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Internal Retardation

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Summary

It is suggested that the quantum electrodynamics of point particles be formulated as the point particle limit of a theory of extended particles, and that this limit be taken on the quantum level rather than the classical level. One is thus required to include the retarded interaction within the extended particle. In the nonrelativistic approximation, this method leads to a vanishing self-energy and to the absence of run-away and preacceleration effects, as has been shown previously. A comparison with the conventional perturbation methods of quantum electrodynamics is made.

I. Introduction

Conventional quantum electrodynamics (QED) treats charged particles as point particles from the very beginning. Proceeding via the conventional perturbation expansion one is led to divergent self-energy effects; specifically to an electromagnetic contribution to the particle mass which is divergent and which is removed by renormalization.

We propose here to treat charged particles, initially, as extended objects and to take the point particle limit at a suitable stage of the calculation, *after* the internal retardation effects have been taken into account to all orders in Planck's constant $\hbar$. We argue that this procedure is an improvement over the conventional one in that it leads (in the nonrelativistic approximation, which is the only one that has been carried through so far [1]) to a convergent electromagnetic self-mass δm even in the point particle limit. In the classical limit δm remains convergent and,

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furthermore, the solutions do not display runaway behavior or preacceleration [2]. The background of the present discussion can be found in Refs. [1, 3–5].

In carrying out the proposed program an approximation method is employed which differs completely from the conventional perturbation expansion. This method is in a sense an expansion in positive powers of $\hbar$ while the conventional one involves powers of α or $1/\hbar$. We shall return to this comparison later.

In order to distinguish the usual point particle theory (PPT) from the one in which the point limit is taken only at the end, we shall call the latter "quasi-point particle theory" (QPPT). The fact that QPPT leads to results different from PPT, i.e., the fact that the convergent δm of the extended particle theory survives as a convergent δm even when the point particle limit is taken, can be compared to the grin of the Cheshire cat that survived after the cat has disappeared.

The mechanism of this phenomenon is not at all mysterious. It is essentially just the observation that a particular limit cannot be interchanged with an integration. Consider a charge distribution ρ centered about the position $\mathbf{R}(t)$ and characterized by a length L. Its point limit is

$$\lim_{L\to 0} \rho(\mathbf{x}-\mathbf{R}(t); L) = \delta_3(\mathbf{x}-\mathbf{R}(t)). \tag{1.1}$$

The retarded potential

$$\phi_{\mathrm{ret}}(\mathbf{R}) = e\int \frac{\rho(\mathbf{x}'-\mathbf{R}(t_{\mathrm{ret}}))\, d^3x'}{|\mathbf{x}'-\mathbf{R}(t)|} \tag{1.2}$$

has the point limit

$$\lim_{L\to 0}\int \frac{\rho(\mathbf{x}'-\mathbf{R}(t_{\mathrm{ret}}))}{|\mathbf{x}'-\mathbf{R}(t)|}\, d^3x' \neq \int \lim_{L\to 0} \frac{\rho(\mathbf{x}'-\mathbf{R}(t_{\mathrm{ret}}))}{|\mathbf{x}'-\mathbf{R}(t)|}\, d^3x'. \tag{1.3}$$

The left side is QPPT, the right side is PPT. The latter gives

$$\int \frac{\delta_3(\mathbf{x}'-\mathbf{R}(t_{\mathrm{ret}}))}{|\mathbf{x}'-\mathbf{R}(t)|}\, d^3x' = \int \frac{\delta_3(\mathbf{x}'-\mathbf{R}(t-|\mathbf{x}'-\mathbf{R}(t)|/c))}{|\mathbf{x}'-\mathbf{R}(t)|}\, d^3x'$$

$$= \int \frac{\delta_3(\mathbf{x}'-\mathbf{R}(t))}{|\mathbf{x}'-\mathbf{R}(t)|}\, d^3x' = \infty. \tag{1.4}$$

In PPT the retardation effect disappears and the result diverges. We shall show in the following section how QPPT leads to a finite result, reviewing some of the key points of Ref. [1]. In Sec. 2 we shall also review some of the assumptions underlying nonrelativistic QED. A comparison with the conventional theory is then made in Sec. 3.

