Mass renormalization in classical electrodynamics

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The electromagnetic mass of a charged object can be calculated by three independent methods: (1) $m_u = u/c^2$, where u is the field energy of the object at rest, (2) $m_p = p/v$, where p is the field momentum when the particle is moving at speed v, and (3) $m_s = F/a$, where F is the self-force when the object has acceleration a. In the context of a simple dumbbell model we demonstrate that $m_p = m_s$, but these in general exceed m_u.

I. INTRODUCTION

In classical electrodynamics the rest mass of a charged particle is greater than that of its uncharged twin; its total mass is the sum of its "mechanical" mass and its "electromagnetic" mass

$$m = m_{\text{mech}} + m_{\text{em}}. \tag{1}$$

The latter term reflects the inertia of the particle's electromagnetic field. There are at least three ways[1] that one might go about calculating this electromagnetic mass for a particle of specified size, shape, and charge: (1) With the particle at rest, find the electrostatic energy (u) and divide by c^2:

$$m_u = u/c^2. \tag{2}$$

(Alternatively, with the particle in motion, we would compute the total electromagnetic energy and divide by γc^2.) We shall call m_u the "energy-derived mass" of the particle. (2) With the particle moving at constant speed v, find the electromagnetic momentum p and divide by v (or, for high velocities, by γv):

$$m_p = p/v. \tag{3}$$

We shall call m_p the "momentum-derived mass." (3) Recognizing that, from a dynamical point of view, the electromagnetic inertia is due to the force exerted on the particle by its own fields, we evaluate this "self-force" F_s, pick out the piece proportional to acceleration (call this piece F), and divide by a:

$$m_s = F/a. \tag{4}$$

(If the particle is moving at high speed, in rectilinear motion,[2] divide by $\gamma^3 a$.) We call this the "self-force-derived mass."

The first method exploits Einstein's formula $E = mc^2$ (or, for high speeds, $E = \gamma mc^2$); the second uses the equation for momentum, $p = mv$ (or, for high speeds, $p = \gamma mv$); the third relies on Newton's second law $F = ma$ (or, for high speeds, $F = \gamma^3 ma$). Offhand, one would certainly expect the three approaches to yield the same answer.[3] But it is a notorious paradox, discovered around the turn of the century,[4] that m_u *and* m_p do *not* in general agree (for a spherical charge distribution they differ by a factor of 4/3)—not, at any rate, if we calculate u and p by the standard rules to be found in any textbook on electromagnetism. Less well known is the fact that m_s agrees with m_p:

$$m_s = m_p \neq m_u. \tag{5}$$

Poincaré[5] attributed the discrepancy to an incompleteness in the model: a charged particle requires the intervention of some *non*electromagnetic force to hold it together—as we "turn on" the charge, we must at the same time turn on this so-called "Poincaré stress" to keep the particle from flying apart. Now this "other" force, whatever its physical nature, will also contribute to the energy, the momentum, and hence the mass, of the particle. Poincaré showed that when such stabilizing forces are included the anomaly disappears.

But Poincaré's resolution has not enjoyed universal acceptance, and over the years a number of authors, from Fermi[6] to Rohrlich,[7] have looked for an answer within electrodynamics itself. They point out that the standard method for calculating electromagnetic energy and momentum, which amounts to integrating the first four elements of the stress tensor $T^{\mu\nu}$ over all space,

$$p^\mu = \int T^{\mu 0}\, dV, \tag{6}$$

yields a 4-vector only when $T^{\mu\nu}$ is divergenceless. At the location of a charge, the electromagnetic stress tensor is *not* divergenceless, and so Eq. (6) cannot be regarded as a proper definition of electromagnetic energy and momentum. ($T^{\mu 0}$ represents the energy and momentum density only in charge-free regions; it is fine for studying radiation at large distances from the source, but unacceptable for calculating the total energy and momentum of a particle.) These authors typically retain $\int T^{00}\, dV$ to define the electrostatic energy in the particle's rest frame, and derive the energy and momentum in any other frame by Lorentz transformation.

In spite of much argument,[8] there is no real incompatibility between Poincaré's resolution and Rohrlich's. The instability of a purely electromagnetic particle and the nonconversation of the stress tensor are in fact two sides of the same coin. Poincaré adds a nonelectromagnetic term to make $T^{\mu\nu}$ divergenceless, Rohrlich sticks with the electromagnetic part and modifies the procedure for calculating momentum. If we insist on defining a electromagnetic energy-momentum 4-vector, then Rohrlich's is the remedy of choice; if we are prepared to forego this, in favor of the *total* energy-momentum 4-vector, and we are willing to include the nonelectromagnetic forces which must, after all, be present to stabilize the particle, then Poincaré's is the appropriate prescription.

Neither of them, however, has anything to say about the self-force-derived mass. Why should m_s agree with m_p, and not with m_u? The answer is far from obvious, though it is suggestive that m_p and m_s can be given a purely classical formulation, whereas m_u (depending as it does on $E = mc^2$) is explicitly relativistic. If, following Rohrlich, we modify p to bring m_p in line with m_u, m_s is unaffected, and the paradox remains. If, on the other hand, we introduce Poincaré

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stress, it is unclear what this will do to m_s.

