of impressed force in the conductors and in the dielectrics require to be distinguished.*

In the Volta-force experiment (say, copper and zinc), we have a feeble thermo-electric force at the metallic junction, feeble thermo-electric forces in the zinc and copper if temperature varies, and two big forces, one of $x + \cdot 8$ volt, say, in a thin layer of the dielectric next the zinc surface, and one of x volt similarly at the other end of the dielectric. On charging, the rise of potential is very nearly as much at the zinc layer, with very nearly as much fall as before at the copper layer.

The theory of impressed force and potential in a dielectric is curiously illustrated by the phenomenon of absorption. The electric elasticity is not perfect; under the action of the stress the dielectric slowly yields, and with it a part of the displacement set up by an impressed force outside ceases to be of the elastic character, becoming intrinsic, and the difference of potential falls, requiring more current to enter to keep up the difference of potential.

The first discharge, like the first charge, is of elastic displacement. What is left, which shows no signs of being there at all, was elastic, but is no longer so. We may regard it as being kept up by uniformly distributed impressed force in the dielectric itself, arising from altered state of the dielectric produced by loss of elasticity. In time it recovers itself, or the impressed force is taken off, when the residual charge shows itself by the difference of potential it can now produce.

To really discharge the condenser at once, we must apply, after the first discharge, an opposite impressed force of the right amount, of course apparently charging the condenser oppositely to before. Leave it to itself, disconnected, and the apparent charge will gradually disappear.†

Residual magnetisation in soft iron is somewhat analogous, but the effect is of far greater magnitude, and there is permanent set as well, which becomes predominant in intrinsic steel magnets. But we can set up permanent set of displacement also in a dielectric, as by passing a current through warm glass and then cooling it. It is then like a permanent magnet.

If we had conductors for magnetic induction (analogous to electric conductors), we, by magnetising a plate of iron, setting up residual magnetisation, could apparently discharge it so as to show no force outside. It would then be like the charged condenser (in which "absorption" has occurred) after its apparent discharge.

* [The above investigation may be compared with that on p. 376 *ante*, relating to condensers in sequence, subjected to a simple-harmonic impressed force.]

† [See Sections x. and xii. of the next Art., XXX.]

XXX. ELECTROMAGNETIC INDUCTION AND ITS PROPAGATION.

[*The Electrician*, 1885-6-7. Section I., Jan. 3, 1885, p. 148; II., Jan. 10, p. 178; III., Jan. 24, p. 219; IV., Feb. 21, p. 306; V., March 14, p. 366; VI., April 4, p. 430; VII., April 25, p. 490; VIII., May 15, p. 6 (vol. 15); IX., June 12, p. 73; X., July 3, p. 134; XI., July 17, p. 170; XII., August 7, p. 230; XIII., August 21, p. 270; XIV., August 28, p. 290; XV., September 4, p. 301; XVI., October 9, p. 408; XVII., Nov. 13, p. 6 (vol. 16); XVIII., Nov. 27, p. 46; XIX., Dec. 11, p. 86; XX., Dec. 18, 1885, p. 106; XXI., Jan. 1, 1886, p. 146; XXII., Jan. 15, p. 186; XXIII., Jan. 22, p. 206; XXIV., March 26, p. 386. The second half of this article is in Vol. II., with the references thereto.]

Section I. Rough Sketch of Maxwell's Theory.

Conductivity, Capacity, and Permeability.

In the electromagnetic scheme of Maxwell there are recognised to be three distinct properties of a body considered with reference to electric force and magnetic force, viz., conductivity, electrostatic capacity, and magnetic permeability. The body may support a conduction current, it may support electric displacement, and it may support magnetic induction. These three phenomena may, and in general do, coexist at any one point. Quantitatively considered, they are all vector magnitudes, having definite directions as well as strengths, which are reckoned per unit area perpendicular to their directions in terms of chosen units.

The facility of supporting the three states of conduction current, electric displacement and magnetic induction varying with the nature of the medium for equal amounts of energy concerned, brings in three coefficients, the electric conductivity k, the electric capacity c, and the magnetic permeability μ. At first sight it might appear as if three other vector magnitudes related to the former by these coefficients were involved; but in reality there are but two, the electric force and the magnetic force, the former being connected with both the conduction current and the displacement.