II. Nonrelativistic Quantum Electrodynamics

The classical electrodynamics (CED) of electrons involves the characteristic length $r_0 = (e^2/mc^2)$ and a corresponding characteristic time interval $\tau_0 = \frac{2}{3}(e^2/mc^3) = \frac{2}{3}r_0/c$. Quantum mechanics involves the characteristic length $\lambda = (\hbar/mc)$. In either theory extended particles are characterized by an independent length L. These three lengths and their relative size are of crucial importance in the following.

The unphysical runaway solutions and preacceleration are believed to be present in CED whenever $L < r_0$ and absent for $L > r_0$, as has been discussed at various times [6, 1]. On the other hand, for point particles ($L = 0$) in nonrelativistic QED, no such unphysical effects occur. Since CED is expected to be a limit of QED there exists an apparent inconsistency. Its resolution was one of the accomplishments of Ref. [1].

Like any quantum mechanical theory, nonrelativistic QED is characterized by a set of operators which satisfy an equal-time commutator algebra, and a Hamiltonian. Here the basic operators are the position ($\mathbf{R}$) and momentum ($\mathbf{P}$) operators of the charged particle, and the quantized potentials of the electromagnetic field, $\mathbf{A}$ and ϕ. The Hamiltonian is

$$H = \frac{1}{2m_0}\left[\mathbf{P} - \frac{e}{c}\mathbf{A}\right]^2 + \frac{1}{8\pi}\int d^3\mathbf{x}(\mathbf{E}^2 + \mathbf{B}^2). \tag{2.1}$$

In addition, for extended particles a static charge distribution $e\rho(\mathbf{x}-\mathbf{R})$ is specified. ρ depends on some length L which characterizes the size of the distribution. We shall not display this dependence explicitly from here on, as we have done in (1.1). By the point particle limit we always mean $L \to 0$.

The charge distribution ρ can be thought of as having its origin in some other interaction (weak interaction for an electron, strong interaction for a proton, etc.). We regard the introduction of ρ as a crude phenomenological way of summarizing the effects of these interactions in an electrodynamic calculation. We take the point limit, $L \to 0$, $\rho \to \delta$ at the end of the calculation so that our results do not depend on the details of the unknown quantity ρ. Finally, we note that treating the charge distribution as static is consistent with our nonrelativistic approximation, in which energies are excluded that are high enough to probe the internal structure of the particle.

The finite extension of the charge affects the interaction in (2.1) in that $\mathbf{A}$ and ϕ in the Hamiltonian are not the fields $\mathbf{A}(\mathbf{x})$ and $\phi(\mathbf{x})$ at a point $\mathbf{x}$, but are "smeared" over the charge distribution:

$$A^\mu \equiv A^\mu(\mathbf{R}) = \int d^3x\, \rho(\mathbf{x}-\mathbf{R}) A^\mu(\mathbf{x}, t). \tag{2.2}$$

The above formulation of the theory permits one to obtain in standard fashion the Maxwell equations and the Lorentz equation of motion for the position $\mathbf{R}(t)$ of the particle center. Taking account of the retardation effects, this equation of motion reduces after a long calculation [1] to

$$m_0 \ddot{\mathbf{R}} = e\mathbf{E}_{in} - \frac{2}{3}\frac{e^2}{c^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}A_n\frac{d^{n+2}}{dt^{n+2}}\mathbf{R}(t), \tag{2.3}$$

where the A_n are known functionals of ρ.

In deriving Eq. (2.3), only terms linear in a time derivative of $\mathbf{R}$ have been retained. This approximation is in keeping with our nonrelativistic treatment of the motion of the charged particle, which requires the neglect of all terms of order

$$(D^m\mathbf{R}/c)(D^n\mathbf{R}/c) \qquad (m+n \geqslant 0), \qquad D \equiv \frac{\lambda\!\!\!^{-}}{c}\frac{d}{dt},$$

or smaller. Thus a consistent application of the nonrelativistic approximation leads to a linearization of the problem. However, it would be incorrect to say that one is attempting a linearization and that *therefore* the higher order terms are being dropped.

