Dumbbell model for the classical radiation reaction

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In his treatise on electron theory, Lorentz showed that the classical radiation reaction is attributable to the breakdown of Newton's Third Law within the structure of an accelerating charge. But Lorentz's calculation, based on a spherical model, is so difficult as to obscure the underlying physical principle, and requires that the charge be instantaneously at rest. Noting that the answer must be structure independent, in the point-charge limit, we develop Lorentz's theory in the context of a more tractable model: the dumbbell. We also compute the self-torque on a rotating dumbell.

I. INTRODUCTION

According to the laws of classical electrodynamics, an accelerating charge radiates. This radiation carries away energy, which, for a structureless particle, must come at the expense of kinetic energy. Under the influence of a given force, therefore, a charged particle accelerates *less* than a neutral one of the same mass. This phenomenon is known as the radiation reaction, or, in the case of oscillatory systems, radiation damping.

There are two ways of calculating the radiation reaction force, which go back respectively to Abraham[1] and Lorentz.[2] Abraham's method[3] exploits conservation laws: at its most primitive level one simply quotes the Larmor formula for the power radiated, and associates this with the rate of energy loss by the charge. This is the approach taken in most textbooks;[4] reflecting the limitation of the Larmor formula, it provides only an approximate equation for the radiation reaction, valid at small velocities:

$$\mathbf{F}_{\rm rad} = (2/3)(e^2/c^3)\dot{\mathbf{a}}. \tag{1}$$

Here e is the charge of the particle, and $\mathbf{a}$ is its acceleration; the dot indicates a time derivative. c is the speed of light, and we use cgs units throughout. In his more refined version Abraham incorporated conservation of momentum and allowed for arbitrary speeds:

$$\mathbf{F}_{\rm rad} = \frac{2}{3}\frac{e^2}{c^3}\gamma^4\left[\mathbf{g} + \frac{1}{c^2}\mathbf{v}\times(\mathbf{v}\times\mathbf{g})\right], \tag{2}$$

where $\mathbf{g} = \dot{\mathbf{a}} + 3\gamma^2[(\mathbf{a}\cdot\mathbf{v})/c^2]\,\mathbf{a}$, $\mathbf{v}$ is the velocity of the particle, and $\gamma = 1/(1-v^2/c^2)^{1/2}$. Today, Abraham's formula is usually written in 4-vector notation as[5]

$$F^{\mu}_{\rm rad} = \frac{2}{3}\frac{e^2}{c^3}\left[\frac{d^2v^{\mu}}{d\tau^2} - \frac{v^{\mu}}{c^2}\left(\frac{dv^{\nu}}{d\tau}\frac{dv_{\nu}}{d\tau}\right)\right], \tag{3}$$

where v^{μ} is the 4-velocity and τ is the proper time. Indeed, Abraham's formula is often "derived" as the simplest permissable covariant generalization of Eq. (1). Abraham's procedure for calculating the energy and momentum radiated by the particle is open to criticism, and alternative methods have been proposed over the years by Sommerfeld,[6] Dirac,[7] Rohrlich,[8] and Teitelboim,[9] to name a few. But these authors do not dispute the validity of Abraham's final formula.

There is, however, a fundamental question to which Abraham's approach does not address itself at all: what is the *cause* of the radiation reaction? It is one thing to say that such a force must be present in order to sustain the conservation laws, but quite another to identify the actual agency responsible. Lorentz showed that the radiation reaction force is attributable to the breakdown of Newton's Third Law in classical electrodynamics.[10] When an extended charge accelerates, the force of one part on another is *not* equal and opposite to the force of the second part on the first. When these imbalances are integrated over the entire configuration, the result is a net force of the charge *on itself.* Lorentz calculated this "self-force," for a spherical charge instantaneously at rest, and obtained an answer consistent with Eq. (1).