Our purpose in this paper is to illuminate these issues by studying the simplest possible extended charge distribution: a dumbbell, consisting of two point charges a short distance apart. The virtue of the dumbbell model[9] is that the calculations are, for the most part, transparent. Moreover, since an arbitrary distribution may be assembled out of interacting pairs of point charges, one can easily generalize the dumbbell results to obtain the electromagnetic mass of a spherical shell, or any other configuration of interest. In Sec. II we calculate the energy-derived mass of a dumbbell, and in Sec. III we work out its momentum-derived mass. Section IV is devoted to the self-force-derived mass of the dumbbell. In Sec. V we exploit these results to determine m_u, m_p, and m_s for a spherical shell, and conclude with some general observations regarding the three electromagnetic masses.[10]

II. ENERGY-DERIVED MASS OF A DUMBBELL

The dumbbell at rest consists of two equal charges, $q_1 = q_2 = e/2$, a distance d apart. The electrostatic energy of such a configuration (in Gaussian units) is

$$u = q_1 q_2 / (|\mathbf{r}_1 - \mathbf{r}_2|) = e^2/4d, \tag{7}$$

and hence

$$m_u = e^2/4dc^2. \tag{8}$$

This result can also be obtained from the general formula[11]

$$u_{\text{total}} = \frac{1}{8\pi} \int E^2 \, dV. \tag{9}$$

The dumbbell's electric field consists of two parts, due to q_1 and q_2, respectively:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2, \tag{10}$$

where (see Fig. 1)

$$\mathbf{E}_1 = (q_1/r_1^3)\mathbf{r}_1, \quad \mathbf{E}_2 = (q_2/r_2^3)\mathbf{r}_2. \tag{11}$$

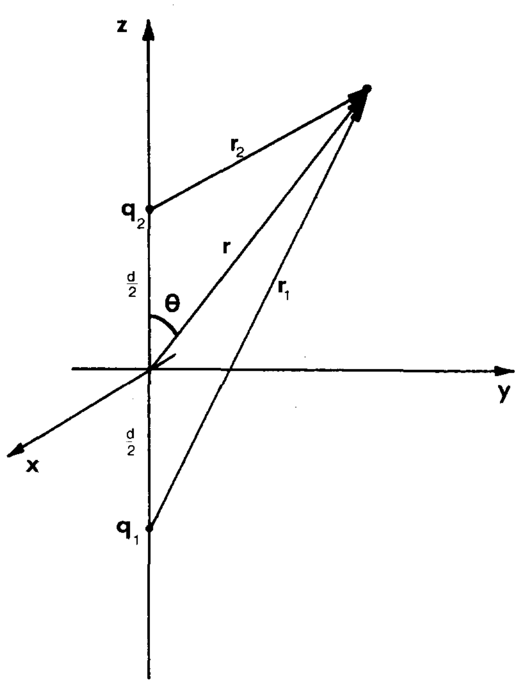

Fig. 1. The dumbbell. Diagram defines the various quantities in Eq. (11).

Thus

$$E^2 = E_1^2 + E_2^2 + 2\mathbf{E}_1 \cdot \mathbf{E}_2. \tag{12}$$

However, we are not concerned here with the first two terms, which correspond to the energy of q_1 alone and q_2 alone (both of which are actually *infinite* for point charges). Rather, we want the *energy of interaction*, which derives from the $\mathbf{E}_1 \cdot \mathbf{E}_2$ term. So for our purposes

$$u = \frac{1}{4\pi} \int \mathbf{E}_1 \cdot \mathbf{E}_2 \, dV. \tag{13}$$

Using spherical coordinates as indicated in Fig. 1, we have

$$\begin{aligned} \mathbf{r}_1 &= (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r\cos\theta + d/2), \\ \mathbf{r}_2 &= (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r\cos\theta - d/2), \\ dV &= r^2 \sin\theta \, dr \, d\theta \, d\phi, \end{aligned} \tag{14}$$

and hence

$$u = \frac{e^2}{16\pi} \int \frac{r^2 - d^2/4}{[(r^2 + d^2/4)^2 - (rd\cos\theta)^2]^{3/2}} \times r^2 \sin\theta \, dr \, d\theta \, d\phi. \tag{15}$$

Integrating ϕ (from 0 to 2π), θ (from 0 to π), and r (from 0 to ∞), we recover Eq. (7).

III. MOMENTUM-DERIVED MASS OF A DUMBBELL

If the dumbbell is now set in motion, at constant velocity $\mathbf{v}$, its fields carry momentum[12] as well as energy:

$$\mathbf{p} = \frac{1}{4\pi c} \int (\mathbf{E} \times \mathbf{B}) dV. \tag{16}$$

Here $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ and $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$, so that

$$\mathbf{E} \times \mathbf{B} = (\mathbf{E}_1 \times \mathbf{B}_1) + (\mathbf{E}_2 \times \mathbf{B}_2) + (\mathbf{E}_1 \times \mathbf{B}_2) + (\mathbf{E}_2 \times \mathbf{B}_1). \tag{17}$$

As before, we are not interested in the first two terms, which correspond to the (infinite) momentum of either end alone—it is the *interaction momentum* $(\mathbf{E}_1 \times \mathbf{B}_2) + (\mathbf{E}_2 \times \mathbf{B}_1)$ that concerns us here. Now, the magnetic field of a point charge in motion is given by[13]

$$\mathbf{B} = (\mathbf{v}/c) \times \mathbf{E}, \tag{18}$$

so

$$\begin{aligned} \mathbf{E}_1 \times \mathbf{B}_2 &= (1/c)\mathbf{E}_1 \times (\mathbf{v} \times \mathbf{E}_2) \\ &= (1/c)[\mathbf{v}(\mathbf{E}_1 \cdot \mathbf{E}_2) - \mathbf{E}_2(\mathbf{v} \cdot \mathbf{E}_1)]. \end{aligned} \tag{19}$$

Thus for our purposes

$$\mathbf{p} = \frac{1}{4\pi c^2} \int [2\mathbf{v}(\mathbf{E}_1 \cdot \mathbf{E}_2) - \mathbf{E}_1(\mathbf{v} \cdot \mathbf{E}_2) - \mathbf{E}_2(\mathbf{v} \cdot \mathbf{E}_1)] dV. \tag{20}$$

For nonrelativistic velocities ($v \ll c$) the electric fields are just the Coulomb fields of Eq. (11). (There are relativistic corrections,[14] but they are second order in v/c.) Accordingly,

$$\mathbf{p} = \frac{e^2}{16\pi c^2} \int \frac{1}{(r_1 r_2)^3} [2\mathbf{v}(\mathbf{r}_1 \cdot \mathbf{r}_2) - \mathbf{r}_1(\mathbf{v} \cdot \mathbf{r}_2) - \mathbf{r}_2(\mathbf{v} \cdot \mathbf{r}_1)] dV, \tag{21}$$

where $\mathbf{r}_1$, $\mathbf{r}_2$, and dV are given by Eq. (14).