First we have Ohm's law. $\mathbf{C}$ being the conduction current-density, $\mathbf{E}$ the electric force, and k the specific conductivity,

$$\mathbf{C} = k\mathbf{E}. \quad \text{.........(Conduction current)} \quad (1)$$

Far more is known about conductivity than about capacity or permeability. In an unstrained isotropic metal, k appears to depend on the temperature only, and not to vary rapidly with it. That is, k is practically a constant, which simplicity is of great utility. Within wide limits k is independent of the current or the electric force.

The range of conductivity in different media is very great. From the conductivity of copper to that of cold glass is such an enormous range as to compare with astronomical ratios, and it speaks well for electrical science that it can compare definitely such widely differing magnitudes.

Dry air in its ordinary state appears to have no conductivity. But it is a vacuum that is the perfect non-conductor in Maxwell's theory. Where there is no matter, in the ordinary sense, there is no dissipation of energy; and ether, whatever it be, is perfectly conservative and non-dissipative, dynamically considered. Dissipation of energy is a necessary accompaniment of a conduction current, so far as is known; though of course a perfect conductor can be imagined in which a continuous current developed no heat. But ether cannot be this perfect conductor consistently with the propagation of magnetic disturbances, for none can be propagated in a perfect conductor. Grant that they are propagated in pure ether (space from which all "matter" has been removed) without loss of energy in the medium, and it follows that ether is the perfect non-conductor. This however, somewhat anticipates electric displacement and magnetic induction.

Equation (1) is a vector equation. In an isotropic medium k is a scalar constant. We may symbolise $\mathbf{E}$, $\mathbf{C}$, or other physical vector magnitudes by geometrical vectors, lines drawn of the proper lengths and in the proper directions. Thus $\mathbf{E}$ is one vector, $\mathbf{C}$ is another, and when, as ordinarily, k is a scalar constant, (1) says simply that $\mathbf{C}$ and $\mathbf{E}$ are parallel, and that $\mathbf{C}$ is k times as long as $\mathbf{E}$. Vector quantities are compounded like velocities; in a vector equation containing n vectors, separated by + or − signs, the n vectors form the n sides of a polygon. But two straight lines cannot enclose a space, so, in equation (1), $\mathbf{C}$ and $k\mathbf{E}$ are parallel and equal.

But in a body eolotropic as regards conductivity, $\mathbf{C}$ and $\mathbf{E}$ are only exceptionally parallel. Using the same equation (1) to represent the relation between them, k, from being a scalar constant, becomes a linear operator; $k\mathbf{E}$ must be regarded as a single symbol, being $\mathbf{E}$ operated upon by k in a certain manner, turning it into a new vector $k\mathbf{E}$. The operation is a little complex when expressed in Cartesian co-ordinates referred to any axes, so it is better to define once for all the meaning to be attached to k when eolotropy is to be included, and then use equation (1), rather than be repeating the Cartesian operations over and over again. The following defines the operation k, and the same will serve for c and μ later. First let there be no rotatory power. Then, in three directions, mutually perpendicular, fixed in a body at the point considered, depending on its structure there, Ohm's law, as ordinarily considered, is obeyed. That is to say, if electric forces $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ act successively parallel to the above mentioned directions of the principal axes of conductivity, and $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ be the corresponding currents, $\mathbf{C}_1$ will be parallel to $\mathbf{E}_1$, $\mathbf{C}_2$ to $\mathbf{E}_2$, and $\mathbf{C}_3$ to $\mathbf{E}_3$, and we shall have

$$\mathbf{C}_1 = k_1\mathbf{E}_1, \quad \mathbf{C}_2 = k_2\mathbf{E}_2, \quad \mathbf{C}_3 = k_3\mathbf{E}_3,$$

and

$$C_1 = k_1E_1, \quad C_2 = k_2E_2, \quad C_3 = k_3E_3, \quad \text{..................(2)}$$

where k_1, k_2, k_3 are scalar constants, being the principal conductivities, and C_1 is the tensor or magnitude of $\mathbf{C}_1$, C_2 of $\mathbf{C}_2$, etc. From these we may find the current when the force acts in any other direction than parallel to one of the principal axes. For if $\mathbf{E}$ be the force, let its components parallel to the axes be $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$; the components of the current will then be $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, as defined by (2). Compounding them, we get $\mathbf{C}$. Thus the relation of $\mathbf{C}$ to $\mathbf{E}$ requires a knowledge of the principal conductivities and the directions of the principal axes.