Although Lorentz's method is physically illuminating, it is in some respects model dependent, and a number of people, most notably Schott,[11] devoted much labor to the computation of the reaction force on ellipsoidal and even more exotic charge configurations, in the not unreasonable belief that they were laying the foundations for the theory of elementary particles. (Indeed, Schott calls his book "prolegomena to any future electron theory." But Schott's prescience was not the equal of Kant's, for with the advent of quantum mechanics, interest in these calculations largely evaporated.) By its nature, Lorentz's method must be carried out for an extended charge distribution; the comparison with Abraham's result is made in the limit as the size of the particle goes to zero. In this limit, of course, all structure dependence must vanish, since we know that conservation laws alone suffice to determine the answer.

Over the last half-century, Lorentz's method has fallen into disfavor, presumably because the calculations involved are extremely difficult.[12] This is unfortunate, because the *idea* is a lovely one, and there is no matter of principle that is beyond the reach of an advanced undergraduate. In the hope of making Lorentz's method more accessible, we present here the simplest possible model which nevertheless permits the essential mechanism (imbalance of internal electromagnetic forces) to function: the dumbbell—two parts, each of charge $(e/2)$, held a fixed distance d apart. Of course, this model does not bear much resemblance to an actual particle, but it does show very clearly where the self-force comes from, and, in the limit $d \to 0$, it reproduces the Abraham-Lorentz radiation reaction force. In fact, the dumbbell model is so simple that we have no need for Lorentz's restriction to charges instantaneously at rest.

In the following section we develop the basic theory of the self-force on a dumbbell in longitudinal motion—that is, moving in the direction of its own axis. We treat three examples exactly, and then derive a power series solution for the general case. In Sec. III we discuss the first term in this series, the electromagnetic mass renormalization. In Sec. IV we consider the second term, which is the radiation

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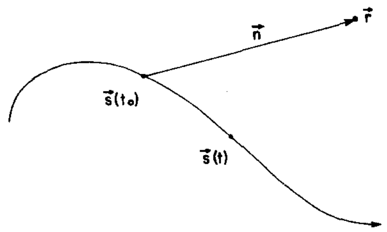

Fig. 1. Particle trajectory and retarded time.

reaction proper, and draw the connection with Abraham's result. In the final section we treat a related matter: the self-torque on a rotating dumbbell, the electromagnetic analog to the much debated problem of the gravitational self-torque on a rotating binary star.

II. SELF-FORCE ON A DUMBBELL IN LONGITUDINAL MOTION

The electric and magnetic fields of a point charge q moving along some specified trajectory $\mathbf{s}(t)$, are obtained from the Liénard-Wiechert potentials.[13] Reflecting the fact that electromagnetic "news" travels at the speed of light c the fields at point $\mathbf{r}$ and time t do not depend on the state of the charge at *that* moment, but rather at some earlier time t_0, when the "message" left the charge, as shown in Fig. 1. As a function of $\mathbf{r}$ and t, this "retarded time" is determined implicitly by the formula:

$$c(t-t_0) = |\mathbf{r} - \mathbf{s}(t_0)|, \tag{4}$$

since $|\mathbf{r} - \mathbf{s}(t_0)|$ is the distance traveled, and $(t - t_0)$ is the time it took to make the trip. The electric field of the particle is then

$$\mathbf{E}(\mathbf{r},t) = \frac{qn}{(\mathbf{n}\cdot\mathbf{m})^3}[(c^2 - v^2)\mathbf{m} + \mathbf{n}\times(\mathbf{m}\times\mathbf{a})], \tag{5}$$

where $\mathbf{v}$ (the particle's velocity), $\mathbf{a}$ (its acceleration), $\mathbf{n} = \mathbf{r} - \mathbf{s}$ (the vector from the particle to point $\mathbf{r}$), and $\mathbf{m} = c\hat{n} - \mathbf{v}$, are all to be evaluated at the retarded time t_0. ($n = |\mathbf{n}|$, and $\hat{n} = \mathbf{n}/n$.) The magnetic field, meanwhile, is given by