A. Longitudinal motion

If $\mathbf{v}$ points along the axis of the dumbbell ($\mathbf{v} = v\hat{z}$, in Fig. 1), Eq. (21) reduces to

$$\mathbf{p} = \frac{e^2\mathbf{v}}{8\pi c^2}\int \frac{r^2\sin^2\theta}{[(r^2+d^2/4)^2-(rd\cos\theta)^2]^{3/2}} \times r^2\sin\theta\,dr\,d\theta\,d\phi, \tag{22}$$

and we obtain, upon performing the integrals,

$$\mathbf{p} = (e^2/2c^2d)\mathbf{v}. \tag{23}$$

The corresponding momentum-derived mass is

$$m_p = e^2/2c^2d, \tag{24}$$

which is *twice* the energy-derived mass.

B. Transverse motion

If the velocity is perpendicular to the dumbbell axis (we may as well take $\mathbf{v} = v\hat{x}$, in Fig. 1), Eq. (21) yields

$$\mathbf{p} = \frac{e^2\mathbf{v}}{8\pi c^2}\int \frac{r^2(1-\sin^2\theta\cos^2\phi)-d^2/4}{[(r^2+d^2/4)^2-(rd\cos\theta)^2]^{3/2}} \times r^2\sin\theta\,dr\,d\theta\,d\phi, \tag{25}$$

leading to

$$\mathbf{p} = (e^2/4c^2d)\mathbf{v}, \tag{26}$$

and hence

$$m_p = e^2/4c^2d. \tag{27}$$

Evidently the momentum-derived mass depends on the direction of the velocity, and for transverse motion it agrees with the energy-derived mass.

C. Arbitrary motion

If $\mathbf{v}$ makes an angle ψ with the dumbbell axis [$\mathbf{v} = v(\sin\psi\hat{x} + \cos\psi\hat{z})$ in Fig. 1], Eq. (21) gives

$$\mathbf{p} = \frac{e^2 v}{8\pi c^2}\left(\sin\psi\hat{x}\int\frac{r^2(1-\sin^2\theta\cos^2\phi)-d^2/4}{(r_1r_2)^3}dV + \cos\psi\hat{z}\int\frac{r^2\sin^2\theta}{(r_1r_2)^3}dV\right), \tag{28}$$

and hence

$$\mathbf{p} = (e^2v/4c^2d)(\sin\psi\hat{x} + 2\cos\psi\hat{z}), \tag{29}$$

or, letting $\hat{d}$ be a unit vector along the dumbbell axis

$$\mathbf{p} = (e^2/4c^2d)[\mathbf{v} + (\mathbf{v}\cdot\hat{d})\hat{d}\,]. \tag{30}$$

The momentum is not in general parallel to the velocity; the components parallel and perpendicular to $\mathbf{v}$ are

$$p_\parallel = (e^2v/4c^2d)(1+\cos^2\psi);$$
$$p_\perp = (e^2v/4c^2d)\sin\psi\cos\psi, \tag{31}$$

so it is only for longitudinal and transverse motion that the notion of momentum-derived mass carries unambiguous meaning.

IV. SELF-FORCE-DERIVED MASS OF A DUMBBELL

When the dumbbell *accelerates*, the electromagnetic force of q_1 on q_2 is not equal and opposite to the force of q_2 on q_1; there is a net force *of the dumbbell on itself*, which we call $\mathbf{F}_{\text{self}}$. Our purpose in this section is to calculate $\mathbf{F}_{\text{self}}$, and obtain from it the "self-force-derived mass," m_s. Now, the electric field of a charge q in arbitrary motion is given by[15]

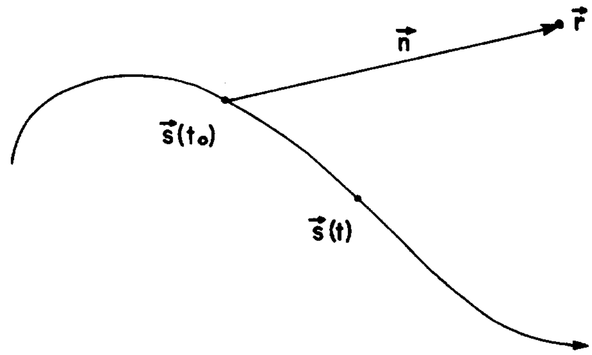

Fig. 2. Retarded position. Fields at $\mathbf{r}$ are determined by the status of the charge at the retarded time t_0, when it was at $\mathbf{s}\,(t_0)$.

$$\mathbf{E}(\mathbf{r},t) = [qn/(\mathbf{n}\cdot\mathbf{m})^3][(c^2-v^2)\mathbf{m} + \mathbf{n}\times(\mathbf{m}\times\mathbf{a})], \tag{32}$$

where $\mathbf{v}$ (the particle's velocity), $\mathbf{a}$ (its acceleration), $\mathbf{s}$ (its position), $\mathbf{n} = (\mathbf{r}-\mathbf{s})$, and $\mathbf{m} = (c\hat{n} - \mathbf{v})$, are all to be evaluated at the retarded time t_0, defined implicitly by

$$c(t-t_0) = |\mathbf{r} - \mathbf{s}(t_0)| = n \tag{33}$$

(see Fig. 2). The magnetic field, meanwhile, is $\mathbf{B}(\mathbf{r},t) = \hat{n}\times\mathbf{E}$.