But should the body possess rotatory power, the above process is incomplete. Let ϵ be a vector, directed parallel to the conductivity axis of rotation, and of length properly chosen; then, to the current as found by the above process must be added another current expressed by

$$\mathrm{V}\epsilon\mathbf{E}, \quad \text{...............(Vector product) (3)}$$

which stands for a vector whose direction is perpendicular to the plane containing ϵ and $\mathbf{E}$, and whose length equals the product of the length of ϵ into that of $\mathbf{E}$, into the sine of the angle between their directions. This also defines the prefix V before two vectors. The + direction is defined thus. Let ϵ and $\mathbf{E}$ be the short and the long hands of a watch. Let ϵ point to XII. and $\mathbf{E}$ anywhere else. The angle between ϵ and $\mathbf{E}$ is measured positive in the usual direction of motion of the hands, and the direction of $\mathrm{V}\epsilon\mathbf{E}$ when positive is from the face to the back.

It is possible, consistent with the linear principle, for k_1, k_2, k_3 to be all zero, and ϵ not zero. Then

$$\mathbf{C} = \mathrm{V}\epsilon\mathbf{E}$$

simply; the current is always perpendicular to the force, of maximum strength when ϵ and $\mathbf{E}$ are perpendicular, and vanishing when they are parallel.

Returning to equation (1), multiply it by $\mathbf{E}$. Then

$$\mathbf{EC} = \mathbf{E}k\mathbf{E} = Q, \text{ say.} \quad \text{......(Dissipativity) (4)}$$

Q is the dissipativity per unit volume. It is, in the first place, the rate of working of the force $\mathbf{E}$, and next, by the experimental law of Joule, the rate of generation of heat per unit volume. (4) is a scalar equation. All our equations will be either wholly scalar or wholly vector. In case of isotropy, with k a scalar constant, we may write

$$Q = kE^2;$$

or, since

$$E = k^{-1}C,$$
$$Q = k^{-1}C^2,$$

where k^{-1} is the specific resistance, a more familiar form of Joule's law. But in general, when k is a linear operator, we must not take $\mathbf{E}k\mathbf{E} = kE^2$, unless $\mathbf{E}$ act parallel to one of the principal axes, when we may do so, with the appropriate value of k for that axis. When $\mathbf{E}$ and $\mathbf{C}$ are not parallel, the product $\mathbf{EC}$ means the strength of $\mathbf{E}$ multiplied by that of $\mathbf{C}$, and by the cosine of the angle between their directions; which of course includes the common algebraic meaning of $\mathbf{EC}$, since when $\mathbf{E}$ and $\mathbf{C}$ *are* parallel, $\cos 0° = 1$. Referred to three rectangular axes, if E_1, E_2, E_3 are the scalar components of $\mathbf{E}$, and C_1, C_2, C_3 those of $\mathbf{C}$, then

$$\mathbf{EC} = E_1C_1 + E_2C_2 + E_3C_3, \quad \text{.....(Scalar product) (5)}$$

which is an equivalent definition of $\mathbf{EC}$.

