The charge distribution on a conductor for non-Coulombic potentials

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We study the distribution of charge on a conductor for Yukawa ($e^{-\mu r}/r$) and power-law ($1/r^n$) potentials. In the Yukawa case some charge goes to the surface, while the remainder distributes uniformly over the volume. In the power-law case no such general result is available, but we obtain the distribution for spheres, cylinders, and slabs, on the range $1 \leqslant n \leqslant 3$. In the Coulomb limit ($n = 1$) the charge all goes to the surface; at the other extreme ($n = 3$) it distributes uniformly over the volume.
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I. INTRODUCTION

In ordinary electrostatics, the charge on a solid conductor flows to the surface. This seems reasonable: Like charges repel, and they are simply getting as far away from one another as possible. On the other hand, it does seem a waste of all the empty space inside—why doesn't *some* of the charge prefer to remain in the interior? In point of fact, it *does*, in other geometries: On two-dimensional conducting plates *most*, but *not all* of the charge goes to the rim,[1] and on a one-dimensional wire the linear charge density is actually *uniform*.[2] Even for three-dimensional conductors, moreover, exclusive accumulation at the surface is an artifact of the $1/r^2$ nature of Coulomb's law.[3]

Our purpose in this paper is to explore the latter observation. Specifically, we ask how charge would distribute itself over a solid conductor if the Coulomb potential of a point charge q were replaced by Yukawa's:

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}e^{-\mu r}, \tag{1}$$

or by a power law:

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^n}. \tag{2}$$

(Of course, ϵ_0 need not have the same value—or even, in the second case, the same *dimensions*—as the ordinary vacuum permittivity; this is just a convenient way to express the constant. Note that as $\mu \to 0$ and $n \to 1$ our results should reduce to the standard ones.) For *spherical* conductors this problem was solved by Spencer a decade ago,[4] and our paper represents an extension of Spencer's work. In Sec. II we provide the general solution to the Yukawa case: *Some* of the charge goes to the surface, and the remainder distributes itself uniformly over the volume. What *fraction* of the total goes to the surface depends on the shape of the conductor (and also, of course, on the parameter μ), but, remarkably, the volume charge density is uniform *regardless* of the shape. As examples we treat the sphere, the infinite cylinder, and the infinite slab. In Sec. III we consider power-law potentials. In this case no general solution is available, but we again consider spheres, cylinders, and slabs. In Sec. IV we note that the results in Sec. III yield solutions to analogous problems for two- and one-dimensional conductors, including the vexed case of the conducting needle.[5]

Before we begin, however, it is appropriate to ask what it *means* to be a "conductor," in this world that is not our own. We shall take a conductor to be a material that has an unlimited[6] supply of charge that is free to move in response to electric fields (balanced, when neutral, by equal amounts of fixed charge of the opposite sign). It follows that in the static case the *net* field is zero inside a conductor, or (equivalently) that the potential is constant:

$$\mathbf{E} = 0, \quad V = \text{constant} \quad (\text{inside a conductor}). \tag{3}$$

So the question becomes: How must charge arrange itself over a conductor in order to achieve this condition?

We do not address here the subtle and elusive problem of uniqueness and stability.[7] In the Yukawa case our conclusions are presumably unambiguous, for the Yukawa potential admits a standard uniqueness theorem. But it is conceivable that there exist other solutions for the power-law potential—that is to say, charge distributions different from ours that also lead to $\mathbf{E} = 0$ inside the conductor. We regard this as unlikely (except in certain anomalous cases), but we cannot absolutely exclude the possibility.

II. YUKAWA POTENTIAL

For the Yukawa potential (1) the electric field of a point charge is

$$\mathbf{E} = -\nabla V = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}(1 + \mu r)e^{-\mu r}\hat{\mathbf{r}}, \tag{4}$$

the analog to Gauss's law is[8]

$$\oint \mathbf{E}\cdot d\mathbf{a} + \mu^2 \int V\, d\tau = \frac{1}{\epsilon_0}Q_{\text{enc}}, \tag{5}$$

and Poisson's equation generalizes to

$$-\nabla^2 V + \mu^2 V = \frac{1}{\epsilon_0}\rho. \tag{6}$$

Inside a conductor, $V(\mathbf{r}) = V_0$ is constant, and it follows immediately that

$$\rho = \epsilon_0 \mu^2 V_0 \tag{7}$$

is *also* constant. This surprising conclusion is entirely independent of the *shape* of the conductor.[9]