A. Longitudinal motion

Suppose the dumbbell is constrained to move along the line of its axis. Let $\mathbf{E}_1(t)$ be the electric field of q_1 at the location of q_2, and $\mathbf{E}_2(t)$ the field of q_2 at q_1. In this case $\mathbf{n}$, $\mathbf{m}$, $\mathbf{v}$, and $\mathbf{a}$, all lie along the z axis, so that the second term in (32) vanishes, and

$$\hat{n}_{1,2} = \pm\hat{z}, \quad \mathbf{m}_{1,2} = (\pm c - v)\hat{z}.$$

Meanwhile, from (33),

$$n_{1,2} = c(t - t_{1,2}),$$

where t_1 and t_2 are the respective retarded times. Thus Eq. (32) reduces to

$$\mathbf{E}_1(t) = \frac{e}{2c^2(t-t_1)^2}\left(\frac{c+v_1(t_1)}{c-v_1(t_1)}\right)\hat{z},$$
$$\mathbf{E}_2(t) = -\frac{e}{2c^2(t-t_2)^2}\left(\frac{c-v_2(t_2)}{c+v_2(t_2)}\right)\hat{z}, \tag{34}$$

where $v_1(t_1)$ is the velocity of end (1) at time t_1, and $v_2(t_2)$ is the velocity of end (2) at time t_2. If $z(t)$ is the position of the center of the dumbbell at time t, then the ends are at $z(t) \pm d/2$, so the retarded time condition (33) becomes

$$c(t-t_1) = |[z(t)+d/2] - [z(t_1)-d/2]| = z(t) - z(t_1) + d,$$
$$c(t-t_2) = |[z(t)-d/2] - [z(t_2)+d/2]| = z(t_2) - z(t) + d. \tag{35}$$

In writing Eq. (35) we assume the dumbbell is rigid *in the lab frame*, with a fixed length d. The model is to this extent nonrelativistic, since it makes reference to a particular inertial frame. A fully relativistic treatment is given in the Appendix. Note that the magnetic field on the axis is zero. The self-force on the dumbbell is therefore

$$\mathbf{F}_{\text{self}} = (e/2)(\mathbf{E}_1 + \mathbf{E}_2). \quad (36)$$

Unfortunately, the retarded time condition [Eq. (35)] cannot be solved in closed form. However, it is easy to obtain the general solution as a power series in d:

$$t_1 = t - \frac{d}{(c-v)} + \frac{ad^2}{2(c-v)^3} + (\;)d^3 + \cdots,$$
$$t_2 = t - \frac{d}{(c+v)} - \frac{ad^2}{2(c+v)^3} + (\;)d^3 + \cdots, \quad (37)$$

where v and a are the velocity and acceleration of the center at time t: $v = dz(t)/dt$, $a = d^2z(t)/dt^2$. This leads to a corresponding expansion of the fields

$$\mathbf{E}_1 = \frac{e}{2c^2}\left(\frac{(c^2-v^2)}{d^2} - \frac{a}{d} + (\;)d^\circ + (\;)d + \cdots\right)\hat{z},$$
$$\mathbf{E}_2 = -\frac{e}{2c^2}\left(\frac{(c^2-v^2)}{d^2} + \frac{a}{d} + (\;)d^\circ + (\;)d + \cdots\right)\hat{z}. \quad (38)$$

The self-force on the dumbbell is therefore

$$\mathbf{F}_{\text{self}} = [-(e^2/2c^2d)a + (\;)d^\circ + (\;)d + \cdots]\hat{z}. \quad (39)$$

The leading term here is proportional to the acceleration[16]; in Newton's second law

$$\mathbf{F}_{\text{external}} + \mathbf{F}_{\text{self}} = m_{\text{mech}}\,\mathbf{a}, \quad (40)$$

it is natural to pull this term over to the right-hand side, where it has the effect of adding to the mass an amount

$$m_s = e^2/2c^2d. \quad (41)$$

Evidently, for longitudinal motion the self-force-derived mass agrees with the momentum-derived mass [Eq. (24)], but not the energy-derived mass [Eq. (8)].

B. Transverse motion

Suppose now that the motion is *perpendicular* to the axis of the dumbbell[17]—say, v is along the x direction in Fig. 1. The magnetic forces are in the z direction, and they cancel. By symmetry, the net self-force is in the x direction, and the two ends contribute equally, so we need only calculate the x component of the electric field at q_2, due to q_1. If $x(t)$ is the position of the center of the dumbbell, at time t, then end (2) is at

$$\mathbf{r}(t) = x(t)\hat{x} + d/2\hat{z},$$

while, at the retarded time t_1, end (1) was at

$$\mathbf{s}(t_1) = x(t_1)\hat{x} - d/2\hat{z}.$$

So the terms in Eq. (32) are

$$\mathbf{n} = [x(t) - x(t_1)]\hat{x} + d\hat{z},$$
$$n = c(t - t_1) = \{[x(t) - x(t_1)]^2 + d^2\}^{1/2},$$
$$\mathbf{v} = v(t_1)\hat{x},$$
$$\mathbf{a} = a(t_1)\hat{x},$$
$$\mathbf{m} = \frac{[x(t) - x(t_1) - (t-t_1)v(t_1)]\hat{x} + d\hat{z}}{(t-t_1)},$$
$$\mathbf{n}\cdot\mathbf{m} = c^2(t-t_1) - [x(t) - x(t_1)]v(t_1),$$

and hence

$$(E_1)_x = \frac{e}{2}c\left(\frac{[c^2 - v(t_1)^2][x(t) - x(t_1) - (t-t_1)v(t_1)] - d^2a(t_1)}{\{c^2(t-t_1) - [x(t) - x(t_1)]v(t_1)\}^3}\right). \quad (42)$$

