This is not the situation in Lucite, since on the polymerization of methyl metacrylate no conjugated double bonds remain; and consequently, it would not be expected to fluoresce in the spectral region investigated here. We have not been able to ascertain the composition of the Paraplex P-43 used in this work; but presumably it does not have an appreciable amount of conjugation after polymerization, since there is no evidence of fluorescence on x-ray excitation. Anthracene in solution has been reported[9] to have a series of intense overlapping absorption bands in the 3500A and 2500A regions. Consequently, the polystyrene fluorescence is absorbed by the anthracene in the phosphor-plastic; and the anthracene in turn gives off its characteristic radiation when it falls from the excited state to a lower state. This interpretation is supported by the fact that even in long exposures where the anthracene emission is strongly overexposed there was no visible evidence of the polystyrene fluorescence on the plate, indicating that a large fraction of it had been absorbed by the solute molecules. A similar situation exists in the case of stilbene. In the case of Lucite and Paraplex the fluorescence is very weak, so there is very little energy transfer from the plastic to the phosphor; and as a result, such solid solutions make very poor counters.

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[9] Radulecu and Ostrogovich, Ber. **64**, 2233 (1931).

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Plasma Oscillations in a Static Magnetic Field*

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A theory of the small-amplitude oscillations of an ionized gas in a static magnetic field is developed, including the effects of temperature motions. The Boltzmann equation is solved for this problem, and exact expressions are obtained for the distribution function and dispersion relation. A general feature of the dispersion relation is the existence of gaps in the spectrum at frequencies which are approximately multiples of $\omega_c = eH/mc$. The magnitude of the gap depends on the temperature of the gas, being proportional to it for long wavelengths. This leads to the prediction of selective reflection of waves impinging on a plasma with frequency in the forbidden range.

For $ck \gg \omega_p$, ω_c the waves split into approximately longitudinal plasma waves and transverse waves. Detailed analysis is made of the plasma waves for ω_c small and ω_c large. At long wavelengths the frequency is $\omega^2 \simeq \omega_p^2 + \omega_c^2 + \beta(\kappa T/m)k^2$, where β depends on ω_p and ω_c. For waves near the Debye length the waves are heavily damped.

Two simplified treatments of plasma oscillations based on transport equations are compared with the above treatment. Expressions of the form $\omega^2 \simeq \omega_p^2 + \omega_c^2 + \beta(\kappa T/m)k^2$ are obtained where the factor β is independent of ω_c and ω_p. In addition, the transport treatments fail to predict the heavy damping near the Debye length and the existence of gaps in the frequency spectrum.

I. INTRODUCTION

The small amplitude vibrations of a plasma oscillating in a static magnetic field are discussed in this paper. For the case in which the thermal velocities of the particles are negligible, or for the limiting case of zero wavelength, the customary[1] theory of the propagation of electromagnetic waves in the ionosphere provides a satisfactory treatment of the vibrations, including the longitudinal plasma oscillations. The customary theory is not adequate to take into account the effects of random thermal motions and to treat these it is necessary to make use of the general methods of kinetic theory. In this type of problem, one is generally interested, from the practical point of view, in the dispersion relation giving the frequency of the possible waves as a function of the wavelength and also in the polarization properties of the waves. From the theoretical viewpoint it is also of great interest to know the velocity distribution of the particles if it is possible to compute it exactly, since this allows one to see what part the individual particles play in sustaining the oscillation. Plasmas are of particular interest in this connection, since they display organized properties depending on the cooperative action of all members of the assembly, but are systems still so simple that one can follow mathematically the detailed motion of the individual members of the assembly to a great extent. A detailed physical picture of the mechanism of ordinary plasma oscillations has been given,[2–4] and in the present

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[1] H. Lassen, Ann. Physik **1**, 415 (1947); H. R. Mimno, Revs. Modern Phys. **9**, 1 (1937).
[2] D. Bohm and E. P. Gross, Phys. Rev. **75**, 1851 (1949).
[3] D. Bohm and E. P. Gross, Phys. Rev. **75**, 1864 (1949).
[4] D. Bohm and E. P. Gross, Phys. Rev. **79**, 992 (1950).

paper we derive exact expressions for the distribution function and dispersion relation in the presence of a magnetic field.

The subject of plasma oscillations in static electric and magnetic fields has been of interest in experimental work with discharge tubes[5] and in the applications of trochotrons.[6] The instability of the oscillations in magnetic fields seems to have been observed, but there does not seem to be much precise information concerning the dependence of instabilities on the static electric fields and density gradients present in the plasma. Electric fields, density gradients, and beams of sharply defined energy provide sources from which energy can be taken to excite oscillations of the plasma. Theoretical calculations have been made on the basis of the gas discharge experiments. Bohm has carried out unpublished calculations showing excitation in magnetic fields when electric fields and drift arising from density gradients are present. Malmfors[7] has discussed the oscillations in a magnetic field of a simplified plasma consisting of electrons of a single velocity; and claims that the system is unstable. Later, we shall treat this case and show that, because of an error in his calculations, Malmfors's result is incorrect.

The same problems have arisen in the studies of electromagnetic propagation in ionized atmospheres, particularly in the explanations of the origin of solar radiation. For example, Bailey[8] has treated some of the problems, taking into account random motions by means of Maxwell's equations of transfer. His results show clearly that growing oscillations may occur in the presence of electric fields. The maxwell transfer equations, however, do not provide an exact description of the oscillating plasma, so that Bailey's results are not quantitatively exact. In view of this previous work it seems as though some of the conditions under which instability occurs are understood, so that we do not aim at a complete discussion of oscillations in electric fields and density gradients. We shall obtain an exact solution for the oscillations in the presence of a static magnetic field only and compare this treatment with approaches based on macroscopic transport equations such as the maxwell transfer equations. It will be shown that these lead to qualitatively correct conclusions, at least at long wavelengths. This is encouraging, since in the general case involving drift and collisions it is extremely difficult to give an exact treatment, while treatment by means of the transfer equations is a matter of a few lines of calculation. We must add, however, that there are some effects not predicted by the transport treatments. Among these are the existence of bands of forbidden frequencies at multiples of the cyclotron frequency, and the heavy damping at wavelengths near the Debye length. In addition the quantitative agreement is poor, particularly when magnetic fields are present.

[5] D. Bohm, in A. Guthrie and R. Wakerling, *Characteristics of Electrical Discharges in Magnetic Fields* (McGraw-Hill Book Company, Inc., New York, 1949), National Nuclear Energy Series, Vol. I, Paper No. 5.

[6] H. Alvén and co-workers, "Theory and applications of trochotrons," Kgl. Tekniska Högskolans Handlingar, No. 22 (1948).

[7] K. Malmfors, Arkiv Fysik **1**, 569 (1950).

[8] V. Bailey, Phys. Rev. **78**, 428 (1950).

II. EQUATIONS OF THE PROBLEM

The state of the plasma can be described by a distribution function[9] $f(\mathbf{x}, \mathbf{v}, t)$, giving the average number of particles in a small range of position and velocity at a time t. The distribution function changes with time in accordance with the Boltzmann equation

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla f+\frac{e}{m}\left(\mathbf{E}+\frac{\mathbf{v}\times\mathbf{H}_0}{c}\right)\cdot\frac{\partial f}{\partial \mathbf{v}}=\left(\frac{\delta f}{\delta t}\right)_{\text{coll}}. \quad (1)$$

The field $\mathbf{H}_0$ is the static magnetic field, while the field $\mathbf{E}$ includes fields arising from the particles of the system as well as impressed electric fields. To understand the physical assumptions[2–4,10] made in this treatment, we note that the forces acting on a given particle can be divided roughly into two types. One type is the ordinary short-range force acting on a particle when it makes a close collision involving a heavy momentum exchange with a neutral atom, and is taken into account by the collision term $(\delta f/\delta t)_{\text{coll}}$. In many low-density plasmas these collisions give rise to a minor damping effect, but in higher-density plasmas they may destroy many of the characteristic plasma properties. The second type of force is the long-range coulomb force, coming from other charged particles of the system. In this type of collision a very small momentum transfer is exchanged between two particles. A given electron is engaged in a many-body collision with a large number of distant particles, the effect of which is taken into account as a smoothed-out force. This latter force depends on the distribution function itself and gives rise to the characteristic plasma properties. The electric field is determined by Maxwell's equations, from which one can derive the equations

$$\nabla\times(\nabla\times\mathbf{E})=-(\partial^2\mathbf{E}/c^2\partial t^2)-(4\pi\partial\mathbf{j}/c^2\partial t);\qquad \mathbf{j}=e\int\mathbf{v}f d\mathbf{v}. \quad (2)$$

Equations (1) and (2) are a set of coupled, nonlinear, integro-differential equations which determine $\mathbf{E}$ and f in a self-consistent field problem.[11]

[9] It is possible to give an alternative description of the plasma which has advantages in understanding the physical processes (reference 1). We shall use the distribution function treatment here, since it has the advantage of mathematical compactness.