A. Example 1: The slab

Imagine first an infinite conducting slab, of thickness d, carrying a total charge per unit area (inside and on the surface) Σ. We would like to know what fraction of the charge goes to the surface, and what fraction remains (uniformly) distributed over the volume. Outside the slab (where $\rho=0$) ''Poisson's equation'' (6) says

$$-\frac{d^2V}{dz^2}+\mu^2V=0, \tag{8}$$

where z runs perpendicular to the slab (with $z=0$ at the center). The general solution is

$$V(z)=C_1e^{-\mu z}+C_2e^{\mu z}. \tag{9}$$

where C_1 and C_2 are constants. For positive z only the first term is physically acceptable (the second blows up at infinity), and the constant C_1 is determined by the continuity of V at the surface; similar considerations apply for negative z. Thus

$$V(z)=\begin{cases} V_0, & |z|\leqslant d/2 \\ V_0e^{-\mu(|z|-d/2)}, & |z|\geqslant d/2. \end{cases} \tag{10}$$

To find V_0 in terms of Σ we apply ''Gauss's law'' (5), using a ''pillbox'' of area A that straddles the slab and extends to infinity in both directions: the surface integral vanishes (E decreases exponentially), and we are left with

$$\frac{\Sigma A}{\epsilon_0}=\mu^2A\int_{-\infty}^{\infty}V(z)dz=\mu AV_0(\mu d+2). \tag{11}$$

Evidently,

$$V_0=\frac{\Sigma}{\epsilon_0\mu(\mu d+2)}, \tag{12}$$

and hence (7),

$$\rho=\frac{\mu\Sigma}{(\mu d+2)}. \tag{13}$$

Multiplying by d we obtain the charge per unit area inside the slab; subtracting this from Σ we get the total charge at the surface (both sides). It is convenient to express the final result in the form of the dimensionless ratio $\mu\sigma$ (where σ is the charge per unit area on each surface) over ρ:

$$\frac{\mu\sigma}{\rho}=1 \quad \text{(slab)}. \tag{14}$$

As μ goes to zero, all the charge goes to the surface (consistent with the Coulomb case); as $\mu\rightarrow\infty$ the charge is all in the interior.

B. Example 2: The sphere

Suppose we put a total charge Q on a sphere of radius R. Again, we would like to know what fraction goes to the surface, and what fraction remains (uniformly) distributed over the volume. Outside the sphere, ''Poisson's equation'' (6) says

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right)+\mu^2V=0, \tag{15}$$

to which the general solution is

$$V(r)=\frac{C_1}{r}e^{-\mu r}+\frac{C_2}{r}e^{\mu r}, \tag{16}$$

where C_1 and C_2 are constants. Only the first term is physically acceptable (the second blows up at infinity), and the constant C_1 is determined by the continuity of V at the surface:

$$V(r)=\begin{cases} V_0, & r\leqslant R \\ V_0\dfrac{R}{r}e^{-\mu(r-R)}, & r\geqslant R. \end{cases} \tag{17}$$

To find V_0 in terms of Q we apply ''Gauss's law'' (5), using a spherical Gaussian surface at infinity; the first term vanishes, and we are left with

$$\frac{Q}{\epsilon_0}=4\pi\mu^2\int_0^{\infty}r^2V(r)dr=4\pi V_0R\left(\frac{\mu^2R^2}{3}+\mu R+1\right). \tag{18}$$

Evidently,

$$V_0=\frac{Q}{4\pi\epsilon_0R(\mu^2R^2/3+\mu R+1)}, \tag{19}$$

and hence (7)

$$\rho=\frac{\mu^2Q}{4\pi R(\mu^2R^2/3+\mu R+1)}. \tag{20}$$

Multiplying by $(4/3)\pi R^3$ we obtain the volume charge; subtracting this from Q we get the surface charge; in this case the dimensionless ratio is

$$\frac{\mu\sigma}{\rho}=1+\frac{1}{\mu R} \quad \text{(sphere)}. \tag{21}$$

C. Example 3: The cylinder

Imagine finally an infinite solid conducting cylinder of radius S and net charge per unit length Λ. Outside the cylinder, ''Poisson's equation'' says