In this case the retarded time condition (33) reduces to

$$c^2(t-t_1)^2 = [x(t) - x(t_1)]^2 + d^2. \quad (43)$$

The net self-force on the dumbbell is then

$$\mathbf{F}_{\text{self}} = 2(e/2)(E_1)_x\hat{x}. \quad (44)$$

As before, we solve the retarded time condition [Eq. (43)] in the form of a power series in d:

$$t_1 = t - \frac{1}{\sqrt{c^2-v^2}}d + \frac{va}{2(c^2-v^2)}d^2 + (\;)d^3 + \cdots, \quad (45)$$

where v and a are the velocity and acceleration of the center at time t: $v = dx(t)/dt$, $a = d^2x(t)/dt^2$. The resulting expansion of $(E_1)_x$ is

$$(E_1)_x = -\frac{e}{2}c\frac{a}{2(c^2-v^2)^{3/2}}\frac{1}{d} + (\;)d^\circ + (\;)d^2 + \cdots, \quad (46)$$

and the self-force on the dumbbell is[16]

$$\mathbf{F}_{\text{self}} = [-\gamma^3(e^2/4c^2d)a + (\;)d^\circ + (\;)d + \cdots]\hat{x}. \quad (47)$$

Evidently the self-force-derived mass for the case of transverse motion is

$$m_s = \gamma^3(e^2/4c^2d). \quad (48)$$

All of our other calculations have been nonrelativistic, and it comes as a surprise to see the correct relativistic factor of γ^3 emerge at this stage. But the reason is simple[18]: The transverse dumbbell is in fact a fully relativistic model, whereas the longitudinal dumbbell, as presented here, is not. For in the longitudinal case we did not take into account the Lorentz contraction of the dumbbell, while in the transverse case there *is* no contraction. In the Appendix we develop the relativistic version of the dumbbell in longitudinal motion, and the appropriate factor of γ^3 then emerges naturally. For the moment, though, we want the nonrelativistic limit of Eq. (48):

$$m_s = e^2/4c^2d. \quad (49)$$

In transverse motion, then, the energy-derived mass [Eq. (8)], momentum-derived mass [Eq. (27)], and self-force-derived mass [Eq. (49)] are all equal.

C. Arbitrary motion

Finally, suppose the motion is along a line which makes an angle ψ with the dumbbell axis, so that the center is at

$$\mathbf{s}(t) = r(t)(\sin\psi\hat{x} + \cos\psi\hat{z}) \quad (50)$$

in Fig. 1 and the two ends are at

$$\mathbf{s}_1(t) = r(t)\sin\psi\hat{x} + [r(t)\cos\psi - d/2]\hat{z},$$
$$\mathbf{s}_2(t) = r(t)\sin\psi\hat{x} + [r(t)\cos\psi + d/2]\hat{z}. \quad (51)$$

The retarded times t_1 and t_2 are determined by the conditions

$$c^2(t-t_1)^2 = [\mathbf{s}_2(t) - \mathbf{s}_1(t_1)]^2$$
$$= [r(t) - r(t_1)]^2 + 2[r(t) - r(t_1)]\cos\psi d + d^2,$$
$$c^2(t-t_2)^2 = [\mathbf{s}_1(t) - \mathbf{s}_2(t_2)]^2$$
$$= [r(t) - r(t_2)]^2 - 2[r(t) - r(t_2)]\cos\psi d + d^2. \quad (52)$$

To simplify the algebra at no real cost, let us assume the dumbbell is instantaneously at rest, at time t. The series solution to Eq. (52) then takes the form

$$t_1 = t - (1/c)d + (a\cos\psi/2c^3)d^2 + (\ )d^3 + \cdots,$$
$$t_2 = t - (1/c)d - (a\cos\psi/2c^3)d^2 + (\ )d^2 + \cdots, \quad (53)$$

where $a = d^2r/dt^2$ is the acceleration of the dumbbell at time t. Expanding Eq. (32) in powers of d, we obtain

$$\mathbf{E}_1(t) = \frac{e}{2}\left(\frac{1}{d^2}\hat{z} - \frac{a}{2c^2d}(\sin\psi\hat{x} + 2\cos\psi\hat{z}) + (\ )d^\circ + (\ )d^1 + \cdots\right), \quad (54)$$

$$\mathbf{E}_2(t) = \frac{e}{2}\left(-\frac{1}{d^2}\hat{z} - \frac{a}{2c^2d}(\sin\psi\hat{x} + 2\cos\psi\hat{z}) + (\ )d^\circ + (\ )d^1 + \cdots\right).$$

Since the charges are at rest, the magnetic forces are zero, and hence

$$\mathbf{F}_{\text{self}} = (e/2)(\mathbf{E}_1 + \mathbf{E}_2)$$
$$= -(e^2a/4c^2d)(\sin\psi\hat{x} + 2\cos\psi\hat{z}) + (\ )d^\circ + (\ )d^1 + \cdots. \quad (55)$$

We are concerned here with the term which is proportional to a (call it $\mathbf{F}$, for short)[16]:

$$\mathbf{F} = -(e^2a/4c^2d)(\sin\psi\hat{x} + 2\cos\psi\hat{z}), \quad (56)$$

or, letting $\hat{d}$ be a unit vector along the dumbbell axis,

$$\mathbf{F} = -(e^2/4c^2d)[\mathbf{a} + (\mathbf{a}\cdot\hat{d})\hat{d}]. \quad (57)$$

$\mathbf{F}$ does not in general lie along the direction of motion; the components parallel and perpendicular to $\mathbf{a}$ are

$$F_\parallel = -(e^2a/4c^2d)(1+\cos^2\psi);$$
$$F_\perp = -(e^2a/4c^2d)\sin\psi\cos\psi. \quad (58)$$

Notice that the dependence of F on a is identical to that of p on v [Eqs. (29)–(31)]. Only $F_\parallel$ can be regarded as a purely inertial term; accordingly

$$m_s = (e^2/4c^2d)(1+\cos^2\psi). \quad (59)$$

V. ELECTROMAGNETIC MASS OF A SPHERICAL SHELL

Given the formulas for energy, momentum, and self-force of a dumbbell, it is a straightforward matter to generalize to arbitrary rigid charge distributions: we simply chop the configuration up into pairs of infinitesimal charges and integrate (dividing by 2 to avoid counting the same pair twice). For example, suppose we have a spherical shell of radius R carrying a uniform surface charge density σ. We divide it into patches of area da (Fig. 3), so that the charge at each end of the "dumbbell" is $e/2 \to \sigma\, da$.