[10] A. Vlasov, J. Phys. (U.S.S.R.) **9**, 25, 130 (1945).

[11] In the static magnetic field problem in general it is not correct to determine $\mathbf{E}$ from f by means of the Poisson equation. As will be seen, this procedure is correct at a limit ck much larger than ω_p and ω_c. In the general case the strong coupling of longitudinal and transverse motions makes it necessary to use the full set of Maxwell's equations, or equivalently, Eqs. (2). In Eq. (1) we have made the customary approximation of neglecting

Equations (1) and (2) are, in general, exceedingly difficult to solve; but, if one is concerned with only small amplitude oscillations, one can put $f=f_0(\mathbf{v})+f_1(\mathbf{x},\mathbf{v},t)$ with $f_1\ll f_0$. The function f_0 represents the distribution in the absence of oscillations. One then neglects terms quadratic in f_1 and $\mathbf{E}$, so that the equations become linear.[12]

The linearized Boltzmann equation, in absence of collisions, is

$$\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\nabla f_1+\frac{e}{m}\mathbf{E}\frac{\partial f_0}{\partial \mathbf{v}}+\frac{e}{mc}(\mathbf{v}\times\mathbf{H}_0)\cdot\frac{\partial f_0}{\partial \mathbf{v}}+\frac{e}{mc}(\mathbf{v}\times\mathbf{H}_0)\cdot\frac{\partial f_1}{\partial \mathbf{v}}=0. \quad (1a)$$

Since $f_0(\mathbf{v})$ is the distribution function in the absence of oscillating electric fields, but in the presence of static magnetic fields, one must have $\mathbf{v}\times\mathbf{H}_0(\partial f_0/\partial\mathbf{v})=0$. If the static magnetic field $\mathbf{H}_0$ is considered to be in the z direction, this condition is $v_y(\partial f_0/\partial v_x)-v_x(\partial f_0/\partial v_y)=0$, so that we must have $f_0=f_0([v_x^2+v_y^2]^{\frac{1}{2}}, v_z)$. The maxwellian velocity distribution is of this type.

The linearized equations may be used to treat various boundary problems, e.g., by means of laplace transformation methods.[13] If we confine ourselves to a discussion of the unbounded plasma, we find that, starting with an arbitrary initial distribution, the electric field has no simple form for small times; but the asymptotic behavior for long times for a wave of wave number k is $e^{i(kx-\omega t)}$, where $\omega(k)$ is, in general, complex. The distribution function has the asymptotic behavior $f\sim A(v)e^{-ikvt}e^{ikx}-e^{i(kx-\omega t)}$. The term $A(v)e^{ik(x-vt)}$ depends on the initial state and describes the "free particle" motions. If one is interested in the dispersion relation and in the part of the distribution function corresponding to the organized motions of particles, it is sufficient to study a very special initial distribution of electrons having the property that for all times the electric field and distribution function vary as $e^{i(kx-\omega t)}$. This solution does not contain the "free-particle" contributions and has been so chosen that all of the electrons are started with motions appropriate to an organized plasma wave.[2] The dispersion relation obtained with this solution is the same as that obtained in the more general case. In the treatment of the magnetic field problem we shall therefore restrict ourselves to a consideration of the special steady-state solutions.

Collisions may be taken into account in the present treatment by adding a term $(\delta f/\delta t)_{\text{coll}}=-(f-f_0)/\tau$ on the right-hand side of Eq. (1a). This represents a damping of the ordered (oscillatory) part of the distribution by collisions with heavy particles. The mean collision time is $\tau(v)$, and in the simplest case is a constant. Since we shall look for steady-state solutions of the type $e^{i(kx-\omega t)}$, collisions may be taken into account by replacing ω by $\omega+i/\tau$ in Eq. (1a). For the present we neglect collisions (see Sec. IV for a discussion of the modifications necessary to take collisions into account).

III. DETERMINATION OF THE DISTRIBUTION FUNCTION

Let us now attempt to solve Eqs. (1a) and (2) in the steady state for plane waves progressing in the x direction. The magnetic field couples motions in the x and y direction so that we expect both longitudinal and transverse components of the electric field. Steady-state solutions are found by setting electric field and distribution function proportional to $e^{i(kx-\omega t)}$, where ω and k are connected by means of the dispersion relation which will be found later. If one makes use of the vector identity $\nabla\times(\nabla\times\mathbf{E})\equiv\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}$, one can write the equations describing the system as

$$i(kv_x-\omega)f_1+\frac{e}{m}\left(E_x\frac{\partial f_0}{\partial v_x}+E_y\frac{\partial f_0}{\partial v_y}\right)+\omega_c\left(v_y\frac{\partial f_1}{\partial v_x}-v_x\frac{\partial f_1}{\partial v_y}\right)=0, \quad (3a)$$

$$\omega^2E_x+4\pi i\omega e\int v_xf_1d\mathbf{v}=0, \quad (3b)$$

$$(\omega^2-c^2k^2)E_x+4\pi i\omega e\int v_yf_1d\mathbf{v}=0, \quad (3c)$$

where $\omega_c=eH_0/mc$.

Let us consider the Boltzmann equation first. It is possible to obtain the solution of this partial differential equation in v_x and v_y from the solution of an ordinary differential equation. We introduce new independent variables ρ, δ (polar coordinates in velocity space) defined by

$$\rho^2=v_x^2+v_y^2,\quad v_x=\rho\cos\delta,$$
$$\tan\delta=v_y/v_x,\quad v_y=\rho\sin\delta. \quad (4a)$$

forces arising from time-varying magnetic fields, since these are of order v/c times the changing electric forces. In Eq. (2) we have neglected currents arising from the motion of the positive ions. To take into account the contribution of these ions one must define a distribution function $g(\mathbf{x},\mathbf{v},t)$ and write the additional equation

$$(\partial g/\partial t)+\mathbf{v}\cdot\nabla g-(e/M)(\mathbf{E}+\mathbf{v}\times\mathbf{H}_0/c)\cdot\partial g/\partial\mathbf{v}=(\delta g/\delta t)_{\text{coll}}.$$

In the present work the positive ions are considered as smeared out into a uniform positive charge continuum.

[12] The main condition of validity of the linear approximation is

$$\mathbf{E}\cdot\partial f_1/\partial\mathbf{v}\ll\mathbf{E}\cdot\partial f_0/\partial\mathbf{v}.$$

This condition holds for sufficiently small amplitudes, and for most particles continues to hold for rather large amplitudes. For particles moving with velocities close to the phase velocity of the oscillation the condition breaks down after a long time. These particles are connected with the excitation and damping of oscillations in the linear approximation, but cease to play this role for larger amplitudes. A discussion of these questions is given in reference 1.

[13] L. Landau, J. Phys. U.S.S.R. 10, 25 (1946).

Then

$$\left.\begin{aligned}\frac{\partial}{\partial v_x}&=\frac{v_x}{\rho}\frac{\partial}{\partial\rho}-\frac{v_y}{v_x^2}\cos^2\delta\frac{\partial}{\partial\delta}=\cos\delta\frac{\partial}{\partial\rho}-\frac{\sin\delta}{\rho}\frac{\partial}{\partial\delta}\\ \frac{\partial}{\partial v_y}&=\frac{v_y}{\rho}\frac{\partial}{\partial\rho}+\frac{\cos^2\delta}{v_x}\frac{\partial}{\partial\delta}=\sin\delta\frac{\partial}{\partial\rho}+\frac{\cos\delta}{\rho}\frac{\partial}{\partial\delta}\\ v_y\frac{\partial f_1}{\partial v_x}&-v_x\frac{\partial f_1}{\partial v_y}=-\frac{\partial f_1}{\partial\delta}\end{aligned}\right\}. \quad (4b)$$

We also note that $E_x(\partial f_0/\partial v_x)+E_y(\partial f_0/\partial v_y)$ can be written as $(E_x\cos\delta+E_y\sin\delta(df_0/d\rho)$. Finally, on introducing the new quantities $F_x=E_x-iE_y$ and $F_y=E_x+iE_y$, the Boltzmann equation becomes

$$i(k\rho\cos\delta-\omega)f_1+\frac{e}{2m}\frac{df_0}{d\rho}(F_xe^{i\delta}+F_ye^{-i\delta})-\omega_c\frac{\partial f_1}{\partial\delta}=0. \quad (5)$$

This is an ordinary linear differential equation for δ in which ρ may be treated as a parameter. Thus, the solutions of the equation contain an arbitrary function of ρ which must be chosen so that the distribution function is periodic in δ with period 2π, as is required by physical considerations.