$$-\frac{1}{s}\frac{d}{ds}\left(s\frac{dV}{ds}\right)+\mu^2V=0, \tag{22}$$

where s is the radial coordinate. The general solution is

$$V(s)=C_1I_0(\mu s)+C_2K_0(\mu s), \tag{23}$$

where I_0 and K_0 are hyperbolic Bessel functions. But I_0 blows up[10] as $s\rightarrow\infty$, so $C_1=0$, while C_2 is determined by the continuity of V at the surface:

$$V(s)=\begin{cases} V_0, & s\leqslant S \\ V_0\dfrac{K_0(\mu s)}{K_0(\mu S)}, & s\geqslant S. \end{cases} \tag{24}$$

To find V_0 in terms of Λ we apply ''Gauss's law'' (5), using a cylindrical Gaussian surface of length L and infinite radius: the first term vanishes, and we are left with[11]

$$\frac{\Lambda L}{\epsilon_0}=2\pi\mu^2L\int_0^{\infty}sV(s)ds$$

$$=\pi V_0\mu SL\left(\mu S+2\frac{K_1(\mu S)}{K_0(\mu S)}\right). \tag{25}$$

Evidently,

$$V_0=\frac{\Lambda}{\pi\epsilon_0\mu S\left(\mu S+2\frac{K_1(\mu S)}{K_0(\mu S)}\right)},\tag{26}$$

and hence (7),

$$\rho=\frac{\mu\Lambda}{\pi S\left(\mu S+2\frac{K_1(\mu S)}{K_0(\mu S)}\right)}.\tag{27}$$

Multiplying by πS^2 we obtain the charge per unit length inside the cylinder; subtracting this from Λ we get the charge per unit length on the surface. In this case the dimensionless ratio is

$$\frac{\mu\sigma}{\rho}=\frac{K_1(\mu S)}{K_0(\mu S)}\quad\text{(cylinder)}.\tag{28}$$

Comparing Eqs. (14), (21), and (28), we see that the fraction of the total charge that resides at the surface depends on the shape of the conductor, except in the limiting cases $\mu=0$ (all charge to the surface) and $\mu\to\infty$ (all charge to the interior).

III. POWER LAWS

For the power law potential (2) the field of a point charge is

$$\mathbf{E}=-\nabla V=\frac{1}{4\pi\epsilon_0}\frac{nq}{r^{n+1}}\hat{\mathbf{r}},\tag{29}$$

but the analogs to Gauss's law and Poisson's equation are not available.[12] The general solution therefore eludes us, and we must resort to special cases.

A. Example 1: The slab

To begin with, let us calculate the field at a height z above an infinite plane with uniform charge density σ:

$$E=\frac{n\sigma z}{2\epsilon_0}\int_0^\infty\frac{s}{(s^2+z^2)^{(n+2)/2}}ds=\frac{\sigma}{2\epsilon_0 z^{n-1}}.\tag{30}$$

The corresponding potential is

$$V(z)=-\int_{z_0}^{z}E\,dz.\tag{31}$$

We may as well choose $z_0=0$ for $n<2$, $z_0=1$ for $n=2$, and $z_0=\infty$ for $n>2$; then

$$V(z)=\begin{cases}\dfrac{\sigma}{2\epsilon_0(n-2)|z|^{n-2}} & (n\neq 2)\\[2ex] -\dfrac{\sigma}{2\epsilon_0}\ln|z| & (n=2).\end{cases}\tag{32}$$

Now imagine a slab of thickness d, carrying a charge density $\rho(z)$. Slicing it into planes, and applying Eq. (32) for $n\neq 2$ (we'll treat the case $n=2$ separately), the potential is

$$V(z)=\frac{1}{2\epsilon_0(n-2)}\int_{-d/2}^{+d/2}\frac{\rho(z')}{|z-z'|^{n-2}}dz'.\tag{33}$$