$$\mathbf{B}(\mathbf{r},t) = \hat{n}\times\mathbf{E}. \tag{6}$$

Suppose now that we have two charges, each $(e/2)$, constrained to move along the x axis in such a way that the distance d between them remains fixed, as shown in Fig. 2. (If you wish, think of them as fastened to the ends of a stick, but notice that there is no Lorentz contraction here. By assumption, the separation distance *in the laboratory* frame is always d.) Let $w(t)$, the position of the midpoint as a function of time, be given. We wish to calculate the net self-force on this dumbbell: the force on (1) due to (2), plus the force on (2) due to (1). If Newton's Third Law held, in classical electrodynamics, this sum would of course be zero. Since the magnetic field vanishes on the axis,

$$\mathbf{F}_{\text{self}} = (e/2)(E_1 - E_2)\hat{i}, \tag{7}$$

where E_1 is the magnitude of the electric field at (1) due to (2), and E_2 is the magnitude of the electric field at (2) due to (1). Noting that $\mathbf{n}$, $\mathbf{m}$, and $\mathbf{v}$ are colinear, in this context, we find

$$E_1 = \frac{e}{2c^2}\left(\frac{c+v(t_1)}{c-v(t_1)}\right)\frac{1}{(t-t_1)^2};$$
$$E_2 = \frac{e}{2c^2}\left(\frac{c-v(t_2)}{c+v(t_2)}\right)\frac{1}{(t-t_2)^2}, \tag{8}$$

where the retarded times t_1 and t_2 are given by

$$c(t-t_1) = w(t) - w(t_1) + d,$$
$$c(t-t_2) = w(t_2) - w(t) + d. \tag{9}$$

(For calculational purposes, notice that the formula for t_2 can be obtained from that for t_1 by simply switching the sign of w; the same goes for E_2 and E_1.)

Example (1): Constant velocity

In the case $w(t) = vt$, Eq. (9) yields

$$t_1 = t - \frac{d}{(c-v)}; \qquad t_2 = t - \frac{d}{(c+v)}; \tag{10}$$

from which it follows that

$$E_1 = E_2 = \frac{e}{2c^2d^2}(c^2 - v^2) \tag{11}$$

so $F_{\text{self}} = 0$. Incidentally, if $v > c$, the "message" from (1) never makes it to (2), and the net self-force is

$$\mathbf{F}_{\text{self}} = -\frac{e}{2}E_2\hat{i} = \frac{e^2}{4d^2}(v^2/c^2 - 1)\hat{i}.$$

This suggests that at superluminary velocities energy is radiated *even at constant speeds*, an observation which led Sommerfeld to a precocious prediction of Čerenkov radiation.

Example (2): Constant acceleration

If $w(t) = 1/2\, at^2$, Eq. (9) gives

$$t_1 = \frac{c - [(c-at)^2 + 2ad]^{1/2}}{a} \tag{12}$$

(t_2 is obtained by changing the sign of a). With this we find

$$E_1 = \frac{e}{4c^2d^2}\left((c^2 - a^2t^2) - ad + \frac{(c^2 - a^2t^2)(c-at) + 2a^2dt}{[(c-at)^2 + 2ad]^{1/2}}\right), \tag{13}$$

and a similar formula for E_2, with the sign of a reversed. Thus

$$F_{\text{self}} = -\frac{e^2}{8c^2d^2}\left(2ad + \frac{(c^2 - a^2t^2)(c+at) + 2a^2dt}{[(c+at)^2 - 2ad]^{1/2}} - \frac{(c^2 - a^2t^2)(c-at) + 2a^2dt}{[(c-at)^2 + 2ad]^{1/2}}\right). \tag{14}$$

At $t = 0$, when the velocity is zero,

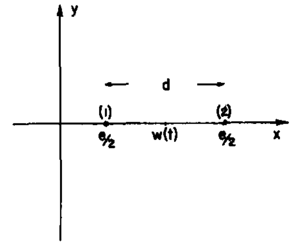

Fig. 2. Dumbbell in longitudinal motion

$$F_{\text{self}} = -\frac{e^2}{8d^2}\left(\epsilon + \frac{1}{(1-\epsilon)^{1/2}} - \frac{1}{(1+\epsilon)^{1/2}}\right), \quad (15)$$

where $\epsilon = 2ad/c^2$.