The general solution of the equation is

$$f_1=A(\rho)\exp\left[+i\left(\frac{k\rho\sin\delta-\omega\delta}{\omega_c}\right)\right]\times\exp\left[i\left(\frac{k\rho\sin\delta-\omega\delta}{\omega_c}\right)\right]\cdot\int_0^\delta\exp\left[-i\left(\frac{k\rho\sin\delta-\omega\delta}{\omega_c}\right)\right]\cdot\frac{e}{2m\omega_c}\frac{df_0}{d\rho}(F_xe^{i\delta}+F_ye^{-i\delta})d\delta. \quad (6)$$

In order to discuss this equation further and to obtain the periodic solutions, we make use of the expansion

$$e^{iz\sin\delta}=\sum_{n=-\infty}^{+\infty}J_n(z)e^{in\delta}.$$

The distribution function becomes

$$f_1=A(\rho)\exp\left[+i\left(\frac{k\rho\sin\delta-\omega\delta}{\omega_c}\right)\right]+\exp\left[i\left(\frac{k\rho\sin\delta-\omega\delta}{\omega_c}\right)\right]\frac{e}{2m\omega_c}\frac{df_0}{d\rho}\times\sum_{n=-\infty}^{+\infty}J_n\left(-\frac{k\rho}{\dot{\omega}_c}\right)\left[\frac{F_x\exp[i(n+1+\omega/\omega_c)\delta]-1}{i(1+n+\omega/\omega_c)}+\frac{F_y\exp[i(n-1+\omega/\omega_c)\delta]-1}{i(n-1+\omega/\omega_c)}\right].$$

To select the solutions periodic in δ of period 2π, one puts

$$A(\rho)=\frac{e}{2m\omega_c}\frac{df_0}{d\rho}\sum_{n=-\infty}^{+\infty}J_n\left(-\frac{k\rho}{\omega_c}\right)\times\left\{\frac{F_x}{i(1+n+\omega/\omega_c)}+\frac{F_y}{i(n-1+\omega/\omega_c)}\right\}. \quad (7)$$

This gives for the velocity distribution

$$f_1=\frac{e}{2m\omega_c}\frac{df_0}{d\rho}\exp\left[i\frac{k\rho\sin\delta}{\omega_c}\right]\sum_{n=-\infty}^{+\infty}J_n\left(-\frac{k\rho}{\omega_c}\right)\times\left\{\frac{F_xe^{i(n+1)\delta}}{i(1+n+\omega/\omega_c)}+\frac{F_ye^{i(n-1)\delta}}{i(n-1+\omega/\omega_c)}\right\}. \quad (8)$$

We take Eq. (8) as the correct distribution function for an oscillating plasma in a static magnetic field. The distribution function may be expressed in terms of v_x and v_y by carrying out the transformation (4a); but the result is, in general, exceedingly complex.

IV. THE DISPERSION RELATION

To obtain the dispersion relation connecting ω and k we must substitute f_1 from Eq. (8) into Eqs. (3) and set the determinants formed by the coefficients of F_x and F_y equal to zero. [We note that $E_x=\frac{1}{2}(F_x+F_y)$, $E_y=-i\frac{1}{2}(F_y-F_x)$ in Eqs. (3).] This will give ω as a function of k, and for each value of ω it will be possible to determine the ratio of amplitudes F_x/F_y. In carrying out the procedure it will be necessary to compute the integrals $\int v_xf_1d\mathbf{v}$ and $\int v_yf_1d\mathbf{v}$. We introduce new quantities α, β, γ, δ by

$$\alpha F_x+\beta F_y=8\pi ei\int v_xf_1d\mathbf{v}=\frac{4\pi e^2}{m\omega_c}\int_0^\infty\rho^2d\rho\frac{df_0}{d\rho}\int_0^{2\pi}\cos\delta d\delta\int_{-\infty}^{+\infty}dv_z\times\exp\left(i\frac{k\rho\sin\delta}{\omega_c}\right)\sum_{n=-\infty}^{+\infty}J_n\left(-\frac{k\rho}{\omega_c}\right)\times\left[\frac{F_xe^{i(n+1)\delta}}{(1+n+\omega/\omega_c)}+\frac{F_ye^{i(n-1)\delta}}{(n-1+\omega/\omega_c)}\right], \quad (9)$$

and

$$\gamma F_x+\delta F_y=8\pi e\int v_yf_1d\mathbf{v}=\frac{4\pi e^2}{m\omega_c}\int_0^\infty\rho^2\frac{df_0}{d\rho}d\rho\int_0^{2\pi}\sin\delta d\delta\int_{-\infty}^{+\infty}dv_z\times\exp\left(i\frac{k\rho\sin\delta}{\omega_c}\right)\sum_{n=-\infty}^{+\infty}J_n\times\left[\frac{F_xe^{i(n+1)\delta}}{i(1+n+\omega/\omega_c)}+\frac{F_ye^{i(n-1)\delta}}{i(n-1+\omega/\omega_c)}\right]. \quad (10)$$

In terms of these quantities the dispersion relation becomes

$$(\omega+\alpha)(-\omega^2+c^2k^2+\omega\delta)-(\omega+\beta)(\omega^2-c^2k^2+\omega\gamma)=0. \quad (11)$$

We now compute α, β, γ, δ, making use of the expressions

$$J_{-n}(z)=(1/2\pi)\int_0^{2\pi} e^{+in\delta}e^{iz\sin\delta}d\delta,$$

$$J_n(-z)=(-)^n J_n(z), \quad J_{-n}(z)=(-)^n J_n(z).$$

Use of the above integral expressions for the bessel function in Eqs. (9) and (10) gives

$$\alpha,\beta=(4\pi e^2/m\omega_c)\pi\int_{-\infty}^{+\infty}dv_z\int^{\infty}\rho^2 d\rho(df_0/d\rho) \times\sum_{n=-\infty}^{+\infty}\frac{J_n(-k\rho/\omega_c)}{(n\pm1+\omega/\omega_c)}\{J_{n+1\pm1}+J_{n-1\pm1}\}. \quad (12)$$

$$\gamma,\delta=-(4\pi e^2/m\omega_c)\pi\int_{-\infty}^{+\infty}dv_z\int_0^{\infty}\rho^2 d\rho(df_0/d\rho) \times\sum_{n=-\infty}^{+\infty}\frac{J_n}{(n\pm1+\omega/\omega_c)}\{J_{n+1\pm1}-J_{n-1\pm1}\}. \quad (13)$$

To obtain the dispersion relation, one must substitute these expressions in Eq. (11) and solve for ω as a function of k. In general, this is not easy since ω is contained in all the terms of the summation in Eqs. (12) and (13). However, at the limit of $kv_{th}/\omega_c\ll1$ one can replace the bessel functions by their values at z's and only a small number of terms remain. Thus by direct expansion the dependence of ω on k can be found for long wavelengths and high magnetic field strengths. We shall carry out this expansion later, and in addition we shall represent the series occurring in the above equations by a definite integral which will permit computation of the dispersion relation in the general case.

In investigating Eq. (11) one makes use of what is known about the dispersion relation in absence of random thermal motions. This dispersion relation can be written as

$$(\omega^2-\omega_p^2)(\omega^2-\omega_p^2-c^2k^2)-\omega_c^2(\omega^2-c^2k^2)=0. \quad (14)$$

Solving for ω^2 we find

$$\omega^2-\omega_p^2=\tfrac{1}{2}(\omega_c^2+c^2k^2)\pm\tfrac{1}{2}[(\omega_c^2-c^2k^2)^2+4\omega_c^2\omega_p^2]^{\frac{1}{2}}. \quad (15)$$

In the limiting case of a zero magnetic field there are two types of solutions: $\omega^2=\omega_p^2$ and $\omega^2=\omega_p^2+c^2k^2$. The first is a "plasma-type" solution in which the oscillations are purely longitudinal; i.e., $E_y=0$. The second solution represents transverse electromagnetic waves traveling through the ionized gas; i.e., $E_x=0$. In the limit $ck\gg\omega_p$ the effect of the presence of ionized gas is negligible, and these latter waves become electromagnetic waves in free space. For non-zero magnetic fields one can again easily separate the types of waves, provided $ck\gg\omega_p$ and $ck\gg\omega_c$. We then expect to have two sets of waves: one set at frequencies comparable to ck (electromagnetic) and one set at much lower frequencies (the plasma waves). From Eq. (14) the plasma waves are found by putting $\omega^2-\omega_p^2-c^2k^2\simeq-c^2k^2$ and $\omega^2-c^2k^2\simeq-c^2k^2$. The result is $\omega^2\simeq\omega_c^2+\omega_p^2$. For the electromagnetic type one finds

$$\omega^2\simeq c^2k^2+\omega_p^2/(1-\omega_c^2/c^2k^2)\simeq c^2k^2+\omega_p^2+(\omega_p^2\omega_c^2/c^2k^2).$$

For the general case in which ck is comparable to ω_p or ω_c, the waves cannot be separated into purely longitudinal and purely transverse sets and the dispersion relation becomes complicated (Eq. 15).