Our problem is to determine ρ such that V is constant inside the slab $(-d/2\leqslant z\leqslant d/2)$. Spencer[13] calls attention to a useful formula due to Auer and Gardner:[14]

$$\int_{-1}^{+1}(1-t^2)^\nu\frac{C_m^\nu(t)}{|x-t|^{1+2\nu}}dt=K_m^\nu C_m^\nu(x),\tag{34}$$

where $C_m^\nu(x)$ are the Gegenbauer polynomials $(-1\leqslant x\leqslant 1)$ and K_m^ν is a constant related to their normalization. In particular, $C_0^\nu(x)=1$, so if we pick $m=0$, $\nu=(n-3)/2$, $t=2z'/d$, and $x=2z/d$, Eq. (34) becomes

$$\int_{-d/2}^{+d/2}\frac{\left[\frac{d^2}{4}-(z')^2\right]^{(n-3)/2}}{|z-z'|^{n-2}}dz'=K_0^{(n-3)/2}.\tag{35}$$

The left-hand side is precisely the form we require [Eq. (33)], and the right-hand side is independent of z; this suggests that

$$\rho(z)=C\left(\frac{d^2}{4}-z^2\right)^{(n-3)/2},\tag{36}$$

where C is a constant. Certainly this $\rho(z)$ yields $\mathbf{E}=0$ inside the slab; as mentioned in Sec. I, it is conceivable (absent a uniqueness theorem) that some other distribution might also work.

To determine C we set the total charge per unit area equal to Σ:

$$\Sigma=C\int_{-d/2}^{+d/2}\left(\frac{d^2}{4}-z^2\right)^{(n-3)/2}dz=C\frac{\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2}{\Gamma(n-1)}d^{n-2},\tag{37}$$

and conclude that

$$\rho(z)=\frac{\Gamma(n-1)\Sigma}{\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2 d^{n-2}}\left(\frac{d^2}{4}-z^2\right)^{(n-3)/2}.\tag{38}$$

For what range of n do we expect this result to hold? Auer and Gardner[15] confine their attention to the range $-1/2<\nu<0$ (which is to say, for us, $2<n<3$), but our own numerical studies indicate that it can be extended down to $\nu=-1$ $(n=1)$. We already know that $n=2$ is a special case, but it is easy to check, using the logarithmic potential in Eq. (32), that (38) holds for this case too:

$$\rho(z)=\frac{\Sigma}{\pi}\frac{1}{\sqrt{d^2/4-z^2}}\tag{39}$$

yields

$$\begin{aligned}V(z)&=-\frac{1}{2\epsilon_0}\int_{-d/2}^{+d/2}\rho(z')\ln|z-z'|dz'\\&=-\frac{\Sigma}{4\pi\epsilon_0}\int_{-d/2}^{+d/2}\frac{\ln[(z-z')^2]}{\sqrt{d^2/4-(z')^2}}dz'\\&=-\frac{\Sigma}{2\epsilon_0}\ln(d/4),\end{aligned}\tag{40}$$

which is indeed independent of z. Evidently, Eq. (38) is valid for $1<n<3$. How about the end points? As $n\to 1$ (the Cou-

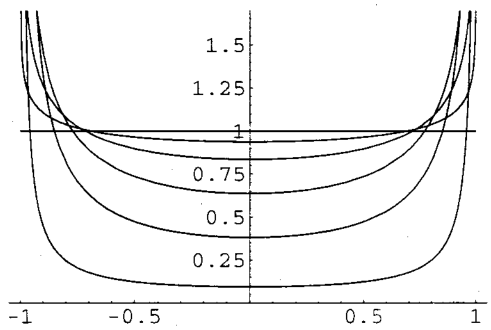

Fig. 1. Charge density on a slab, for the power law potential (38), using $d=2$, $\Sigma=2$, and $n=1.1$, 1.5, 2.0, 2.5, 2.8, and 3.0 (reading up from the lowest curve).

lombic limit) the charge density drops to zero within the slab, and goes over to a delta function at $\pm d/2$: All the charge now resides at the surface.[16] As $n\to 3$ the charge density goes to a constant, and, although the potential within the slab is now infinite, the limit would appear to be nonproblematic. In Fig. 1 we plot $\rho(z)$ for typical values of n between 1 and 3.