Another aspect of this approximation bears mentioning. Operator products may be neglected only in appropriate Hermitean combinations; otherwise an approximation may result which is not internally consistent. However, if this caution is observed, the indicated approximation is unique.

Finally, we note that in the quantum mechanical context, the statement that terms such as $\dot{\mathbf{R}}^2/c^2$ are negligible means that there is a subset of states in Hilbert space for which the matrix elements $\langle m|\dot{\mathbf{R}}^2/c^2|n\rangle$ are "small", and that one can work consistently to a given level of accuracy within this set of states. It is a crucial assumption of this calculation that such a set of states exists.

The operator equation (2.3) is formally identical to the classical result for an extended charge [7]; the coefficients A_n of course depend differently on ρ. All the results to be discussed now follow from this equation.

The electromagnetic mass δm is identified as the negative coefficient of $\ddot{\mathbf{R}}$ in the sum (2.3); for a spherically symmetric charge distribution:

$$\delta m = \frac{2}{3}\frac{e^2}{c^3}A_0$$

$$= \frac{2}{3}\frac{e^2}{c^3}\left(1 + \frac{\lambda\!\!\!^{-}_0}{6}\frac{\partial}{\partial \lambda\!\!\!^{-}_0}\right)\left(1 + \lambda\!\!\!^{-}_0\frac{\partial}{\partial \lambda\!\!\!^{-}_0}\right)\frac{2}{\pi}P\int_0^{\infty}\frac{\tilde{\rho}^2(k)\,dk}{1-(\lambda\!\!\!^{-}_0 k/2)^2} < \infty, \tag{2.4}$$

where $\lambda\!\!\!^{-}_0 = (\hbar/mc)$. The convergence of δm depends crucially on the fact that (2.3) was carried through to all orders in $\hbar$, resulting in the particular dependence of the integral for δm on λ_0.

A mass renormalization must be carried out since an inertial term occurs on the right-hand side of (2.3)

$$m = m_0 + \delta m. \tag{2.5}$$

The renormalized equation of motion now reads

$$\ddot{\mathbf{R}} = \frac{e}{mc}\mathbf{E}_{in} - m\tau_0\sum_{1}^{\infty}\frac{(-1)^n}{n!c^n}A_n\frac{d^n\ddot{\mathbf{R}}}{dt^n}. \tag{2.6}$$

The point particle limit leads to

$$\delta m = 0 \qquad (L \to 0), \tag{2.7}$$

and the A_n vanish for n even; for n odd they are proportional to $\lambda\!\!\!^{-\,n-1}$; where $\lambda\!\!\!^{-} = \lambda\!\!\!^{-}_0$ because of (2.7). Therefore, when the classical limit ($\lambda\!\!\!^{-} \to 0$ equivalent to $\hbar \to 0$) is carried out *now*, only the $\dot{\mathbf{R}}$ term survives and one obtains in fact exactly the Lorentz equation with radiation reaction. One notes the order of limits: $L \to 0$ first, $\hbar \to 0$ second, which explains the difference between this result and the well-known divergent δm of the classical theory. The latter is recovered with $\hbar \to 0$ first, $L \to 0$ second.

In order to study the free particle in this theory one cannot put the in-fields equal to zero since they are operators. One must take the subspace Φ of the Hilbert space $\mathscr{H}$ which is spanned by all vectors which describe the photon vacuum (no incident or outgoing photons).

Consider the quantum mechanical free *point* particle. Runaway solutions exist only when for some states $|m\rangle$ and $|n\rangle$ the matrix elements

$$\dot{\mathbf{R}}(t)_{mn} = e^{iE_{mn}t/\hbar}\dot{\mathbf{R}}(0)_{mn} \tag{2.8}$$

do not vanish and E_{mn} has a negative imaginary part. One can show [1, 4] that this is not the case for the physical value of the fine structure constant α (and in fact it is not the case for any $\alpha < \alpha_{\rm crit} \sim 1$). One also notes that the series which gives (2.8) converges only for $|E_{mn}| < \frac{1}{2}mc^2$, i.e., everywhere in the nonrelativistic domain.