Our general dispersion relation can also be simplified in the limit of large ck's. We find that the plasma-type solutions are given by

$$\omega\simeq-\tfrac{1}{2}(\alpha+\beta). \quad (16)$$

This will be evaluated in later sections, and we shall find that Eq. (14) is obtained in the absence of random motions. From Eqs. (3) it is seen that these oscillations are longitudinal. The electromagnetic type solutions are given by

$$\omega^2\simeq c^2k^2+ck(\delta-\gamma). \quad (17)$$

Finally, we discuss the modifications which must be made in Eqs. (3) and (11) in order to take into account the effects of collisions. In the model adopted here $(\delta f/\delta t)_{\text{coll}}=(f_0-f)/\tau=-f_1/\tau$, so that one must replace ω by $\omega+i/\tau$ in the Boltzmann equation (3a) and in consequence in the quantities α, β, γ, δ. On the other hand, this replacement for ω must not be made where ω appears explicitly in Maxwell's equations (3b), (3c), and hence in Eq. (11). Thus, the proper procedure is to replace ω by $\omega+i/\tau$ only when ω occurs in α, β, γ, δ. Actually, the model adopted here does not represent collisions but describes a process in which electrons are absorbed by atoms at a rate depending on the prevalent distribution at the given time and are re-emitted at a rate depending on the equilibrium distribution. As a result, in computing the charge density of the system one must take into account the changing net charge of the trapping centers. This charge, however, does not contribute to the currents so that the procedure described above for computing collision effects may be employed. The present model serves only to take into account damping arising from destruction of phase relationships at collisions.

V. DISPERSION RELATION FOR A PEAKED VELOCITY DISTRIBUTION

In this section we study the special plasma treated by Malmfors,[7] an electron gas with a sharp distribution in a plane perpendicular to H_0. We investigate whether excitation is possible, point out the existence of gaps in the frequency spectrum, and replace Malmfors's results by what we believe to be correct expressions.

Let us take for the velocity distribution

$$f_0(\rho, v_z) = (\rho_0/2\pi v_0)\delta(\rho - v_0)\cdot g(v_z)/g_0,$$

where δ is the Dirac delta-function. Here

$$\int_{-\infty}^{+\infty} g(v_z)dv_z = g_0,$$

v_0 is the absolute velocity of the particles in the plane perpendicular to H_0, and $\rho_0 = \int_{-\infty}^{+\infty} f_0 d\mathbf{v}$ is the density of the ion gas. Then

$$df_0/d\rho = (\rho_0/2\pi v_0)\delta'(\rho - v_0)g(v_z)/g_0,$$

where the δ'-function has the property

$$\int_{-\infty}^{+\infty} f(x)\delta'(x-a) = -f'(a).$$

With these assumptions and with the help of Eqs. (12), (13), and (42), the following expressions for α, β, γ, δ are obtained:

$$\begin{aligned} \alpha, \gamma &= -(\omega_p^2/2\omega_c v_0)(d/dv_0)\{v_0^2(K_0 \pm K_2)\}, \\ \beta, \delta &= -(\omega_p^2/2\omega_c v_0)(d/dv_0)\{v_0^2(L_0 \pm K_2)\}. \end{aligned} \tag{18}$$

As discussed in Sec. IV, one finds the plasma-type solutions for large ck from the simplified dispersion relation $\omega \simeq -\frac{1}{2}(\alpha+\beta)$. The dispersion relation for the plasma-type solutions is

$$\omega = \frac{\omega_p^2}{4\omega_c v_0}\frac{d}{dv_0}\left[v_0^2 \sum_{n=-\infty}^{+\infty} \frac{(J_n + J_{n+2})^2}{(n+1+\omega/\omega_c)}\right].$$

This can be put in the form obtained by Malmfors by writing

$$1 = \frac{\omega_p^2}{4\omega_c^2}\frac{1}{\lambda}\frac{d}{d\lambda}\left[\lambda^2 \sum_{n=-\infty}^{+\infty} \frac{2n}{\lambda}\frac{J_n(J_{n-1}+J_{n+1})}{(\omega/\omega_c)(n+\omega/\omega_c)}\right],$$

where we have put $\lambda = kv_0/\omega_c$ and have used the relation $J_{n-1}(\lambda) + J_{n+1}(\lambda) = 2nJ_n(\lambda)/\lambda$. Finally, we have, using the identity

$$\sum_{n=-\infty}^{+\infty} nJ_n^2(\lambda) \equiv 0,$$

$$1 = -\frac{\omega_p^2}{2\omega_c^2}\frac{1}{\lambda}\frac{d}{d\lambda}\left[\lambda \sum_{n=-\infty}^{+\infty} \frac{J_n(J_{n-1}+J_{n+1})}{n-\omega/\omega_c}\right]. \tag{19}$$

It is to be noted that this dispersion relation differs from that of Malmfors in that a factor i is not present and in addition the sum, rather than the difference, of J_{n-1} and J_{n+1} appears on the right-hand side. It is possible to check the correctness of the above expression if one remembers that in the limit of zero velocity; i.e., $\lambda = 0$, the system becomes a plasma at zero temperature, for which the dispersion relation is $\omega^2 = \omega_p^2 + \omega_c^2$. The above relation reduces to this expression while that of Malmfors does not.

The most interesting feature of Eq. (19) is the behavior when ω is approximately a multiple of ω_c. We shall now show that there is a jump in the frequency in this region. It is particularly easy to study the behavior in the limit $\lambda \ll 1$. We note that in the absence of thermal motions, i.e., $\lambda = 0$, one can always choose ω_p so that ω is a multiple of ω_c; e.g., with $\omega_p^2 = 3\omega_c^2$ one finds $\omega = 2\omega_c$. In the following discussion it will be assumed that $\omega_p^2 \simeq 3\omega_c^2$. One may write Eq. (19) in the form

$$1 = -\frac{\omega_p^2}{\omega_c^2}\frac{1}{\lambda}\frac{d}{d\lambda}\left[\sum_{n=-\infty}^{+\infty} \frac{nJ_n^2}{n-\omega/\omega_c}\right]. \tag{20}$$

Expanding J_n and retaining only powers of λ up to the fourth, one finds

$$1 \simeq -\frac{\omega_p^2}{\omega_c^2}\frac{1}{2}\left[\frac{1}{1-\omega/\omega_c} + \frac{1}{1+\omega/\omega_c}\right] - \frac{\omega_p^2\lambda^2}{8\omega_c^2(2-\omega/\omega_c)}.$$

For $\lambda = 0$ this gives $\omega^2 = \omega_p^2 + \omega_c^2$. With $\omega \simeq 2\omega_c$ the equation simplifies to

$$1 \simeq \frac{\omega_p^2}{\omega_c^2}\frac{1}{(\omega^2/\omega_c^2 - 1)} - \frac{\omega_p^2}{\omega_c^2}\cdot\frac{1}{8}\frac{\lambda^2}{(2-\omega/\omega_c)}. \tag{21}$$

When ω/ω_c is less than 2, the second term on the right-hand side is large and negative; when ω/ω_c is greater than 2, it is large and positive. The first term on the right-hand side is slowly varying so that the dispersion relation can be satisfied only if ω/ω_c does not pass through the value 2. We can show that ω/ω_c jumps discontinuously as ω_p is varied continuously. Let us put $\omega/\omega_c = 2 + \epsilon\lambda$ and expand $\omega^2/\omega_c^2 - 1$. The dispersion relation becomes

$$\omega_c^2 \simeq \tfrac{1}{3}\omega_p^2 - \tfrac{4}{9}\omega_p^2\epsilon\lambda + \tfrac{1}{8}(\omega_p^2\lambda/\epsilon).$$

Thus, as ω_p^2 varies continuously in the vicinity of 3, the dispersion relation can be satisfied by taking $\epsilon = \pm(\frac{9}{32})^{\frac{1}{2}}$. The magnitude of the jump in ω is $\Delta(\omega/\omega_c) \simeq (\frac{9}{8})^{\frac{1}{2}}\lambda$ and is proportional to the velocity of the beam. A similar behavior occurs when λ is no longer small; but, in this case, many terms of the same order of magnitude enter in the dispersion relation so that it is more difficult to find an expression for the magnitude of the gap.