Example (3): Hyperbolic motion

In classical mechanics, a constant force (say, the electric force on a charge in the region between the plates of a large parallel-plate capacitor) produces a constant acceleration. In special relativity such a force yields "hyperbolic" motion[14]:

$$w(t) = [b^2 + (ct)^2]^{1/2} - b; \qquad v(t) = \frac{c^2 t}{[b^2 + (ct)^2]^{1/2}}. \quad (16)$$

(The constant b represents the rest energy divided by the force: $b = mc^2/F$.) For this case we find

$$t_1 = \frac{-(d-ct)(2b^2 - d^2 + 2dct) + d(d-2ct)[b^2 + (ct)^2]^{1/2}}{2c(b^2 - d^2 + 2dct)}; \quad (17)$$

t_2 can be obtained from this by changing the sign of the radical. It follows that

$$E_1 = \frac{2eb^2}{d^2\{d + 2[b^2 + (ct)^2]^{1/2}\}^2}. \quad (18)$$

(Again, E_2 can be found by switching the sign of the square root.) Thus

$$F_{\text{self}} = -\frac{8e^2b^2[b^2 + (ct)^2]^{1/2}}{d[4b^2 - d^2 + 4(ct)^2]^2}. \quad (19)$$

As these examples illustrate, Eqs. (7), (8), and (9) determine the electromagnetic self-force on a dumbbell undergoing any given longitudinal motion $w(t)$. Unfortunately, the implicit equation for the retarded times [Eq. (9)] is in general difficult, if not impossible, to solve in closed form. We can, however, provide a *series* solution, as follows. Write $w(t_1)$ as a Taylor expansion about the point t:

$$w(t_1) = w(t) + \dot{w}(t)(t_1 - t) + \frac{1}{2}\ddot{w}(t)(t_1 - t)^2 + \frac{1}{3!}\dddot{w}(t)(t_1 - t)^3 + \cdots. \quad (20)$$

The dots denote time derivatives; for simplicity we shall suppress the argument t from now on. Putting this into Eq. (9):

$$d = (\dot{w} - c)(t_1 - t) + \frac{1}{2}\ddot{w}(t_1 - t)^2 + \frac{1}{3!}\dddot{w}(t_1 - t)^3 + \cdots. \quad (21)$$

Equation (21) can be solved for $(t_1 - t)$ by reverting the series[15]:

$$(t_1 - t) = A_1 d + A_2 d^2 + A_3 d^3 + \cdots \quad (22)$$

with

$$A_1 = \frac{1}{(\dot{w} - c)}; \qquad A_2 = \frac{-\ddot{w}}{2(\dot{w} - c)^3};$$

$$A_3 = \frac{3\ddot{w}^2 - (\dot{w} - c)\dddot{w}}{6(\dot{w} - c)^5}; \textit{etc.} \quad (23)$$

Equation (22) enables us to write the expansion for E_1 in powers of d:

$$E_1 = \frac{e}{2c^2}\left[\frac{(c^2 - \dot{w}^2)}{d^2} - \frac{\ddot{w}}{d} + \left(\frac{\dddot{w}(2c - \dot{w})}{3(c - \dot{w})^2} + \frac{\ddot{w}^2(3c - \dot{w})}{4(c - \dot{w})^3}\right) + (\ )d + (\ )d^2 + \cdots\right]. \quad (24)$$

The corresponding formula for E_2 is obtained by changing the sign of all the w's. Inserting this into (7), finally,

$$F_{\text{self}} = \frac{e^2}{c^2}\left[-\frac{\ddot{w}}{2d} + \frac{c^3}{(c^2 - \dot{w}^2)^3}\left(\frac{\dddot{w}}{3}(c^2 - \dot{w}^2) + \dot{w}\ddot{w}^2\right) + (\ )d + (\ )d^2 + \cdots\right]. \quad (25)$$

We have carried the series to order d^0 only, because our real interest is in the point-charge limit, $d \to 0$. The higher terms are in any event model dependent, in the sense that they apply only to the dumbbell configuration.

