\right)+\sin\left(\frac{r}{\lambda\!\!\!^{-}}-\omega t\right)}\,d\left(\frac{r}{\lambda\!\!\!^{-}}\right). \tag{5}$$

Logarithmic forms are found on both sides; thus[3]

$$\sin^2\theta = \frac{K}{\left(\frac{\lambda\!\!\!^{-}}{r}\right)\cos\left(\frac{r}{\lambda\!\!\!^{-}}-\omega t\right)+\sin\left(\frac{r}{\lambda\!\!\!^{-}}-\omega t\right)}, \tag{6}$$

where K is a constant of integration. This one relatively simple equation generates all of the seemingly complex phenomena seen.

In the film different values of ωt are used for different frames; the entire cycle $0 \leqslant \omega t \leqslant 2\pi$ is divided into 64 approximately equal parts. Within each frame different values of $|K|$ produce different field lines, here 0.15, 0.45, 0.75, 1.05, 1.35 are used. And finally, for a given field line, many different values of $r/\lambda\!\!\!^{-}$ are introduced into Eq. (6) to find corresponding values of θ and thus locate points on the field line.

It is interesting that for $1 < |K| < 1.15$, field lines form loops which break away but eventually dwindle and vanish; whereas for smaller $|K|$ the loops break away and grow, and for larger $|K|$ no loops break away. All three cases are observable in the film.

The process of separation of a loop is shown in detail in a special section of the film, for $|K| = 0.75$. The process is enlarged by a factor of 2, and slowed by a factor of 16, so that minimum time between pictures represents less than 1/1000 of a full cycle.

Within each part of the film, the order of presentation is (a) the process is shown; (b) a noteworthy aspect is pointed out; (c) the process is repeated in a slow-motion "instant replay."

The density of field lines shown gives only a rough indication of field strength, because their representation is two-dimensional, whereas the field itself spreads out in three dimensions. Also the idea of moving field lines is *a priori* suspect because field should be regarded not as moving relative to the observer, but rather as changing in magnitude

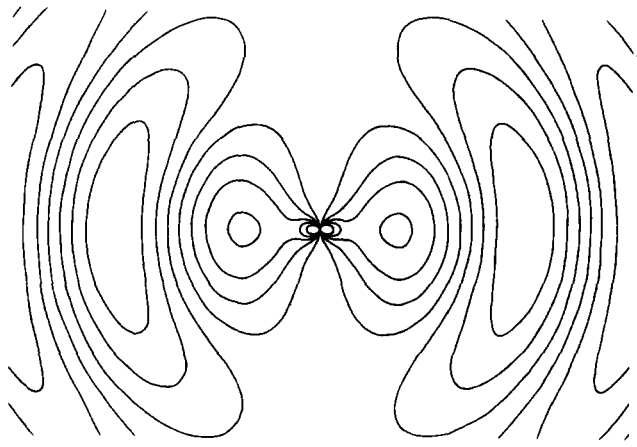

Fig. 3. Dipole radiates.

and direction at fixed points, as is proper. However, the field lines shown do follow the direction of the field and so they are helpful in visualizing it.

[1]John C. Hamilton and Judah L. Schwartz, Am. J. Phys. **39**, 1540 (1971); Roger Y. Tsien, Am. J. Phys. **40**, 46 (1972). Field of a charge that starts or stops is also discussed at an elementary level in Edward M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1965), Vol. 2, pp. 163–167.

[2]Paul Lorrain and Dale Corson, *Electromagnetic Fields and Waves*, 2nd ed. (Freeman, New York, 1970), p. 601; John R. Reitz, Frederick J. Milford, and Robert W. Christy, *Foundations of Electromagnetic Theory*, 3rd ed. (Addison-Wesley, New York, 1979), pp. 454–455.

[3]Heinrich Hertz, *Electric Waves* (Dover, New York, 1962), p. 142. (Originally published by Macmillan, New York, 1893); Lorrain and Corson, Ref. 2, p. 607.