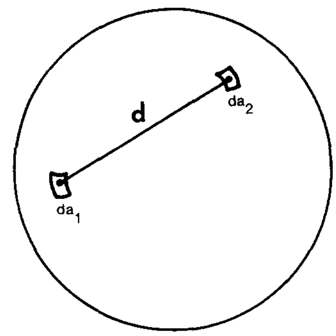

Fig. 3. Spherical shell. Surface is divided into infinitesimal patches, and the dumbbell results are applied to each pair.

A. Energy

From Eq. (7) we have

$$u = \frac{\sigma^2}{2}\int\int\frac{1}{d}\,da_1da_2 = \frac{\sigma^2}{2}\int\left(\int\frac{1}{d}\,da_1\right)da_2. \quad (60)$$

We may as well orient our x_1, y_1, z_1 axes so that da_2 lies at the "north pole"; then

$$d = \sqrt{2}R\sqrt{1-\cos\theta_1}, \quad (61)$$

and hence

$$\int\frac{1}{d}\,da_1 = \int\frac{R^2\sin\theta_1}{\sqrt{2}R\sqrt{1-\cos\theta_1}}\,d\theta_1\,d\phi_1 = 4\pi R, \quad (62)$$

so that

$$u = \frac{\sigma^2}{2}(4\pi R)\int da_2 = 8\pi^2\sigma^2R^3. \quad (63)$$

Or, in terms of the total charge $Q = 4\pi R^2\sigma$,

$$u = \tfrac{1}{2}(Q^2/R). \quad (64)$$

This familiar result can of course be obtained by simpler means[19]; we present it here only as an illustration of method. Evidently the energy-derived mass of a spherical shell is

$$m_u = Q^2/2Rc^2. \quad (65)$$

B. Momentum

Suppose now that the sphere is moving at constant speed v in the z direction. Although each infinitesimal dumbbell has a component of momentum perpendicular to the velocity, symmetry dictates that these components cancel when we integrate over the sphere. Accordingly, the total momentum of the sphere [quoting Eq. (31)] is

$$\mathbf{p} = \frac{\sigma^2\mathbf{v}}{2c^2}\int\int\frac{(1+\cos^2\psi)}{d}\,da_1\,da_2, \quad (66)$$

where ψ is the angle between $\mathbf{d}$ and $\mathbf{v}$:

$$\cos^2\psi = d_z^2/d^2. \quad (67)$$

Now, it is clear from symmetry that

$$\int\int\frac{d_x^2}{d^3}\,da_1\,da_2 = \int\int\frac{d_y^2}{d^3}\,da_1\,da_2 = \int\int\frac{d_z^2}{d^3}\,da_1\,da_2, \quad (68)$$

while their *sum* is simply $\int\int(1/d)\,da_1\,da_2$. Evidently, then,

$$\int\int\frac{\cos^2\psi}{d}\,da_1\,da_2 = \frac{1}{3}\int\int\frac{1}{d}\,da_1\,da_2, \quad (69)$$

and hence

$$\mathbf{p} = \frac{4}{3}\frac{\mathbf{v}}{c^2}\frac{\sigma^2}{2}\int\int\frac{1}{d}\,da_1\,da_2 = \frac{4}{3}\frac{\mathbf{v}}{c^2}u. \tag{70}$$

Consequently, the momentum-derived mass of a spherical shell is 4/3 its energy-derived mass[20]:

$$m_p = \frac{4}{3}m_u = \frac{2}{3}\frac{Q^2}{Rc^2}. \tag{71}$$

C. Self-force

The self-force calculation [starting from Eq. (58)] is identical to that for momentum:

$$\mathbf{F} = -\frac{4}{3}(\mathbf{a}/c^2)u, \tag{72}$$

and so

$$m_s = m_p = \frac{4}{3}m_u. \tag{73}$$

VI. CONCLUSION

We have calculated the electromagnetic energy, momentum, and self-force for dumbbell and spherical shell configurations. Each result yields a particular expression for the electromagnetic mass of the object; however, the three methods do not agree. Whereas the momentum- and self-force-derived masses are always equal, they exceed the energy-derived mass by an amount which depends on how the charge is distributed along the line of motion:

$$m_s = m_p \geqslant m_u. \tag{74}$$

Only when the charge is confined to the plane perpendicular to the motion are all three masses in agreement. Moreover, since the momentum is not in general parallel to the velocity, nor the self-force parallel to the acceleration, the corresponding masses lose their scalar character (the sphere is an exception). The discrepancy between m_p and m_u is best resolved by Rohrlich's method,[7] which disqualifies our calculation of **p**. But Rohrlich's procedure does not address the problem of self-force-derived mass, and to our knowledge $m_s \neq m_u$ remains an unsolved paradox.[21]

