$$\phi = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}_1,t^*)}{|\mathbf{r}-\mathbf{r}_1|}\,dv_1, \tag{8}$$

$$A = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}_1,t^*)}{|\mathbf{r}-\mathbf{r}_1|}\,dv_1, \tag{9}$$

and $t^* = t-(1/c)|\mathbf{r}-\mathbf{r}_1|$ is the retarded time.

The above potentials satisfy the Lorentz condition $\nabla\cdot\mathbf{A} + (1/c)^2(\partial\phi/\partial t) = 0$, and one can show that this condition is fulfilled if the densities of charge and current satisfy the equation of continuity. Thus we find, in general,

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\nabla\int \frac{\rho(\mathbf{r}_1,t^*)}{|\mathbf{r}-\mathbf{r}_1|}\,dv_1 - \frac{\partial}{\partial t}\frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}_1,t^*)}{|\mathbf{r}-\mathbf{r}_1|}\,dv_1. \tag{10}$$

We observe that even if the retardation effects are neglected (and then of course the derivation cannot claim to be exact), the electric field is still *not* given by Eq. (6). In this case, replacing t^* by t, one finds

$$\mathbf{D} = \epsilon_0\mathbf{E} = \left(\frac{1}{4\pi}\right)\int \frac{\rho(\mathbf{r}_1,t)(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^3}\,dv_1$$

$$-\left(\frac{1}{4\pi c^2}\right)\int \frac{[\partial\mathbf{J}(\mathbf{r}_1,t)/\partial t]}{|\mathbf{r}-\mathbf{r}_1|}\,dv_1.$$

Thus, even the quasistatic electric field is not given by (6). An additional integral over the time derivative of the current distribution must be included.

In conclusion,

$$\frac{1}{4\pi}\int \frac{\rho(\mathbf{r}_1,t)(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^3}\,dv_1$$

can only represent $\mathbf{D}$ if ρ is strictly constant, and in this case $\partial\mathbf{D}/\partial t = 0$, showing that the Biot–Savart equation, following the steps of Biswas up to his Eq. (7), leads to $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$, as it should.

This example shows how very cautious one must be in interpreting certain electrodynamic expressions in their integral form.

[a] Deceased. At the time of Professor Namias' death, this note had been tentatively accepted, pending a few minor modifications; the necessary final revisions were made by the editor. Please address all correspondence concerning this article to: P. D. Gupta, Chemistry and Physics Department, Purdue University Calumet, Hammond, IN 46323.

Note on "Field versus action-at-a-distance in a static situation," by N. L. Sharma [Am. J. Phys. 56, 420–423 (1988)]

D. J. Griffiths
Physics Department, Reed College, Portland, Oregon 97202

(Received 9 June 1988; accepted for publication 5 July 1988)

In a recent article[1] N. L. Sharma presents a variation on the "Feynman disk paradox."[2] Sharma considers a uniformly magnetized charged conducting sphere. When the charge is drained off (by touching a grounding wire to the south pole) the sphere begins to rotate, in apparent violation of conservation of angular momentum. The point of the "paradox" is to demonstrate that even *static* electromagnetic fields can carry angular momentum—in this instance, the angular momentum initially stored in the fields is

$$L_{\text{em}} = \tfrac{2}{9}\mu_0 Qa^2/M,$$

where M is the magnetization, Q is the charge, and a is the radius of the sphere. Sharma demonstrates that this is precisely the angular momentum picked up by the sphere when it discharges. (As the current flows over the surface to the south pole, it experiences a magnetic force in the azimuthal direction; it is the torque associated with this force that causes the sphere to rotate.)

Now, the angular momentum density stored in the electromagnetic fields,

$$\epsilon_0\mathbf{r}\times(\mathbf{E}\times\mathbf{B}),$$

can be removed *either* by turning off $\mathbf{E}$ (discharging the sphere) *or* by eliminating $\mathbf{B}$ (demagnetizing the sphere), and my purpose here is to note that you get the same answer either way, although the *mechanisms* are entirely different. Suppose that instead of draining off the charge we heat up the sphere, so that (passing through the Curie temperature) it gradually loses its magnetization. The magnetic field inside the sphere is uniform:

$$\mathbf{B} = \tfrac{2}{3}\mu_0 M\hat{z} \quad (r<a);$$

as $\mathbf{B}$ decreases, it will induce an azimuthal electric field, in accordance with Faraday's law:

$$\mathbf{E} = -\frac{1}{3}\mu_0 r\sin\theta\frac{dM}{dt}\hat{\phi} \quad (r\leqslant a).$$

This field exerts a torque on the surface charge $\sigma = Q/4\pi a^2$:

$$\mathbf{N} = \int(\mathbf{r}\times\mathbf{E})\sigma\,dS = -\frac{1}{6}\mu_0 Qa^2\frac{dM}{dt}\hat{z}\int\sin^3\theta\,d\theta$$

$$= -\frac{2}{9}\mu_0 Qa^2\frac{dM}{dt}\hat{z},$$

which causes the sphere to rotate. The final angular momentum is evidently

$$L_{\text{mech}} = \tfrac{2}{9}\mu_0 Qa^2 M,$$

the same as the angular momentum originally stored in the fields.

[1] N. L. Sharma, Am. J. Phys. **56**, 420 (1988). A similar model was discussed by E. M. Pugh and G. E. Pugh, Am. J. Phys. **35**, 153 (1967) and R. H. Romer, Am. J. Phys. **35**, 445 (1967). See also R. H. Romer, Am. J. Phys. **53**, 15 (1985). For further references, see T.-C. E. Ma, Am. J. Phys. **54**, 949 (1986).

[2] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1964), Vol. II, p. 17-5.