Computation of eigenvalues and eigenvectors of a symmetric matrix

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The power method for producing the dominant eigenvalue and eigenvector of a symmetric matrix A is well known.[1] Most textbooks, however, fail to mention the fact that, instead of using the so-called Rayleigh quotient, it is much better for quick convergence to repeatedly square the matrix A. After n iterations the matrix

$$B = A^{2^n} \tag{1}$$

gives an approximation of the dominant eigenvector in the form

$$V_k = \sum_l B_{kl}. \tag{2}$$

The dominant eigenvalue is then obtained multiplying the eigenvector (2) by the matrix A. So far, so good. What about eigenvectors and eigenvalues other than the dominant one? Here all textbooks become rather confusing for a student. To quote from Ref. 1, "The difficulty is to sidetrack the dominant eigenvalue itself and to keep it sidetracked. Roundoff errors have spoiled several theoretically sound methods by returning the dominant eigenvalue to the main line of computation and obscuring the next dominant. . .." Most textbooks give a special recipe to compute the absolutely smallest eigenvalue by using the power method with the matrix A^{-1}, although, to quote again from Ref. 1, "Finding A^{-1} is ordinarily no simple task, but this method is sometimes the best approach to the absolutely smallest eigenvalue."

To make it short, no one seems aware of what I had considered till now an "obvious" extension of the power method to compute *any* eigenvalue and eigenvector.[2] *To find the eigenvector corresponding to the eigenvalue nearest to* λ *use the power method above with the matrix*

$$C = (A - \lambda I)^2 - (|\lambda| + \Lambda)^2 I, \tag{3}$$

where I *is the unit matrix and* Λ *any number equal or greater than the absolute value of the dominant eigenvalue.*

The proof is trivial. The matrix $(A - \lambda I)^2$ has only positive eigenvalues and its absolute smallest eigenvalue corresponds to the eigenvalue of A nearest to λ. Subtracting from this matrix a sufficiently high positive multiple of the unit matrix, all eigenvalues become negative with the absolute smaller becoming the absolute greater. To it the power method applies.

To get a feeling of what is going on, let me give a simple example. Take the matrix

$$A = \begin{pmatrix} 0 & 6 & -10 & -8 \\ 6 & 0 & 8 & 10 \\ -10 & 8 & 15 & 6 \\ -8 & 10 & 6 & 15 \end{pmatrix}. \tag{4}$$

Suppose you want to find the smallest eigenvalue. By simply looking at the matrix, and remembering the very definition of eigenvalue you know that by putting $\Lambda = 100$, you are on the safe side. The smallest eigenvalue can then be taken as the one nearest to $\lambda = -100$. Iterating C 10 times on my programmable pocket calculator I get the eigenvector $(1,-1,0.5,0.5)$ and the eigenvalue -15 exactly.[3] Of course, the convergence is improved if you take $\Lambda = 30$, the dominant eigenvalue, that you could have computed beforehand using matrix A instead of matrix C. The same procedure to find the eigenvalue nearest to 7.6 (almost half way between the other two eigenvalues 5 and 10 of matrix A) gives the "exact" eigenvalue 10 with eigenvector $(1,1,-2,2)$ after 20 iterations instead of 10. Six minutes instead of three on my pocket calculator.

The method is precise and simple both theoretically and practically (you only need a subroutine to square a matrix and divide all the elements when they get too big). I found it particularly useful to teach quantum mechanics. You can often get very good approximations to some of the lowest energy levels of assorted Hamiltonians simply by truncating the matrix. Try, for example, with a matrix representation of the anharmonic oscillator $p^2 + x^4$.

[1]See, for example, F. Scheid, *Numerical Analysis*, Schaum's Outline Series (McGraw-Hill, New York, 1968).

[2]I have been teaching this extension of the power method to my students for many years. Eventually, someone asked me for a reference and with amazement I discovered that I could not find one!

[3]This is, of course, due to the roundoff effect of my calculator that uses twelve digits, but displays only ten. It means that after 10 iterations you get correctly 10 significant figures.

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