MASS RENORMALIZATION

The first term in Eq. (25) is proportional to the acceleration, $\ddot{w}$; if we pull it over to the other side of Newton's Second Law, it simply adds to the dumbbell's mass. In effect, the total inertia of the charged dumbbell is

$$m = 2m_0 + m_{\text{int}}, \quad (26)$$

where m_0 is the mass of either end alone, and

$$m_{\text{int}} = e^2/2dc^2 \quad (27)$$

is the extra contribution due to the electromagnetic interaction between the charges.[16]

In the context of special relativity it is not surprising that the electrical repulsion of the charges should enhance the mass of the dumbbell. For the potential energy of this configuration is (in the static case) $e^2/4d$, and according to the Einstein formula $E = mc^2$, this potential energy should be reflected in the mass of the object. Unfortunately, such reasoning indicates that the mass increase should be $e^2/4dc^2$, disagreeing by a factor of 2 with Eq. (27). This notorious paradox[17] (for a spherical charge the factor is 4/3 instead of 2, but the *cause* is the same) was first resolved by Poincaré,[18] who pointed out that as it stands the particle is unstable: it will fly apart, by virtue of the electrical repulsion of its constituants, unless we provide some *other* force to hold it together. The inclusion of Poincaré's cohesive force does remove the anomaly in the electromagnetic mass, but the argument has never enjoyed universal acceptance, and a quite different explanation of the paradox has been given by Rohrlich and others.[19]

IV. RADIATION REACTION

The second term in Eq. (25) represents the true radiation reaction force; it alone (apart from the mass renormalization) survives in the "point dumbbell" limit $d \to 0$.[20] Let us write this term in a more suggestive form, using $v = \dot{w}$, $a = \ddot{w}$, $\dot{a} = \dddot{w}$, and $\gamma = 1/(1 - v^2/c^2)^{1/2}$:

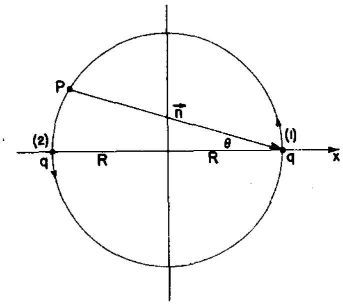

Fig. 3. Rotating dumbbell.

$$F_{\text{rad}}^{\text{int}} = \frac{e^2}{3c^3}\gamma^4\left(\dot{a} + 3\frac{\gamma^2 v a^2}{c^2}\right). \quad (28)$$

Except for an overall factor of 2, this agrees precisely with the relativistic Abraham formula [Eq. (2)], for the case of rectilinear motion. But what about that factor of 2? Simple: we have not yet included the self-force of each end on itself. The *total* radiation reaction, using Eq. (2) for each end (with charge $e/2$), and adding the "interaction term" (28), is

$$F_{\text{rad}} = \left\{2\left[\frac{2}{3}\left(\frac{e}{2}\right)^2\right] + \frac{1}{3}e^2\right\}\frac{\gamma^4}{c^3}\left(\dot{a} + 3\frac{\gamma^2 v a^2}{c^2}\right)$$

$$= \frac{2}{3}\frac{e^2}{c^3}\gamma^4\left(\dot{a} + 3\frac{\gamma^2 v a^2}{c^2}\right). \quad (29)$$

Indeed, if we apply Eq. (28) to a *line* charge, for which the "end" terms do not occur, Eq. (2) emerges automatically with the correct coefficient of 2/3.[21]

It may surprise you that so unlikely a model for a charged particle yields the correct formula for the radiation reaction. The reason is that the limit $d \to 0$ washes out the structure dependence. All *other* terms (even the mass renormalization) are different in the more "realistic" spherical model. But in the point limit *all* models must produce the same answer, since conservation laws alone dictate the result.