It might be thought possible to satisfy Eq. (20) by letting ω be a complex number, representing damping or instability of the oscillations. We shall now show that, at least for $\lambda \ll 1$, this cannot be done. In examining complex solutions one writes $\omega = \omega_R + i\sigma$, where ω_R and σ are real, and one separates the real and imaginary parts of Eq. (20). For $\lambda \ll 1$ and $\omega/\omega_c \simeq 2$ one can use

Eq. (21) instead. One finds for the imaginary part

$$-\frac{\omega_p^2}{3\omega_c^2}\frac{1}{(\omega_R/\omega_c-1)^2+(\sigma^2/\omega_c^2)}$$
$$-\frac{\omega_p^2}{8\omega_c^2}\frac{\lambda^2}{(2-\omega_R/\omega_c)^2+(\sigma^2/\omega_c^2)}=0.$$

With $\omega_R/\omega_c=2$ one obtains $\sigma^2=-\frac{3}{8}\lambda^2$, showing that it is not possible to satisfy the dispersion relation with real values of σ. We have not undertaken a complete study of the dispersion relation (20), but the same type of consideration would seem to hold for all values of λ.

The behavior of the dispersion relation for a plasma in a magnetic field is similar to the behavior of the energy-wave vector relation for an electron moving in a solid. It will be remembered that the permissible values for the energy of the electron lie in bands, the edges of the bands corresponding to wavelengths at which Bragg reflection of the electron waves occurs. It is possible to have energy values in the forbidden region only by allowing k to take on complex values. This behavior refers to the experimental situation in which electrons of the forbidden energies are reflected from the surface of the solid. One is therefore led to expect that electromagnetic waves impinging on a plasma in a magnetic field will be reflected in selective frequency regions. Since the width of the gap depends on the temperature of the electrons of the plasma, it is possible that the theory may be useful in analyzing plasma. It is to be realized that in actual practice one may have to take into account effects of varying plasma density and absorption because of electron-atom collisions. The latter effect can be treated by the methods developed in this paper. It is to be noted that the underlying reason for the band structure is the same for the plasma as for the solid. In the solid-state problem one studies the propagation of an electron wave in the periodic field of the lattice, while in the plasma one studies the propagation of waves in an electron gas in which all electrons are rotated with a period equal to the cyclotron period.

VI. DISPERSION RELATION FOR A SMOOTH, MAXWELLIAN DISTRIBUTION

The maxwellian distribution for the unperturbed system is particularly important in comparing the exact method with the transport solutions. The distribution function is

$$f_0=N_0(m/2\pi\kappa T)^{\frac{3}{2}}\exp[-(m/2\kappa T)(v_z^2+\rho^2)].$$

In the Appendix expressions are given for α, β, γ, δ. These enable one to study the general dispersion relation. Here we confine our discussion to the plasma-type solutions ($ck\gg\omega_c$, ω_p), and find for the dispersion relation:

$$\omega=-\frac{1}{2}\cdot\frac{4\pi e^2}{m}\frac{\pi}{\omega_c}\int_{-\infty}^{+\infty}dv_z\int_0^\infty\rho^2d\rho\frac{df_0}{d\rho}\times\sum_{n=-\infty}^{+\infty}\frac{(J_n+J_{n+2})^2}{n+1+\omega/\omega_c}.\quad(22)$$

Substituting the maxwell distribution for f_0, and using the identity $2nJ_n(z)/z\equiv J_{n+1}+J_{n-1}$, one obtains

$$\omega=+4\omega_p^2\frac{\omega_c}{k^2}\left(\frac{m}{2\kappa T}\right)^2\int_0^\infty\rho\exp\left[-\frac{m}{2\kappa T}\rho^2\right]d\rho\times\sum_{n=-\infty}^{+\infty}\frac{n^2J_n^2}{n+\omega/\omega_c}.$$

With $\lambda=k\rho/\omega_c$, $1/\mu=m\omega_c^2/\kappa Tk^2$, this can be put in the form:

$$\omega=4\omega_p^2\frac{\omega_c^3}{k^4}\left(\frac{m}{2\kappa T}\right)^2\int_0^\infty\lambda\exp\left(-\frac{\lambda^2}{2\mu}\right)d\lambda\times\sum_{n=-\infty}^{+\infty}\frac{n^2J_n^2(\lambda)}{n+\omega/\omega_c}.\quad(23)$$

We shall study this dispersion relation in the two limits $\mu\ll1$ and $\mu\gg1$. The first case corresponds to the limit of low temperatures and long wavelengths, while the second case represents the behavior in the limit of weak magnetic fields.

(A) When $\mu\ll1$

To see how to handle this case we first evaluate Eq. (23) in the limit $T\to0$, using the expressions for the bessel functions of small argument,

$$J_0(\lambda)\simeq1-\tfrac{1}{4}\lambda^2,\quad J_n(\lambda)\simeq\lambda^n/2^nn!,$$

$$\omega=\frac{\omega_p^2}{2\omega_c}\left\{\frac{1}{1+\omega/\omega_c}+\frac{1}{-1+\omega/\omega_c}\right\},\quad\text{or}\quad\omega^2=\omega_p^2+\omega_c^2.$$

The dispersion relation at low temperatures can be found by carrying out the expansion so that the dispersion relation contains terms of the order of μ. The result is

$$\omega^2=\omega_p^2+\omega_c^2+\left(\frac{\kappa Tk^2}{m\omega_c^2}\right)\frac{\omega_p^2\omega_c}{\omega}\left(\frac{\omega^2}{\omega_c^2}-1\right)\cdot\frac{1}{8}\left(\frac{1}{-3+\omega/\omega_c}-\frac{2}{-1+\omega/\omega_c}+\frac{2}{\omega/\omega_c}-\frac{3}{1+\omega/\omega_c}+\frac{2}{2+\omega/\omega_c}\right).\quad(24)$$

This equation is still quite complicated; and one must take care not to introduce irrelevant roots, i.e., roots which do not tend to the correct values as $T\to 0$. To obtain the information which interests us we study several special cases by substituting the value $\omega^2 \simeq \omega_p^2+\omega_c^2$ in the last term of Eq. (24). We find that

$$\omega^2=\omega_p^2+\omega_c^2+\tfrac{1}{8}\omega_c\omega_p(\kappa Tk^2/m\omega_c^2) \quad \text{for} \quad \omega_p\gg\omega_c, \quad (25)$$

$$\omega^2=\omega_p^2+\omega_c^2-\tfrac{1}{2}\omega_c^2(\kappa Tk^2/m\omega_c^2) \quad \text{for} \quad \omega_p\simeq\omega_c, \quad (26)$$

$$\omega^2=\omega_p^2+\omega_c^2-\tfrac{1}{2}\omega_p^2(\kappa Tk^2/m\omega_c^2) \quad \text{for} \quad \omega_p\ll\omega_c. \quad (27)$$

We see from these examples that the coefficient of $\kappa Tk^2/m$ depends, in general, on the parameters ω_p and ω_c. As we shall see later, the transport treatments give results of the correct general form; but the coefficients are fixed numbers. The same type of difference can be found if one does not restrict the discussion to the plasma-type solutions, but considers the strongly coupled waves when $ck\simeq\omega_c$, ω_p, for which it is necessary to use the complete dispersion relation (11).

In addition to the above-mentioned discrepancies between the transport and the exact theories, there is a more serious difference when ω is approximately a multiple of ω_c. As in the previous section, this will always occur, since ω_p varies continuously and gives rise to discontinuous jumps for ω at multiples of ω_c. We find from Eq. (24), with the help of the analysis of Sec. V, that the gap width when $\omega\simeq 3\omega_c$ is $\Delta\omega=(16/9)^{\frac{1}{2}}(\kappa T/m)^{\frac{1}{2}}k$. This is proportional to the mean velocity of the plasma electrons and to the wave vector.