Coming next to specific capacity, although there are media, as air, which appear to have no conductivity, yet, by the continuity of the electric current, they can support current; not steady, but transient, and stopped elastically. By an obvious mechanical analogy the integral current is termed the electric displacement. Let this be **D**, and let **E**, as before, be the electric force. We have then

$$\mathbf{D} = c\mathbf{E}/4\pi. \quad \text{.....(Electric displacement) (6)}$$

The excrescence 4π is a mere question of units, and need not be discussed here. The 4π's are particularly obnoxious and misleading in the theory of magnetism. Privately I use units which get rid of them completely, and then, for publication, liberally season with 4π's to suit the taste of B.A. unit-fed readers. Of course, if it comes to numerical comparisons we should have to consider the ratios of units in the ordinary to what I may call the rational system. Sometimes it is $\sqrt{4\pi}$, sometimes $(4\pi)^{-\frac{1}{2}}$, sometimes 4π, sometimes unity, but in the mere algebra it is simply a matter of putting in 4π's here and there in translating from rational to ordinary units. [*See* pp. 199, 262.]

In a dielectric medium, the force and the displacement are simultaneous, like the force and the current in a conductor. Time does not appear in the equations. In an isotropic dielectric, c is simply a scalar constant; in an eolotropic dielectric it is, as described above for k, a linear vector operator, with this difference, however, that there is no rotatory vector ϵ, so that the relation of **D** to **E** is settled by the values of the principal capacities, and their axes.

Multiply (6) by $\frac{1}{2}\mathbf{E}$; then

$$\tfrac{1}{2}\mathbf{ED} = \mathbf{E}c\mathbf{E}/8\pi = U, \text{ say. (Electric energy) (7)}$$

U is the electric energy per unit volume, the work done by the force on the displacement as they rise from 0 to their final values, or the final displacement multiplied by the mean force which produced it. This energy is stored, and is recoverable in work like the energy of a perfectly elastic strained spring. It is unnecessary to assume that there is any real displacement of anything in the direction of the electric displacement. All the electric and magnetic quantities are more or less abstractions, measurable abstractions, whose real signification is as yet unknown.

Far less is known of c than of k, and it is not so agreeably definite as k. Solid dielectrics appear to have imperfect electric elasticity, as they have imperfect mechanical elasticity. The bent spring, with the applied force removed, and brought quietly to rest, is not exactly in its equilibrium position. A small part of the displacement remains, and slowly disappears. This is easily shown when not visible to the eye by using a microphonic contact; though, by the way, the variability of the contact itself makes it a bad method. Most likely there is no such thing as a perfect return even with small displacements; we cannot draw a hard and fast line to mark the limit of perfect elasticity.

All non-conductors are dielectrics. Bad conductors are also dielectrics. Good conductors, even the best, may be dielectrics as well, so that with a force **E** we shall have a conduction current $k\mathbf{E}$ and a

displacement $(4\pi)^{-1}c\mathbf{E}$ co-existing. But in such case, as well as in the case of known dielectric power of bad conductors, $k\mathbf{E}$ is not the complete or true current, unless the displacement remains steady. The time-variation of the displacement is itself an electric current, and the true current is the sum of the conduction current and of the rate of increase of the displacement. Let Γ be the true current; we then have, in a conducting dielectric, or dielectric conductor,

$$\mathbf{C} = k\mathbf{E}, \qquad \mathbf{D} = c\mathbf{E}/4\pi,$$

$$\Gamma = \mathbf{C} + \dot{\mathbf{D}} = k\mathbf{E} + c\dot{\mathbf{E}}/4\pi. \quad \text{......(True current) (8)}$$

Put $c=0$ in a pure conductor, and $k=0$ in a pure dielectric. It is the true current that is "the current" when we come to induction and variable states.

In the equation $\Gamma = \mathbf{C} + \dot{\mathbf{D}}$ we have three vectors. They form the three sides of a triangle, unless $\dot{\mathbf{D}}$ should be parallel to **C**. But $\dot{\mathbf{D}}$ may not be parallel to **C**, nor need it be parallel to **D**. If we charge a condenser formed of two large flat opposed conductors very close together, the displacement current, when setting up the displacement, is, by general reasoning, parallel to the displacement—at least away from the edges. But this is not invariable. When charged conductors are discharged, the displacement current does not in general follow the tubes of displacement. To do so would require instantaneous propagation of the disturbances to infinite distances. The displacement current may be perpendicular to the displacement—viz., when the displacement at a certain place changes its direction without changing in amount.