Incidentally, for a *dipole*, instead of a dumbbell (that is, *opposite* charges at the two ends) the sign of Eq. (28) is reversed. In this case the "interaction" term and the two "end" terms cancel, so that the radiation reaction is zero in the limit $d \to 0$—as of course it should be. The lowest-order contribution to the self-force on a dipole comes from the d^1 term in Eq. (25).[22]

V. SELF-TORQUE ON A ROTATING DUMBBELL

Suppose now that instead of moving along the x axis, our dumbbell is *rotating* counterclockwise at constant angular velocity ω. (For this problem let the charge at either end be q, and the separation distance $2R$.) We wish to calculate the self-torque on this object; we may as well do it at time $t = 0$, when the dumbbell is coincident with the x axis. By symmetry, the total torque is twice the torque on particle (1). Now the electromagnetic information which reaches (1) at $t = 0$ left (2) at a somewhat earlier time, when it was at point P, an angle θ up from the axis as shown in Fig. 3. In terms of the Cartesian unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$, we have:

$$\hat{n} = \cos\theta\,\hat{i} - \sin\theta\,\hat{j}, \quad (30)$$

while $n = 2R\cos\theta$. Meanwhile, the velocity of (2) at P was

$$\mathbf{v} = -\omega R(\sin 2\theta\,\hat{i} + \cos 2\theta\,\hat{j}) \quad (31)$$

and its acceleration was

$$\mathbf{a} = \omega^2 R(\cos 2\theta\,\hat{i} - \sin 2\theta\,\hat{j}). \quad (32)$$

With this it is a straightforward matter, using Eq. (5), to compute the electric field at particle (1) due to particle (2):

$$\mathbf{E} = \frac{q}{(2R\cos\theta)^2(c + \omega R\sin\theta)^3}$$
$$\times \{c\cos\theta(c^2 + 2c\omega R\sin\theta - \omega^2R^2\cos 2\theta)\hat{i}$$
$$+ [-c^3\sin\theta + c^2\omega R\cos 2\theta$$
$$+ c\omega^2R^2\sin\theta\,(2 + \cos 2\theta) + \omega^3R^3]\hat{j}\}. \quad (33)$$

The torque on the dumbbell, then, is $\mathbf{N} = 2(R\hat{i} \times q\mathbf{E})$, or

$$\mathbf{N} = \frac{q^2}{2R\cos^2\theta(c + \omega R\sin\theta)^3}[-c^3\sin\theta + c^2\omega R\cos 2\theta$$
$$+ c\omega^2R^2\sin\theta(2 + \cos 2\theta) + \omega^3R^3]\hat{k}. \quad (34)$$

[Inasmuch as $\mathbf{E}$ and $\hat{n}$ lie in the xy plane, $\mathbf{B} = \hat{n} \times \mathbf{E}$ is parallel to the z axis. Consequently, the *magnetic* force on (1), $(\mathbf{v}/c) \times \mathbf{B}$, points in the radial direction, and contributes nothing to the torque.]

Now the angle θ is determined by the retarded time condition, Eq. (4):

$$-ct_0 = n, \quad (35)$$

where t_0 is the (negative) time when particle (2) was at P. Since it will rotate through angle 2θ by time $t = 0$, $2\theta = -\omega t_0$, and hence:

$$\theta = \beta\cos\theta, \quad (36)$$

where $\beta = \omega R/c$. Equation (36) implicitly determines θ; before proceeding, let us exploit it to eliminate the trigonometric quantities in Eq. (34):

$$\mathbf{N} = \frac{q^2\beta}{2R\theta^2[1 + (\beta^2 - \theta^2)^{1/2}]^3}[(\beta^4 - \beta^2 + 2\theta^2)$$
$$+ (\beta^2 - \theta^2)^{1/2}(\beta^2 + 2\theta^2 - 1)]\hat{k}. \quad (37)$$

Finally, we must eliminate θ, which [according to Eq. (36)] is a function of β. Of course, it is not possible to *solve* Eq. (36) in closed form, so we must resort to a series solution, bearing in mind that for nonrelativistic velocities $\beta \ll 1$.