(B) When $\mu\gg 1$, (i.e., $m\omega_c^2/k^2\kappa T\ll 1$)

This case represents the limit of a weak magnetic field and we should be able to recover the dispersion relation for plasma oscillations in the absence of a magnetic field when $\omega_c=0$. We notice that in this case ω_c/ω is also expected to be very small. If this occurs in such a way that $m\omega^2/k^2\kappa T\gg 1$, we obtain the ordinary expression, $\omega^2=\omega_p^2+(3\kappa Tk^2/m)$; but if $m\omega^2/k^2 2\kappa T\simeq 1$ we should find that ω is heavily damped,[2,13] expressing the fact that waves shorter than the Debye length cannot exist.

Our starting point is Eq. (23). This equation may be transformed by using the identity

$$\sum_{n=-\infty}^{+\infty}\frac{n^2J_n^2(\lambda)}{n+\omega/\omega_c}\equiv\frac{\omega^2}{\omega_c^2}\sum_{n=-\infty}^{+\infty}\frac{J_n^2(\lambda)}{n+\omega/\omega_c}-\frac{\omega}{\omega_c}\sum_{n=-\infty}^{+\infty}J_n^2(\lambda),$$

where

$$\sum_{n=-\infty}^{+\infty}J_n^2(\lambda)=1.$$

If we introduce the quantity $\Lambda=4\omega_p^2\omega_c^3/k^4(m/2\kappa T)^2$, Eq. (23) becomes

$$\left(\omega+\frac{\omega}{\omega_c}\Lambda\mu\right)=\Lambda\frac{\omega^2}{\omega_c^2}\int_0^\infty\lambda\exp\left(-\frac{\lambda^2}{2\mu}\right)d\lambda\sum_{n=-\infty}^{+\infty}\frac{J_n^2}{n+\omega/\omega_c}.$$

With the help of the methods developed in Appendix 1 we may write

$$\sum_{n=-\infty}^{+\infty}\frac{J_n^2(\lambda)}{n+\omega/\omega_c}=\frac{i}{\exp(i\omega 2\pi/\omega_c)-1}\int_0^{2\pi}J_0 \times(\sqrt{2}\lambda[1-\cos\gamma]^{\frac{1}{2}})\exp\left(i\frac{\omega}{\omega_c}\gamma\right)d\gamma. \quad (28)$$

If we invert the order of integration in Eq. (28) and make use of the expression

$$\int_0^\infty J_0(at)\exp(-\rho^2t^2)dt=(1/2\rho^2)\exp(-a^2/4\rho^2),$$

we find the dispersion relation

$$\left(\omega+\frac{\omega}{\omega_c}\Lambda\mu\right)=\mu\Lambda\frac{\omega^2}{\omega_c^2}\frac{i}{\exp(i\omega 2\pi/\omega_c)-1} \times\int_0^{2\pi}\exp\left(i\frac{\omega}{\omega_c}\gamma\right)\cdot e^{-\mu(1-\cos\gamma)}d\gamma. \quad (29)$$

This form of the dispersion relation holds, of course, for μ small as well as μ large; and, if we wished, we could derive the formulas of the previous case by expansion of the integral in powers of μ. However, the main interest in the present form lies in the fact that it is possible to derive asymptotic expansions for the integral when ω_c/ω and $1/\mu$ are small.

The integral

$$I=\int_0^{2\pi}\exp[i(\omega/\omega_c)\gamma]e^{-\mu(1-\cos\gamma)}$$

is studied in Appendix 2. For $\omega_c^2/\omega_p^2\ll 1$ and $(\kappa T/m)(k^2/\omega_p^2)\ll 1$, one finds for the dispersion relation

$$\omega^2=\omega_p^2+\omega_c^2+(3\kappa T/m)k^2+9(\omega_c^2\kappa T/\omega_p^2m)k^2+(6/\omega_p^2)(\kappa Tk^2/m)^2. \quad (30)$$

Thus, in first approximation one finds that the contributions from the magnetic field and thermal motions are not coupled, while in second approximation they are. The case where $(\kappa T/m)(k^2/\omega_p^2)\simeq 1$ leads to complex values for ω, representing heavy damping of waves near the Debye length. The expression for the damping is that found by Landau,[13] slightly modified by the effects of the magnetic field (see Appendix 2).

VII. TRANSPORT EQUATIONS

Finally, we discuss the macroscopic treatments of the plasma oscillation problem and compare them with the exact solution of the Boltzmann equation. The transport treatments aim at taking into account effects of random motions, i.e., finite temperature, using a set of equations for the moments of the distribution function. Physically, these equations represent continuity, momentum, and energy transport equations, etc. The infinite set is terminated after a few equations by assuming some expression for the highest moments in terms of the lower ones. This termination makes the equations a determined set, and should be accomplished according to a definite procedure so that higher approximations can be carried out. It will develop that if one restricts himself to a low order of approximation, these treatments give qualitatively correct results in many cases, although quantitative accuracy cannot be expected. The advantage of the macroscopic treatments is that they can be applied to many cases in which the exact solution of the Boltzmann equation cannot be obtained, particularly when this contains the integral terms resulting from considering collisions. It is therefore of interest to check the results of the macroscopic treatment with the exact solution when the latter can be found.

The first step in the transport method is to write the correct Boltzmann equation for the problem under discussion. For our problem this is Eq. (1), where in the present analysis we neglect collisions. The moments of f (density, mean velocity, etc.) are defined as

$$\rho(\mathbf{x},t)=\int f d\mathbf{v},\quad \rho\langle v_i\rangle=\int v_i f d\mathbf{v},\quad \rho\langle v_i v_j\rangle=\int v_i v_j f d\mathbf{v}.$$

We then take the zero and first moments of Eq. (1). This gives

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho\langle v_x\rangle)=0, \tag{31a}$$

$$\frac{\partial}{\partial t}(\rho\langle v_x\rangle)+\frac{\partial}{\partial x}(\rho\langle v_x\rangle^2)-\frac{eE_x}{m}\rho-\omega_c\rho\langle v_y\rangle=0, \tag{31b}$$

$$\frac{\partial}{\partial t}(\rho\langle v_y\rangle)+\frac{\partial}{\partial x}(\rho\langle v_x v_y\rangle)-\frac{eE_y}{m}\rho+\omega_c\rho\langle v_x\rangle=0, \tag{31c}$$

$$\frac{\partial}{\partial t}(\rho\langle v_z\rangle)+\frac{\partial}{\partial x}(\rho\langle v_x v_z\rangle)=0. \tag{31d}$$

Equation (31a) is the equation of continuity and Eqs. (31b, c, d) are the equations of momentum transfer. Instead of writing the second moment equations separately, we shall find it sufficient to obtain an expression for $\partial[\rho\langle v_x^2+v_y^2+v_z^2\rangle]/\partial t$. Using Eq. (1) and introducing the quantity $T(x,t)$ by means of the definition $\frac{3}{2}\kappa T(x,t)=\frac{1}{2}m\langle(v_x^2+v_y^2+v_z^2)\rangle$, we have

$$\frac{\partial}{\partial t}\left(\rho\frac{3\kappa T}{m}\right)+\frac{\partial}{\partial x}[\rho\langle v_x^3+v_xv_y^2+v_xv_z^2\rangle]$$
$$-\frac{2e}{m}E_x\bar{v}_x\rho-\frac{2e}{m}E_y\bar{v}_y\rho=0. \tag{31e}$$

It will be noticed that the magnetic field terms do not occur in Eq. (31e), and that this equation contains third moments, so that there are too many unknowns.

We shall describe two different methods of treating the above set of equations. The first is equivalent to the use of the maxwell transfer equations and was first used to describe plasma oscillations by the Thomsons.[14] It has recently been extended by Bailey[8] to describe excitation of plasma oscillations in the presence of static electric and magnetic fields. For the discussion of small amplitude oscillations we may write Eqs. (31) as

$$\frac{\partial\langle v_x\rangle}{\partial t}+\frac{\partial}{\partial x}\langle v_x\rangle^2+\frac{v_x^2}{\rho_0}\frac{\partial\delta\rho}{\partial x}-\frac{eE_x}{m}-\omega_c\langle v_y\rangle=0, \tag{32a}$$

$$\frac{\partial\langle v_y\rangle}{\partial t}+\frac{\partial}{\partial x}\langle v_xv_y\rangle+\frac{\langle v_xv_y\rangle}{\rho_0}\frac{\partial\delta\rho}{\partial x}-\frac{eE_y}{m}+\omega_c\langle v_x\rangle=0, \tag{32b}$$

with $\rho=\rho_0+\delta\rho$. If one now assumes with the Thomsons that $\langle v_x^2\rangle$ is a constant equal to $\kappa T_0/m$ (since $\frac{1}{2}m\langle v_x^2\rangle=\frac{1}{2}\kappa T_0$), where T_0 is the constant temperature of the plasma and assume with Bailey that $\langle v_xv_y\rangle=0$, Eqs. (31a), (32a), and (32b) become a determined set of equations. For if we note that E_x and E_y are determined in terms of ρ, $\langle v_x\rangle$, $\langle v_y\rangle$ with the help of Maxwell's equations (3b) and (3c), we see that we have three equations to solve for the unknowns ρ, $\langle v_x\rangle$, $\langle v_y\rangle$.