B. Example 2: The sphere

This is the case analyzed by Spencer,[17] and it will suffice to summarize his results. The potential of a spherical *shell* with radius r' and uniform surface charge density σ is

$$V(r)=\begin{cases}\dfrac{\sigma r'}{2\epsilon_0(n-2)r}[|r-r'|^{2-n}-(r+r')^{2-n}], & n\neq 2\\ \dfrac{\sigma r'}{4\epsilon_0 r}\ln\left[\left(\dfrac{r+r'}{r-r'}\right)^2\right], & n=2.\end{cases} \tag{41}$$

It follows that the potential inside a *solid* sphere of radius R, with charge density $\rho(r)$, can be written in a form reminiscent of Eq. (33):

$$V(r)=\frac{1}{2\epsilon_0(n-2)r}\int_{-R}^{+R}\frac{r'\rho(r')}{|r-r'|^{n-2}}dr' \quad (n\neq 2) \tag{42}$$

(again, we must handle the case $n=2$ separately). Our problem is to determine $\rho(r)$ such that V is independent of r; it can be solved in the same way as before, only using $C_1^{\nu}(x)=2\nu x$ in the Auer/Gardner formula. The result is

$$\rho(r)=\frac{2nQ\Gamma(n-1)}{\pi(2R)^n\left[\Gamma\left(\dfrac{n-1}{2}\right)\right]^2}(R^2-r^2)^{(n-3)/2}, \tag{43}$$

where Q is the total charge. This holds for the range $1<n<3$ (including the special case $n=2$, which can be checked by hand); as $n\to 1$, $\rho(r)$ goes to zero inside and a delta function at the surface (the Coulomb case), whereas for $n\to 3$ the charge distributes itself uniformly over the volume of the sphere. The functional form (43) is identical to that for the slab (38), and Fig. 1 (with the obvious modifications) shows the charge distribution for various values of n.

C. Example 3: The cylinder

First, we find the potential at a distance s from an infinite line with uniform charge density λ:

$$V(s)=\frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{1}{(s^2+z^2)^{n/2}}dz=\frac{\lambda}{4\epsilon_0\sqrt{\pi}}\frac{\Gamma\left(\dfrac{n-1}{2}\right)}{s^{n-1}\Gamma(n/2)}. \tag{44}$$

Using this, we determine the potential at a distance s from the axis of an infinite cylindrical shell (of radius s') with uniform surface charge σ:[18]

$$\begin{aligned}V(s)&=\frac{\sigma s'\Gamma\left(\dfrac{n-1}{2}\right)}{4\epsilon_0\sqrt{\pi}\Gamma(n/2)}\int_0^{2\pi}\frac{d\phi}{[s^2+(s')^2-2ss'\cos\phi]^{(n-1)/2}}\\&=\frac{\sqrt{\pi}\sigma\Gamma\left(\dfrac{n-1}{2}\right)}{2\epsilon_0\Gamma(n/2)}\\&\quad\times\begin{cases}\dfrac{s'}{s^{n-1}}F\left(\dfrac{n-1}{2},\dfrac{n-1}{2};1;(s'/s)^2\right), & s\geqslant s'\\ \dfrac{1}{(s')^{n-2}}F\left(\dfrac{n-1}{2},\dfrac{n-1}{2};1;(s/s')^2\right), & s\leqslant s'.\end{cases}\end{aligned} \tag{45}$$

Finally, we construct the potential at a distance s from the axis of an infinite cylinder of radius S carrying a volume charge density $\rho(s)$:

$$\begin{aligned}V(s)&=\frac{\sqrt{\pi}\Gamma\left(\dfrac{n-1}{2}\right)}{2\epsilon_0\Gamma(n/2)}\\&\quad\times\left\{\int_0^s\frac{s'}{s^{n-1}}F\left(\frac{n-1}{2},\frac{n-1}{2};1;(s'/s)^2\right)\rho(s')ds'\right.\\&\quad\left.+\int_s^S\frac{1}{(s')^{n-2}}F\left(\frac{n-1}{2},\frac{n-1}{2};1;(s/s')^2\right)\rho(s')ds'\right\}.\end{aligned} \tag{46}$$

As always, our problem is to find $\rho(s)$ such that $V(s)$ is independent of s.

We have not discovered a way to do this deductively. However, our experience with the slab (38) and the sphere (43) suggests that we try the form

$$\rho(s)=\frac{(n-1)\Lambda}{2\pi S^{n-1}}(S^2-s^2)^{(n-3)/2} \tag{47}$$

(where Λ is the total charge per unit length), and exhaustive numerical checking (using MATHEMATICA) confirms that this does indeed lead to a uniform potential inside the cylinder, for $1<n<3$.