Multiply (8) by **E**; then

$$\mathbf{E}\Gamma = \mathbf{E}k\mathbf{E} + \mathbf{E}c\dot{\mathbf{E}}/4\pi = Q + \dot{U}. \quad \text{....................(9)}$$

The rate of working of the force is accounted for partly in heating (Q per second), and partly in the increase in the energy U of the displacement. (Equations (4) and (7)). The first is lost from the system, the latter is stored.

Whilst conductivity depends on the presence of matter, the existence of capacity is independent of matter, though modified in amount by its presence. That is, capacity is a function of the ether, which is the standard dielectric medium of least capacity. Ether is a very wonderful thing. It may exist only in the imaginations of the wise, being invented and endowed with properties to suit their hypotheses; but we cannot do without it. How is energy to be transmitted through space without a medium? Yet, on the other hand, gravity appears to be independent of time. Perhaps this is an illusion. But admitting the ether to propagate gravity instantaneously, it must have wonderful properties, unlike anything we know.

Coming next to permeability, all bodies sustain magnetic induction, and most of them to nearly the same degree. **H** being the magnetic force, **B** the induction, and μ the permeability,

$$\mathbf{B} = \mu\mathbf{H}. \quad \text{.........(Magnetic induction) (10)}$$

μ is taken as unity in ether (in the "electromagnetic" system of units), and is either a little greater or a little less in most bodies. But in some bodies it, very singularly, runs up to large numbers. Iron is the principal offender; then come nickel and cobalt, minor magnetics, but far removed from the crowd of almost unmagnetisable substances. Fe = 56, Ni and Co about 58·5. What can it be?

The linear connection between **H** and **B** is very unsatisfactory. Not merely does μ vary with the temperature, and enormously from one piece of iron to another, being, with moderate strength of magnetic force, largest in the softest iron and smallest in hard steel, but it varies with the magnetic force, first increasing with the force, and then, more importantly, decreasing greatly; how far down is unknown. To make matters worse, part of the induction produced by applied magnetising force becomes fixed, for the time, remaining after the removal of the force. Thus the linear connection between **H** and **B** must be taken with salt. But within moderate limits, and excluding permanent magnetisation, which requires separate consideration, μ in equation (10) may be taken to be, like k and c before, a scalar constant in case of isotropy, and a linear vector operator in eolotropic media, being then, like c, self-conjugate, or without the rotatory power.

μ in soft iron is said to run up to 5,000 or 10,000 (Rowland's experiments. I forget the exact figures). But in general it is very far lower than these tremendous figures. From experiments on the retardation of coils made some years ago, including straight solenoids, I concluded that μ = from 50 to 200 was safe, [for small forces].

Not **B**, but **B**$/4\pi$ should be the magnetic induction to compare with **D**, the electric induction, or displacement. So, dividing (11) by 4π, and then multiplying by $\frac{1}{2}\mathbf{H}$, we have

$$\tfrac{1}{2}\mathbf{HB}/4\pi = \mathbf{H}\mu\mathbf{H}/8\pi = T, \text{ say. (Magnetic energy)} \quad (11)$$

T is the energy of the magnetic induction per unit volume, when wholly induced, and acting conservatively, [within the elastic limits].

Section II. On the Transmission of Energy through Wires by the Electric Current.

Consider the electric current, how it flows. From London to Manchester, Edinburgh, Glasgow, and hundreds of other places, day and night, are sent with great velocity, in rapid succession, backwards and forwards, electric currents, to effect mechanical motions at a distance, and thus serve the material interests of man.