APPENDIX: LORENTZ-CONTRACTED DUMBBELL

In calculating the self-force on a dumbbell in longitudinal motion we assumed that it is rigid in the "lab" frame—that is, its length is d from *our* point of view. This gives special importance to a particular inertial system, and it is not surprising that the resulting mass [Eq. (41)] is missing the relativistic factor of γ^3. Rigidity is a notoriously slippery concept in relativity, but here we have the advantage that there are only two points to worry about: the two ends of the dumbbell. With this in mind we define the covariant dumbbell as follows: Let $z(t)$ be the position of the "center," and let S' be the inertial system in which the center is instantaneously at rest. *We require that the dumbbell have a length d in S'*. Specifically, at time t_0 (in the lab frame, S) the coordinates of the center are

$$\text{center:}\quad z_0 = z(t_0),\quad t_0. \tag{75}$$

By Lorentz transformation, the coordinates of the center in S' are

$$\text{center:}\quad z_0' = \gamma_0(z_0 - v_0 t_0),\quad t_0' = \gamma_0[t_0 - (v_0/c^2)z_0], \tag{76}$$

where $v_0 = (dz/dt)$ at time t_0, and $\gamma_0 = 1/\sqrt{1 - v_0^2/c^2}$. At this instant (in S') the coordinates of the two ends are

$$\begin{aligned} q_1:&\quad z_1' = z_0' - d/2,\quad t_0', \\ q_2:&\quad z_2' = z_0' + d/2,\quad t_0'. \end{aligned} \tag{77}$$

Transforming back to S, the coordinates are

$$\begin{aligned} q_1:&\quad z_1 = \gamma_0(z_1' + v_0 t_0') = z_0 - \gamma_0(d/2), \\ &\quad t_{(1)} = \gamma_0[t_0' + (v_0/c^2)z_1'] = t_0 - \gamma_0 v_0(d/2c^2), \end{aligned} \tag{78}$$

$$\begin{aligned} q_2:&\quad z_2 = \gamma_0(z_2' + v_0 t_0') = z_0 + \gamma_0(d/2), \\ &\quad t_{(2)} = \gamma_0[t_0' + (v_0/c^2)z_2'] = t_0 + \gamma_0 v_0(d/2c^2). \end{aligned} \tag{79}$$

Equation (78) specifies the position (z_1) of q_1 at lab time $t_{(1)}$. Unfortunately, it is expressed in terms of the parameter t_0. We would like to eliminate t_0 and write z_1 directly as a function of lab time t. To begin with we must solve for t_0 in terms of $t = t_{(1)}$:

$$t = t_0 - \gamma_0 v_0(d/2c^2). \tag{80}$$

As always, we expand in powers of d:

$$(t_0 - t) = \frac{d}{2c^2}\left[\gamma v + (t_0 - t)\frac{d}{dt}(\gamma v) + \cdots\right] \tag{81}$$

(unsubscripted quantities are evaluated at time t). Now

$$\frac{d}{dt}(\gamma v) = \gamma^3 a, \tag{82}$$

so

$$(t_0 - t)[1 - (d/2c^2)\gamma^3 a + \cdots] = (d/2c^2)\gamma v, \tag{83}$$

and hence

$$t_0 = t + (\gamma v/2c^2)d + (\gamma^4 va/4c^4)d^2 + (\ )d^3 + \cdots. \tag{84}$$

This gives us t_0 in terms of t; returning to Eq. (78) we have

$$z_1(t) = z(t_0) - \gamma_0(d/2). \tag{85}$$

But

$$z(t_0) = z + (t_0 - t)v + \tfrac{1}{2}(t_0 - t)^2 a + \cdots \tag{86}$$

and

$$\gamma_0 = \gamma + (t_0 - t)\frac{d\gamma}{dt} + \cdots. \tag{87}$$

Putting this together, and using (84), we conclude

$$z_1(t) = z(t) - \frac{1}{\gamma}\frac{d}{2} + \frac{av^2\gamma^2}{8c^4}d^2 + (\ )d^3 + \cdots. \tag{88}$$

The corresponding formula for the location of q_2 is obtained from (79)—or, more simply, by changing the sign of d in Eq. (88):

$$z_2(t) = z(t) + \frac{1}{\gamma}\frac{d}{2} + \frac{av^2\gamma^2}{8c^4}d^2 + (\ )d^3 + \cdots. \tag{89}$$

Notice that $z_2(t) - z_1(t) = d/\gamma$ to this order, so the dumbbell has the appropriately contracted length in the lab.

The next step is to compute the retarded time t_1, given by

$$\begin{aligned} c(t - t_1) &= z_2(t) - z_1(t_1) \\ &= [z(t) - z(t_1)] + \left(\frac{1}{\gamma} + \frac{1}{\gamma_1}\right)\frac{d}{2} \\ &\quad + (av^2\gamma^2 - a_1 v_1^2\gamma_1^2)(d^2/8c^4) + \cdots. \end{aligned} \tag{90}$$

Now

$$[z(t) - z(t_1)] = (t - t_1)v - \tfrac{1}{2}(t - t_1)^2 a + \cdots,$$

$$\left[\frac{1}{\gamma} + \frac{1}{\gamma_1}\right] = \frac{1}{\gamma} + \left[\frac{1}{\gamma} + (t_1 - t)\frac{d}{dt}\left(\frac{1}{\gamma}\right) + \cdots\right] = \frac{2}{\gamma} + \frac{\gamma v a}{c^2}(t - t_1) + \cdots,$$

and

$$(av^2\gamma^2 - a_1 v_1^2 \gamma_1^2) = (t - t_1)\frac{d}{dt}(av^2\gamma^2) + \cdots. \quad (91)$$

Since $t - t_1$ is of order d, the last term contributes to Eq. (90) only in third order, and can be dropped, leaving

$$c(t - t_1) = v(t - t_1) - \tfrac{1}{2}a(t - t_1)^2 + \frac{d}{\gamma} + \frac{d\gamma v a}{2c^2}(t - t_1) + \cdots. \quad (92)$$

This is a quadratic equation for $(t - t_1)$; expanding the solution in powers of d, we obtain

$$t_1 = t - \frac{1}{\gamma(c - v)}d + \frac{a}{2c(c - v)^2}d^2 + (\ )d^3 + \cdots. \quad (93)$$