D. The special case $n=3$

When $n=1$ the charge flows to the surface: that's no surprise, of course—it's the familiar Coulomb case. What *is* surprising is the other limit, $n=3$, for which in each example the charge distributes itself *uniformly* over the volume. Is this perhaps a *general* result, independent of the shape of the conductor? It is. For if we take $n=(3-\epsilon)$, and assume ρ is constant, the potential inside a conductor is

$$V(\mathbf{r})=\frac{\rho}{4\pi\epsilon_0}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|^{3-\epsilon}}d^3\mathbf{r}', \quad (48)$$

and we need only show that this integral is independent of $\mathbf{r}$ in the limit $\epsilon\to 0$. (We dare not set $\epsilon\to 0$ at the start, because the integral blows up.) Calculating the divergence in spherical coordinates, we find that

$$\frac{1}{r^{3-\epsilon}}=\frac{1}{\epsilon}\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^{2-\epsilon}}\right), \quad (49)$$

and hence

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|^{3-\epsilon}}=-\frac{1}{\epsilon}\nabla'\cdot\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^{3-\epsilon}}\right). \quad (50)$$

Therefore (invoking the divergence theorem)

$$V(\mathbf{r})=-\frac{\rho}{4\pi\epsilon_0}\frac{1}{\epsilon}\oint\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^{3-\epsilon}}\cdot d\mathbf{a}'. \quad (51)$$

The integral itself is now perfectly finite as $\epsilon\to 0$, and in fact Gauss's law tells us that its value is 4π, so

$$V(\mathbf{r})=-\frac{\rho}{\epsilon_0\epsilon}, \quad (52)$$

which is indeed independent of $\mathbf{r}$. qed[19]

IV. ANALOGS IN ONE AND TWO DIMENSIONS

Finally, we exploit our results from Sec. III to solve some related problems involving one- and two-dimensional conductors.

A. The needle

Consider an infinitesimally thin "needle" of length d carrying charge density $\lambda(z)$. The potential at a point z on the axis is

$$V(z)=\frac{1}{4\pi\epsilon_0}\int_{-d/2}^{+d/2}\frac{\lambda(z')}{|z-z'|^n}dz'. \quad (53)$$

If the needle is a conductor, then $\lambda(z)$ must be such that V is independent of z. This is mathematically identical to the problem of the slab [Eq. (33)], only with $n\to n+2$, and we can simply quote the solution (38):

$$\lambda(z)=\frac{Q\Gamma(n+1)}{\left[\Gamma\left(\frac{n+1}{2}\right)\right]^2 d^n}\left(\frac{d^2}{4}-z^2\right)^{(n-1)/2}, \quad (54)$$

where Q is the total charge on the needle. This time the formula is valid for $-1<n<1$.[20]

In the limit $n\to -1$ all the charge goes to the ends. This makes sense: The electric field of a point charge is a constant, so a charge placed anywhere on the wire will be repelled toward the end with the lesser charge. In the limit $n\to 1$ (the Coulomb case) charge distributes itself uniformly along the length of the needle (this corresponds to the case $n=3$ in Sec. III), confirming—from yet another perspective—the emerging consensus that the electrostatic charge density on a conducting needle is constant.[21]

B. The ribbon

Imagine an infinite conducting ribbon of width d, stretching along the y axis and lying in the xy plane. We can think of it as made up of a collection of infinite parallel wires, each carrying a uniform line charge $\sigma(x)dx$. The potential at a point x on the ribbon can be obtained from Eq. (44):

$$V(x)=\frac{\Gamma\left(\frac{n-1}{2}\right)}{4\epsilon_0\sqrt{\pi}\Gamma(n/2)}\int_{-d/2}^{+d/2}\frac{\sigma(x')}{|x-x'|^{n-1}}dx'. \quad (55)$$

Our problem is to determine $\sigma(x)$ such that V is independent of x. This is again mathematically identical to the slab [Eq. (33)], with $n\to n+1$, and we read off the solution (38):

$$\sigma(x)=\frac{\Lambda\Gamma(n)}{[\Gamma(n/2)]^2 d^{n-1}}\left(\frac{d^2}{4}-x^2\right)^{(n-2)/2}, \quad (56)$$

where Λ is the total charge per unit length on the ribbon. This time the formula is valid for $0<n<2$, and the limiting cases are $n=0$ (all charge to the edges—though since $\mathbf{E}\equiv 0$ it's not clear what we are to make of this one) and $n=2$ (uniform distribution over the surface). For the Coulomb case ($n=1$) we recover the standard result[22]

$$\sigma(x)=\frac{\Lambda}{\pi\sqrt{(d^2/4)-x^2}}. \quad (57)$$

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[2]J. D. Jackson, "Charge density on thin straight wire, revisited," Am. J. Phys. **68**, 789–799 (2000).