By the way, is there such a thing as an electric current? Not that it is intended to cast any doubt upon the existence of a phenomenon so called; but is it a current—that is, something moving through a wire? Now, although nothing but very careful inculcation at a tender age, continued unremittingly up to maturity, of the doctrine of the materiality of electricity, and its motion from place to place, would have made me believe it, still, there is so much in electric phenomena to support the idea of electricity being a distinct entity, and the force of habit is so great, that it is not easy to get rid of the idea when once it

has been formed. In the historical development of the science, static phenomena came first. In them the apparent individuality of electricity, in the form of charges upon conductors, is most distinctly indicated. The fluids may be childish notions, appropriate to the infancy of science; but still electric charges are easily imaginable to be quantities of a something, though not matter, which can be carried about from place to place. In the most natural manner possible, when dynamic electricity came under investigation, the static ideas were transferred to the electric current, which became the actual motion of electricity through a wire. This has reached its fullest development in the hands of the German philosophers, from Weber to Clausius, resulting in ingenious explanations of electric phenomena based upon forces acting at a distance between moving or fixed individual elements of electricity. It so happened that my first acquaintance with electricity was with the dynamic phenomena, and after I had read with absorbed interest that instructive book, Tyndall's "Heat as a Mode of Motion." This may explain why, when it came later to book-learning regarding electricity, I had the greatest possible repugnance to all the explanations, and could not accept the electric current to be the motion of electricity (static) through a wire, but thought it something quite different. I simply did not believe, except so far as mere statements of experimental facts were concerned. This had its disadvantages; one can get on faster if one has sufficient faith—which we know moves mountains—to accept a certain hypothesis unhesitatingly as a fact, and work out its consequences undoubtingly, regardless of the danger of fixing one's ideas prematurely.

As Maxwell remarked, we know nothing about the velocity of electricity; it may be an inch in a year or a million miles in a second. Following this up, it may be nothing at all. In fact, it is only on the hypothesis that the electric current is something moving, a definite quantity in a given space, that we can entertain the idea of its possessing velocity. Then, the product of its hypothetical density into its velocity is the measure of the current; but, being a mere hypothesis, unless we chose to accept it, to talk of the velocity of electricity in the electric current becomes meaningless. On the other hand, when we apply the ideas of abstract dynamics to electricity, and compare the electric current to a velocity, it is not the above supposititious velocity of electricity that is referred to in any way. It has no meaning now. It is the supposed velocity of electricity in the electric current; whereas, in the dynamical theory, it is the electric current itself that is a velocity, in the generalized sense, with the electromotive force as the generalized force; so that force × velocity = activity. In only one sense do I think we can speak of the velocity of electricity, consistent with Maxwell's theory, viz., by the hypothesis that the electric current in a wire is the continuous discharge of contiguous charged molecules, when plainly we can call the velocity of motion of a molecule the velocity of the charge it carries. As between the molecules we have the electric medium the ether, this view of the conduction current ultimately resolves itself into "displacement" currents in a dielectric.

But is there not the fact that we can send a current into a long circuit, and that it plainly travels along the wire, taking some time to arrive at the other end? Does that not show that electricity travels through the wire? To this I should have answered formerly, when filled with "Heat as a Mode of Motion," that it is a fact that there is a transformation of energy in the battery, and that this energy is transmitted through the wire, there suffering another transformation, viz., into heat; that when the current is set up steadily, the heat is generated uniformly; that the electric current in the wire is therefore some kind of stationary motion of the particles of the wire, not exactly like heat, but having some peculiarity of a directional nature making the difference between a positive and a negative current; but that there was no evidence in the closed circuit of any motion of electricity through the wire, but only of a transfer of energy through the wire.

However, leaving personal details of no importance to anyone but myself, let us consider the transmission of energy through a wire. To fix ideas, let our circuit be an insulated suspended wire from London to Edinburgh, and that we transmit energy to Edinburgh from a battery in London, the circuit being completed through the earth. Let the current be kept on. In the first place the phenomenon is steady. It does not change with the time. Next we find that the magnetic force about the wire is the same everywhere at the same distance, or the wire is in the same condition as regards the magnetic induction outside it, and when we apply our knowledge to the interior of the wire, regarded as a bundle of smaller wires, we find that the magnetic force *in* the wire does not vary along its length. Again, heat is being generated within the wire at a uniform rate (a part of the steadiness above mentioned), and next, this phenomenon is also the same all along the wire. Heat is undoubtedly a kinetic phenomenon, hence the electric current is also, at least in part, a kinetic phenomenon. The electric current is not itself heat; but as its existence in the wire involves the continued production of heat, we conclude that some kind of motion is necessarily involved in the electric current apart from the heat produced, and from the uniformity of effect in different parts of the wire, that it is a kind of stationary motion. Again, the electric force is the same all through the wire. There seems no difference between one part and another. Outside the wire, in the dielectric, however, there is a difference, for the electric force varies not only at different distances from the wire but also at the same distance outside different parts of the wire. (We disregard here all irregularities due to other conductors and currents.)