To calculate $\mathbf{E}_1(t)$—Eq. (34)—we need to know $v_1(t_1)$, the velocity of end (1) at time t_1. According to (78), the speed of q_1 at any lab time $t_{(1)}$ is

$$v_1(t_{(1)}) = \frac{dz_1}{dt_{(1)}} = \frac{dz_1}{dt_0}\frac{dt_0}{dt_{(1)}}, \quad (94)$$

while

$$\frac{dz_1}{dt_0} = \frac{dz_0}{dt_0} - \frac{d}{2}\frac{d\gamma_0}{dt_0} = v_0\left(1 - \frac{d}{2}\frac{\gamma_0^3 a_0}{c^2}\right), \quad (95)$$

and

$$\frac{dt_{(1)}}{dt_0} = 1 - \frac{d}{2c^2}\frac{d}{dt_0}(\gamma_0 v_0) = \left(1 - \frac{d}{2}\frac{\gamma_0^3 a_0}{c^2}\right). \quad (96)$$

Thus

$$v_1(t_1) = v(\bar{t}_0). \quad (97)$$

Surprisingly, the velocity of end (1) at time t_1 is the same as the velocity of the *center* at time $\bar{t}_0$, defined implicitly by the equation

$$t_1 = \bar{t}_0 - \bar{\gamma}_0 \bar{v}_0 (d/2c^2). \quad (98)$$

The series solution to Eq. (98) [quoting (84)] is

$$\bar{t}_0 = t_1 + \gamma(t_1)v(t_1)(d/2c^2) + (\ )d^2 + \cdots, \quad (99)$$

so

$$v_1(t_1) = v(t_1) + a(t_1)[\gamma(t_1)v(t_1)/2c^2]d + (\ )d^2 + \cdots. \quad (100)$$

But

$$v(t_1) = v + (t_1 - t)a + \cdots = v - [a/\gamma(c - v)]d + \cdots, \quad (101)$$

and hence

$$v_1(t_1) = v - (a\gamma/2c^2)(2c + v)d + (\ )d^2 + \cdots. \quad (102)$$

The resulting expansion of $\mathbf{E}_1(t)$ is

$$\mathbf{E}_1(t) = (e/2d^2)[1 - (a\gamma^3/c^2)d + (\ )d^2 + \cdots]\hat{z}, \quad (103)$$

while the corresponding calculation for $\mathbf{E}_2(t)$ yields

$$\mathbf{E}_2(t) = -(e/2d^2)[1 + (a\gamma^3/c^2)d + (\ )d^2 + \cdots]\hat{z}. \quad (104)$$

Finally, the self-force on the dumbbell is

$$\mathbf{F}_{\text{self}} = (e/2)(\mathbf{E}_1 + \mathbf{E}_2) = [-\gamma^3(e^2/2dc^2)a + \cdots]\hat{z}, \quad (105)$$

and the self-force-derived mass

$$m_s = \gamma^3(e^2/2dc^2) \quad (106)$$

now includes the appropriate factor of γ^3.[22]

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[1]A fourth technique is suggested by the equivalence principle. See T. H. Boyer, Am. J. Phys. **47**, 129 (1979).

[2]For motion in a straight line, the relativistic formula $F = dp/dt$ yields $F = d/dt(\gamma m v) = d\gamma/dt\, mv + \gamma m\, dv/dt = \gamma^3 ma$.

[3]This expectation is skillfully exploited in T. H. Boyer, Am. J. Phys. **46**, 383 (1978). However, in the light of the present paper Boyer was fortunate to have chosen transverse motion.

[4]F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1965), Chap. 2; A. Pais, "The Early History of the Theory of the Electron: 1897–1947," in *Aspects of Quantum Theory*, edited by A. Salam and E. Wigner (Cambridge U. P., London, 1972).

[5]H. Poincaré, Co. R. Acad. Sci. Paris **140**, 1504 (1905); Rend. Circ. Mater. (Palermo) **21**, 129 (1906).

[6]See F. Rohrlich, Ref. 4, p. 17.

[7]F. Rohrlich, Am. J. Phys. **28**, 639 (1960); **38**, 1310 (1970).

[8]J. W. Zink, Am. J. Phys. **34**, 211 (1966); A. Gamba, *ibid.* **35**, 83 (1967).

[9]D. J. Griffiths and E. W. Szeto, Am. J. Phys. **46**, 244 (1978). See also Boyer (Ref. 3) and Pearle (Ref. 10).

[10]Two outstanding articles on classical electron theory have recently appeared in *Electromagnetism: Paths to Research*, edited by D. Teplitz (Plenum, New York, 1982). Chapter 6 is S. Coleman's widely known but hitherto unpublished monograph *Classical Electron Theory from a Modern Standpoint*, and Chap. 7 (*Classical Electron Models*) is a review by P. Pearle. The latter includes an extensive bibliography.

[11]E. M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1963), p. 51.

[12]J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), p. 239.

[13]E. M. Purcell, Ref. 11, p. 214.

[14]E. M. Purcell, Ref. 11, p. 160.

[15]D. J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Englewood Cliffs, NJ, 1981), p. 372.

[16]The d^0 term in the expansion for $\mathbf{F}_{\text{self}}$ gives the radiation reaction force (Griffiths and Szeto, Ref. 9).

[17]This case was analyzed by G. S. Cardell, thesis, Reed College, 1981 (unpublished). See also Boyer (Ref. 3).

[18]This argument and the idea for the Appendix are due to D. G. Hoffman.

[19]E. M. Purcell, Ref. 11, p. 51.

[20]This is the famous factor of 4/3 discussed in Refs. 4 and 10.

[21]Note added in proof: Pearle (Ref. 10) has recently shown that the discrepancy is resolved in the case of the spherical shell by inclusion of Lorentz contraction in the self-force calculation.

[22]Most of the material in this paper appears in more detailed form in R. E. Owen, thesis, Reed College, 1982 (unpublished).