[3]Ross L. Spencer, "If Coulomb's law were not inverse square: The charge distribution inside a solid conducting sphere," Am. J. Phys. **58**, 385–390 (1990).

[4]Reference 3.

[5]David J. Griffiths and Ye Li, "Charge density on a conducting needle," Am. J. Phys. **64**, 706–714 (1996). See also Ref. 2.

[6]By assuming an *unlimited* supply of free charge we avoid an awkward asymmetry that affects even the Coulomb case: If the excess charge is *positive* (same sign as the fixed charge), then the surface charge necessarily has a finite thickness, since the maximum positive charge density is determined by the spacing between atoms; but if the excess charge is *negative* it accumulates as a skin of (in principle) infinitesimal thickness. Spencer (Ref. 3) treats the two cases separately, but we shall accept the usual idealization and handle the two signs symmetrically.

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[8]David J. Griffiths, *Introduction to Electrodynamics* (Prentice–Hall, Upper Saddle River, NJ, 1999), 3rd ed., Problem 2.49.

[9]Spencer (Ref. 3) obtained this result for a *spherical* conductor, by an entirely different route, and he noted that it could also be obtained from Eq. (6), but he did not generalize the result to arbitrary shapes.

[10]M. Abromowitz and Irene A. Stegun, eds., *Handbook of Mathematical Functions* (Dover, New York, 1964), p. 374.

[11]I. S. Gradsteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Academic, San Diego, CA, 1994), 5th ed., 5.56.2 and 8.407.

[12]Landy suggests a way to generalize Gauss's law by going to an $(n+1)$-dimensional space, but although we have used his formula to check some of our results, we have not found a way to exploit it systematically. See Steven B. Landy, ''Gauss' law for noninverse square forces,'' Am. J. Phys. **64**, 816–818 (1996).

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[15]Reference 14.

[16]A careful treatment of Eq. (38) in this limit is provided by D. Z. Uvanović, ''How would charge distribute inside a conductor if Coulomb's law were not inverse-square?,'' Reed College senior thesis, 2000.

[17]Reference 3.

[18]Reference 11, 3.665.

[19]One is tempted to conjecture that ρ should remain constant for *all* $n>3$, because physically, the larger n becomes, the more dominant are the nearby charges, and hence the greater the tendency toward a uniform distribution. By the same token, one might suppose that for $n<1$ the charge should remain on the surface, because the distant charges are now dominant. But Spencer's analysis of the sphere (Ref. 3) indicates that there is no unique stable solution for $n<1$, a conclusion also reached on more general grounds by Shaw (Ref. 7). In fact, it is not clear that the problem is even well-defined outside the range $1 \leqslant n \leqslant 3$.

[20]The case $n=0$ is anomalous, for according to Eq. (53) $V(z)$ is independent of z *regardless* of the functional form of $\lambda(z)$, and the solution suggested by Eq. (54) is clearly not unique.

[21]See Ref. 2, and articles cited there. The idea of regularizing the integral by modifying the power in Coulomb's law and then taking the limit is due to N. A. Wheeler, who proved by this means that the charge density is constant; see ''Construction and physical application of the fractional calculus,'' Reed College, unpublished report, 1997.

[22]Reference 5.

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In the late 1920s and early 1930s, at the zenith of his professional life, Nernst was the director of the Physics Institute at the Berlin University. At the time, many students who were to become eminent representatives of the new quantum physics saw Nernst as one of the grand old men of the field. It was in his institute, formerly directed by W. Rubens, that the famed physics colloquia, inaugurated by Gustav Magnus . . . , took place every Friday afternoon. Regardless of the presentations, ''the main performance'' was provided by the audience. The front rows were occupied by Einstein, Max von Laue, Planck, Erwin Schrödinger, Gustav Hertz, and Nernst, joined by Otto Hahn, Lise Meitner, and many younger luminaries.

Diane Kormos Barkan, *Walther Nernst and the Transition to Modern Physical Science* (Cambridge U.P., New York, 1999), p. 25.