$$\beta = \theta\sec\theta = \theta + \frac{\theta^3}{2} + \frac{5}{24}\theta^5 + \frac{61}{720}\theta^7 + \cdots. \quad (38)$$

Reverting this series, we find

$$\theta = \beta - \frac{1}{2}\beta^3 + \frac{13}{24}\beta^5 - \frac{541}{720}\beta^7 + \cdots. \quad (39)$$

Inserting this into Eq. (37) yields

$$\mathbf{N} = \frac{4}{3}\frac{q^2\beta^3}{R}\left(1 - \frac{14}{5}\beta^2 + \cdots\right)\hat{k}. \quad (40)$$

It may come as a surprise that the dominant term in this torque is in such a direction as to *increase* the angular velocity. But, of course, this is not the whole story, for once again we must include the torque attributable to the self-force of each end *on itself*. Referring back to Eq. (2), we find that the radiative self-force on particle (1) at $t = 0$ is[23]

$$\mathbf{F}_{\rm rad} = -(2/3)(q^2/c^3)\gamma^4 R\omega^3 \hat{j}, \qquad (41)$$

so the torque resulting from the radiation reaction on the two ends is

$$\mathbf{N}^{\rm ends} = -(4/3)(q^2\beta^3/R)\gamma^4\hat{k}. \qquad (42)$$

When this is combined with the "interaction" torque (40), perfect cancellation occurs in lowest order, and the leading nonzero term becomes

$$\mathbf{N}_{\rm self} = -(32/5)(q^2\beta^5/R)\hat{k}. \qquad (43)$$

For a rotating *dipole* (oppositely charged ends) the sign of the "interaction" torque (40) is reversed, and the total self-torque is much greater:

$$\mathbf{N}_{\rm self}^{\rm dip} = -(8/3)(q^2\beta^3/R)\hat{k}. \qquad (44)$$

This formula can be checked by computing the fields of a rotating dipole, and integrating the Poynting vector over a large sphere, to find the rate of energy loss. The enhancement of the dipole self-torque reflects the greater efficiency of dipole, as opposed to quadrupole, radiation.

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[16]The electromagnetic contribution to the mass of a charged particle was first discussed by J. J. Thomson, Ref. 10.

[17]See F. Rohrlich, Ref. 5, Chap. 2, for a historical survey.

[18]H. Poincaré, Rend. Circ. Mat. Palermo **21**, 129 (1906). For a recent discussion see G. Horwitz and J. Katz, Nuovo Cimento B **3**, 245 (1971).

[19]For history and bibliography, see F. Rohrlich, Ref. 5, and Am. J. Phys. **38**, 1310 (1970). A delightful and illuminating discussion is given by A. Gamba, Am. J. Phys. **35**, 83 (1967).

[20]The limit $d \to 0$ has an embarrassing effect on the electromagnetic mass (27). Of course, one can say that this term is not separately observable; only the *total* mass (26) is of dynamical significance, and perhaps m_0 contains a compensating infinity. But no one could call this a pretty argument. For an interesting commentary, see R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1964), Sec. II-28.

[21]If a uniform line charge λ of length L is divided into equal infinitesimal segments dx, the radiation reaction for a given pair is obtained from (28) by the substitution $(e/2)^2 \to (\lambda dx_1)(\lambda dx_2)$. Integrating dx_1 and dx_2 from 0 to L, and attaching an overall factor of $1/2$, to correct for counting the same pair twice, we obtain $(\lambda L)^2/2 = e^2/2$. The self-force on each segment is a second-order differential, $(\lambda dx)^2$, and contributes nothing in the limit. We thank Brooke Gregory for suggesting this approach.

[22]This term has been calculated by E. W. Szeto, senior honors thesis (Mount Holyoke College, 1977) (unpublished).

[23]For an *ab initio* derivation of this formula, in the nonrelativistic limit, see T. H. Boyer, Am. J. Phys. **40**, 1843 (1972).