If we set all quantities proportional to $e^{i(kx-\omega t)}$, we easily find the dispersion relation

$$(\omega^2-\omega_p^2-k^2\kappa T_0/m)(\omega^2-\omega_p^2-c^2k^2)-\omega_c^2(\omega^2-c^2k^2)=0. \tag{33}$$

In comparing this with our previous results we note first that the exact treatment indicates that this type of relation cannot hold for large values of k when damping effects become appreciable in the linear approximation. Second, the term $\kappa T_0k^2/m$ replaces our previous terms in Eqs. (25), (26), (27), and (30) in the limit $ck\gg\omega_p$, ω_c. This shows that for long wavelengths the transport dispersion relation has only semiquantitative accuracy. One can see the reason for this discrepancy most easily in the case treated by the Thomsons, i.e., plasma oscillations in the absence of magnetic fields. Then the assumption was that to first-order terms the term $\partial\langle v_x^2\rangle/\partial x$ is negligible compared to the other terms in Eq. (7). With the help of the distribution function

$$f=f_0(v)+[eE_x/mi(kv_x-\omega)]\partial f_0e^{i(kx-\omega t)}/\partial v_x,$$

we see that this is not the case and that the contribution of this term is, in general, of the same order as those of the terms retained by the Thomsons. We remark in passing that the transfer treatment corresponds to the lowest approximation in the method of treating thermal

[14] J. J. and G. P. Thomson, *Conduction of Electricity in Gases* (Cambridge University Press, London, 1933), Vol. 2, p. 353.

motions by expanding the distribution function in spherical harmonics, but a study of the effects of higher approximations has not yet been carried out. Although the complexity increases greatly for the higher approximations, this procedure is still easier than finding exact solutions.

The second method of solving the transport equations is an adaptation of the Hilbert-Enskog zero approximation and is somewhat more systematic though more complicated. In this method one uses Eqs. (2) through (6) and takes the distribution function to be of the form

$$f=\rho(m/2\pi\kappa T)^{\frac{3}{2}}\exp(-m/2\kappa T)[(v_x-\langle v_x\rangle)^2+(v_y-\langle v_y\rangle)^2+(v_z-\langle v_z\rangle)^2], \quad (34)$$

where ρ, T, $\langle v_x\rangle$, $\langle v_y\rangle$, $\langle v_z\rangle$ are functions to be determined by Eqs. (2) through (6). One can easily verify, by forming moments with the above distribution function, that these are the same as those used in setting up Eqs. (2) through (6). Furthermore, the above distribution function allows one to express the higher moments in terms of the five unknown functions, so that Eqs. (2) through (6) become a determined set of five equations in five unknowns. We have, in fact,

$$\rho\langle v_iv_k\rangle=\rho\langle v_i\rangle\langle v_k\rangle+\delta_{ik}\rho\kappa T/m, \quad (35a)$$

$$\rho\langle v_xv_y^2\rangle=\rho\langle v_x\rangle\langle v_y\rangle^2+\rho\langle v_x\rangle\kappa T/m, \quad (35b)$$

$$\rho\langle v_x^3\rangle=\rho(\langle v_x\rangle)^3+3\rho\kappa T\langle v_x\rangle/m. \quad (35c)$$

If we substitute these equations into Eqs. (31), put $\rho=\rho_0+\delta\rho$, $T=T_0+\delta\tau$, and simplify the equations for the case of small oscillations, we find the dispersion relation

$$[\omega^2-\omega_p^2-(5\kappa T_0k^2/3m)](\omega^2-\omega_p^2-c^2k^2)-\omega_c^2(\omega^2-c^2k^2)=0. \quad (36)$$

This differs from the previous treatment in that $\kappa T_0k^2/m$ is multiplied by 5/3. Comparing these results with the discussion given in Sec. VI, we see that the transport treatments do not predict bands of forbidden frequencies and a variation with magnetic field of the coefficient of the temperature term. The precise value of this coefficient will be important in all questions where the group velocity of the waves is significant. These results lead us to expect only qualitative accuracy in treating the excitation of plasma-type oscillations in a plasma with static electric fields as has been done by Bailey.[8]

The author is happy to express appreciation for the interest and encouragement of Professor A. von Hippel. He is also indebted to Professor D. Bohm for several stimulating discussions.

APPENDIX 1

Let us now evaluate the series occurring in Eqs. (12) and (13). We have to deal with series of the types

$$\left.\begin{aligned}K_p&=\sum_{n=-\infty}^{+\infty}\frac{J_{m+p}(a)J_m(a)}{m+1+\omega/\omega_c},\\ L_p&=\sum_{n=-\infty}^{+\infty}\frac{J_{m+p}J_m}{m-1+\omega/\omega_c}.\end{aligned}\right\} \quad (37)$$

We start from the addition theorem for bessel functions[15]

$$J_p(c)e^{ip\beta}=\sum_{n=-\infty}^{+\infty}J_{p+m}(a)J_m(b)e^{im\gamma}. \quad (38)$$

This relation is valid if the five quantities a, b, c, β, γ are all real and are restricted by the relation $ce^{i\beta}=a-be^{-i\gamma}$. For our purposes we shall want to put $a=b$ and to let γ range along the real axis from 0 to 2π. Then, if c and β are to be real for all γ, we must have $c^2=2a^2(1-\cos\gamma)$ and $e^{i\beta}=(1-e^{-i\gamma})/\sqrt{2}(1-\cos\gamma)^{\frac{1}{2}}$, or $\tan\beta=\sin\beta/(1-\cos\gamma)$. Equation (38) becomes

$$\frac{J_p(\sqrt{2}a[1-\cos\gamma]^{\frac{1}{2}})(1-e^{-i\gamma})^p}{(\sqrt{2})^p([1-\cos\gamma]^{\frac{1}{2}})^p}=\sum_{n=-\infty}^{+\infty}J_{p+m}(a)J_m(a)e^{im\gamma}. \quad (39)$$

To sum a series of type K_p we multiply both sides by $\exp\{i[(\omega/\omega_c)+1]\gamma\}$ and integrate γ from 0 to 2π. The result is

$$K_p=\frac{i}{\exp(i\omega 2\pi/\omega_c)-1}\times\int^{2\pi}\frac{J_p(\sqrt{2}a[1-\cos\gamma]^{\frac{1}{2}})(1-e^{-i\gamma})^p\exp\{i[(\omega/\omega_c)+1]\gamma\}d\gamma}{(\sqrt{2})^p(1-\cos\gamma)^{p/2}}. \quad (40)$$

Series of type L_p may be summed by multiplying by $\exp\{i[(\omega/\omega_c)-1]\gamma\}$ and integrating. Since we are concerned only with L_0, we find

$$L_0=\frac{i}{\exp(i\omega 2\pi/\omega_c)-1}\times\int_0^{2\pi}J_0(\sqrt{2}a[1-\cos\gamma]^{\frac{1}{2}})\exp\left[i\left(\frac{\omega}{\omega_c}-1\right)\gamma\right]d\gamma. \quad (41)$$

In the present notation the expressions for α, β, γ, δ are

$$\left.\begin{aligned}\alpha,\gamma&=\frac{4\pi e^2}{m\omega_c}\pi\int_{-\infty}^{+\infty}dv_z\int_0^{\infty}\rho^2d\rho\frac{df_0}{d\rho}(K_0\pm K_2),\\ \beta,\delta&=\frac{4\pi e^2}{m\omega_c}\pi\int_{-\infty}^{+\infty}dv_z\int_0^{\infty}\rho^2d\rho\frac{df_0}{d\rho}(K_2\pm L_0).\end{aligned}\right\} \quad (42)$$

For the case where f_0 is maxwellian it is possible to simplify the preceding expressions. This procedure is illustrated in Sec. VI for the special case of the plasma-type solutions (Eq. (28)), and consists of inversion of the orders of integration and subsequent use of the relation

$$\int_0^{\infty}J_v(at)\exp(-p^2t^2)t^{v+1}dt=\frac{a^v}{(2p^2)^{v+1}}\exp\left(\frac{-a^2}{4p^2}\right). \quad (43)$$