This system (based on the subspace Φ) can be generalized to include a *classical* (c-number) external force $\mathbf{F}(t)$. The response function G defined by

$$m\ddot{\mathbf{R}} = \int dt'\,G(t-t')\mathbf{F}(t') \tag{2.9}$$

determines whether preacceleration effects are present. One shows easily [1, 4] that the above theory allows no such effects either on the quantum level or in the classical limit.

III. Comparison of QPPT with PPT

The domain of validity of classical physics (as compared to quantum physics) is expected to be characterized by classical lengths a which are large compared to the Compton wavelengths of the participating particles. In our language

$$\frac{\lambda}{a} \ll 1, \qquad \text{for the classical domain of validity.} \tag{3.1}$$

An expansion in powers of λ/a, i.e., in powers of $\hbar$, should therefore have the classical description as its zeroth approximation. The results summarized in Sec. 2 were obtained using an expansion of essentially this kind; in particular Eq. (2.6) is an expansion of the form $(\lambda/c)^{n-1}(d^{n-1}/dt^{n-1})$ in the point charge limit. Note, however, that since the main results depend crucially on resumming the series in $\hbar$, the method cannot properly be described as a semi-classical one.

By comparison, the conventional perturbation expansion of QED is an expansion in powers of α,

$$\alpha = \frac{r_0}{\lambda} \tag{3.2}$$

i.e., the ratio of r_0, characteristic of CED, to λ, characteristic of quantum mechanics. This ratio is small and according to (3.1) cannot have the classical description as a zeroth order approximation. In fact, it is believed that this expansion is asymptotic, rather than convergent. It describes a bare particle-field to which the electromagnetic self-field is added "photon by photon".

The expansion in powers of α is on one hand an expansion in powers of $1/\hbar$ (rather than $\hbar$) and on the other hand an expansion in powers of e^2. No expansion in e^2 is involved in deriving the results in Sec. 2. In particular, in Eq. (2.6) e occurs explicitly only to the first and second power.

The reason for this is the following. In the work described in Sec. 2, the gauge invariant expression

$$\dot{\mathbf{R}} = \left(\mathbf{P} - \frac{e}{c}\mathbf{A}\right)\Big/ m_0 \tag{3.3}$$

which occurs in the Hamiltonian is never broken up into the separately gauge dependent terms $\mathbf{P}$ and $(e/c)\mathbf{A}$. Equation (2.6) contains only derivatives of $\dot{\mathbf{R}}$, so that the factor e multiplying $\mathbf{A}$ never occurs explicitly. Moreover, the nonrelativistic approximation which makes (2.6) linear in the time derivatives of $\dot{\mathbf{R}}$ prevents the occurrence of even an implicit infinite expansion in powers of e.

The gauge invariant Hamiltonian (2.1), which can be written as

$$H = \tfrac{1}{2} m_0 \dot{\mathbf{R}}^2 + H_{\text{field}}, \tag{3.4}$$

where $\dot{\mathbf{R}}$ is defined by (3.3), gives no indication of an interaction between the particle and the field. That interaction lies entirely in the operator algebra which will yield a commutator $[\dot{\mathbf{R}}, H_{\text{field}}]$ that will vanish if there is no interaction, and that will not vanish if there is an interaction. In the latter case e will measure the deviation of this commutator from zero.

IV. Conclusion

It is clear from the foregoing discussion that the QPPT formulation of nonrelativistic QED followed by the point particle limit involves approximations which are very different from those of conventional perturbative QED. The key *physical* difference lies in the internal retardation, which is treated exactly in QPPT and which does not occur in PPT. PPT only describes the retarded interaction between different particles.

The consequences of including internal retardation are striking in that they remove the key objections usually leveled against the classical theory: we find that runaway solutions and preacceleration do not appear in the classical limit of QPPT. Nor do these effects occur at the quantum level in QPPT.

Among the open questions concerning this work, two stand out as particularly difficult and important: (i) Can our procedure of introducing a form factor for the charged particle, and ultimately taking the point limit, be validated from the first principles? (ii) Can our calculations be extended to the domain of *relativistic* quantum electrodynamics? It would seem that a positive answer to both of these questions is a necessary condition for future progress along the lines discussed here.

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