The resulting expressions are

$$\alpha,\gamma=-\frac{\omega_p^2}{\omega_c}\frac{i}{\exp[i2\pi(\omega/\omega_c)]-1}\int_0^{2\pi}d\gamma\exp\left[i\left(\frac{\omega}{\omega_c}+1\right)\gamma\right]\times[(d/d\mu)(\mu e^{-\mu(1-\cos\gamma)})\pm\tfrac{1}{2}\mu(1-e^{-i\gamma})^2e^{-\mu(1-\cos\gamma)}], \quad (44a)$$

$$\beta,\delta=-\frac{\omega_p^2}{\omega_c}\frac{i}{\exp[i2\pi(\omega/\omega_c)]-1}\int_0^{2\pi}d\pi\exp\left[i\left(\frac{\omega}{\omega_c}+1\right)\gamma\right]\times[\tfrac{1}{2}\mu(1-e^{-i\gamma})^2e^{-\mu(1-\cos\gamma)}\pm e^{-2i\gamma}(d/d\mu)(\mu e^{-\mu(1-\cos\gamma)})]. \quad (44b)$$

APPENDIX 2

Let us consider the integral

$$I=\int_0^{2\pi}\exp[i(\omega/\omega_c)\delta]\exp[-\mu(1-\cos\delta)]d\delta,$$

where ω is assumed to be real. We find that the stationary points (in the complex plane) of $i(\omega/\omega_c)\delta-\mu(1-\cos\delta)$ are given by $\delta_0=i\phi_0+m\cdot 2\pi$, where $\sin\phi_0=\omega/\omega_c\mu$, so that ϕ_0 is real and positive. In our problem $\mu=(k^2/\omega_c^2)\kappa T/m$, so that $\omega/\omega_c\mu=(m/\kappa T)(\omega\omega_c/k^2)\to 0$ as $\omega_c\to 0$ and $\omega^2/\omega_c^2\mu=(m\omega^2/\kappa Tk^2)$ is constant. The next step is to look for contours along which the imaginary part of the argument is constant. For the contours passing through the first saddle point and the origin we have

[15] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, New York, 1945), p. 144.

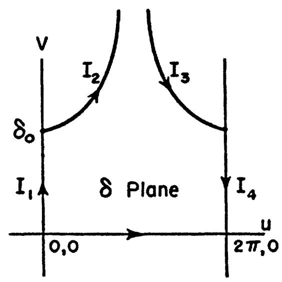

FIG. 1. Paths of constant imaginary argument.

$I[i(\omega\delta/\omega_c)-\mu(1-\cos\delta)]=0$. With $\delta=u+iv$ this becomes $(\omega u/\omega_c)-\mu\sin u\sinh v=0$. By Cauchy's theorem we may deform the path from 0 to 2π to pass along the paths of constant imaginary argument indicated in Fig. 1. On the portions I_1 and I_2 the value of the imaginary argument is zero, while on I_3 and I_4 the value is $2\pi i\omega/\omega_c$. Thus, the integration along I_3 and I_4 gives $-\exp[2\pi i(\omega/\omega_c)](I_1+I_2)$. Furthermore, the contribution to the integral is greatest near the origin and decreases as one proceeds along I_1 and I_2. As a result, the main contribution comes from the integrals along the straight portions I_1 and I_4. We now study I_1 in more detail. For $\omega_c/\omega\ll 1$

$$I_1=i\int_0^{\omega/\omega_c\mu}\exp[-(\omega/\omega_c)\phi]\exp[-\mu(1-\cosh\phi)]d\phi.$$

Putting $\zeta=\omega_c\phi/\omega$, we find

$$I_1=i\frac{\omega}{\omega_c}\int_0^{\omega^2/\omega_c^2\mu}e^{-\zeta}\exp\left[-\mu\left(1-\cosh\frac{\omega_c}{\omega}\zeta\right)\right]d\zeta.$$

We now note that the factor $e^{-\zeta}$ starts at 1 and decreases to $\exp(-\omega^2/\omega_c^2\mu)$ at the upper limit, while the other factor starts at 1 and increases to only $\exp[\frac{1}{2}(\omega^2/\omega_c^2\mu)]$ when ω_c is small. One can therefore evaluate I_1 by expanding the second factor in a convergent series about $\zeta=0$ and integrating term by term. The result is

$$I_1=i\frac{\omega_c}{\omega}\int_0^{\omega^2/\omega_c^2\mu}e^{-\zeta}\left[1+\frac{\mu\omega_c^2}{\omega^2}\frac{\zeta^2}{2!}+\frac{\mu\omega_c^2}{\omega^2}\left(\frac{\omega_c^2}{\omega^2}+\frac{3\mu\omega_c^2}{\omega^2}\right)\frac{\zeta^4}{4!}\right.$$
$$\left.+(\mu+15\mu^2+15\mu^3)\frac{\omega_c^6}{\omega^6}\frac{\zeta^6}{6!}+\cdots\right]d\zeta. \quad (45)$$

The upper limit gives rise to terms of order $\exp(-m\omega^2/\kappa Tk^2)$; and if one is not interested in studying the behavior of the frequency for waves of the Debye length, such terms may be neglected by letting the upper limit tend to infinity. The sum of the contributions along the two straight portions then becomes

$$I_1+I_4=i(\omega_c/\omega)[1-\exp(2\pi i\omega/\omega_c)]$$
$$\times\left\{1+\frac{\mu\omega_c^2}{\omega^2}+\frac{\omega_c^4}{\omega^4}(\mu+3\mu^2)+\frac{\omega_c^6}{\omega^6}(\mu+15\mu^2+15\mu^3)\right\}. \quad (46)$$

When this value is substituted in the expression (29), one finds the dispersion relation

$$\frac{\omega^2}{\omega_p^2}=1+\frac{\omega_c^2}{\omega^2}+\frac{3}{\omega^2}k^2\frac{\kappa T}{m}+\frac{\omega_c^4}{\omega^4}+15\frac{\omega_c^2}{\omega^4}\left(\frac{\kappa T}{m}k^2\right)+\frac{15}{\omega^4}\left(\frac{\kappa T}{m}k^2\right)^2. \quad (47)$$

Equation (47) can be solved by successive approximations when ω_c^2/ω_p^2 and $(\kappa T/m)(k^2/\omega_p^2)$ are small. In the first approximation one puts $\omega=\omega_p$ on the right-hand side and obtains $\omega^2=\omega_p^2+(3\kappa Tk^2/m)+\omega_c^2$. This reduces to the formula for the frequency of plasma oscillations in the absence of a magnetic field. The second approximation to the frequency may be found by substituting the first approximation into the right-hand side. It is

$$\omega^2=\omega_p^2+(3k^2\kappa T/m)+\omega_c^2+(9\omega_c^2\kappa Tk^2/\omega_p^2m)$$
$$+6/(\kappa Tk^2/m)^2\omega_p^2. \quad (48)$$

We must now consider the contribution of the curved portions to the integral I. As one proceeds along the curve out from the saddle-point δ, the contribution to the integral rapidly becomes negligible, so that one may use the first approximation in the method of steepest descents.[16] We note that since we are integrating along a path in the complex plane, we are to expect a contribution which will make ω complex. The contribution to I is

$$I_2+I_3=\left[1-\exp\left(2\pi i\frac{\omega}{\omega_c}\right)\right]\frac{(2\pi)^{\frac{1}{2}}}{2}\frac{e^{[i(\omega\delta_0/\omega_c)-\mu(1-\cos\delta_0)]}}{\mu^{\frac{1}{2}}(\cos\delta_0)^{\frac{1}{2}}}. \quad (49)$$

If this expression is inserted in the dispersion relation (29), one finds for the imaginary part of $\omega=\omega_R+i\sigma$,

$$\sigma=-\frac{\pi}{2}\frac{\omega_p^4}{(2\pi)^{\frac{1}{2}}k^3}\left(\frac{m}{\kappa T}\right)^{\frac{1}{2}}\frac{\exp(-m\omega^2/2\kappa Tk^2)}{(1+\omega^2\omega_c^2m/k^2\kappa T)^{\frac{1}{2}}}. \quad (50)$$

By a comparison with Eq. (19) of reference 1 we see that this is only the Landau damping slightly modified by the presence of the magnetic field. It is to be emphasized that these results hold only for the case of small damping and small magnetic fields. If ω has a sizable imaginary part, the assumption that ω is real which has been made in evaluating I will break down.

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[16] H. and B. S. Jeffreys, *Methods of Mathematical Physics* (Cambridge University Press, London, 1946), p. 473.