Passing to the battery, the complexity of conditions makes it more difficult to follow, though the state of electric force and magnetic force and heat generation is reducible to the same, and may be made identically the same as in the wire by properly choosing its shape, etc. But in the battery there is a very remarkable thing taking place, namely, the loss of chemical energy at a steady rate; and in the system generally, a still more remarkable thing, an exactly equivalent steady gain of heat. Heat that might have been produced on the spot by the chemical

action, otherwise conducted, appears all over the circuit. How does it get there? The natural answer is, through the wire. But to get to the further parts of the wire it must go through the nearer, hence there must be what we may call an energy-current, which, in the wire, at a given place, would be the rate of transfer of energy through a cross section there. Now, which way is the energy-current directed? It would seem only fair to let it go both ways equally from the battery. Let it be so first. Then there is an energy-current entering the wire, equal to one-half the dissipativity, which falls in strength regularly up to the middle of the wire, where it is zero. It falls in strength on account of the heat generation. Similarly the other energy-current goes through the earth to Edinburgh almost unabated in strength, and is then directed from Edinburgh to the middle of the wire, where its strength also falls to nothing. This seems absurd. Then let the energy-current be directed one way only, say with the positive current. If the positive pole of the battery is to line, we have an energy-current in one direction all round the circuit, London to Edinburgh, and back through earth. If of maximum strength at the battery it falls nearly to nothing at the distant end, and quite to nothing through the earth up to the other pole of the battery. But we have no data whatever to fix whereabouts the place of maximum energy-current is. It requires a second assumption. The reader may similarly consider the effect of reversing the battery, or of making the energy-current be directed with the negative current. There is no getting at anything definite, except that the energy-current must vary very widely, though regularly, in strength, whilst there is nothing to fix which way it is directed, or where the maximum strength is. Again, the energy-current is a kinetic phenomenon, and as it varies so widely in different parts, we might expect different parts of the wire itself to be in different electrical states, which is exactly what we do not do; for though its potential varies, yet potential is not a physical state, but a mere scientific concept.

Had we not better give up the idea that energy is transmitted through the wire altogether? That is the plain course. The energy from the battery neither goes through the wire one way nor the other. Nor is it standing still. The transmission takes place entirely through the dielectric. What, then, is the wire? It is the sink into which the energy is poured from the dielectric and there wasted, passing from the electrical system altogether. All [the above mentioned] difficulties now disappear.

That the energy of the battery passes into heat immediately would require its instantaneous transmission to all parts of the wire, which cannot be entertained. There must be an intermediate state or states, after leaving the battery and before becoming heat. And there must be a definite amount of energy in transit at a given moment; in the steady state this must be of constant amount, just as the total rate of transmission is of constant amount. We must not, however, individualize particular elements of energy, and follow their motions, but regard the matter quantitatively only. The energy in transit may be compared to the energy of a machine which is transmitting motion; if done at a steady rate, it remains constant and definite, and the rate of transmission is definite.

Now, in Maxwell's theory there is the potential energy of the displacement produced in the dielectric parts by the electric force, and there is the kinetic or magnetic energy of the magnetic induction due to the magnetic force in all parts of the field, including the conducting parts. They are supposed to be set up by the current in the wire. We reverse this; the current in the wire is set up by the energy transmitted through the medium around it. The sum of the electric and magnetic energies is the energy of the electric machinery which is transmitting energy from the battery to the wire. It is definite in amount, and the rate of transmission of energy (total) is also definite in amount.

It becomes important to find the paths along which the energy is being transmitted. First define the energy-current at a point to be the amount of energy transferred in unit time across unit area perpendicular to the direction of transmission. As the present section is argumentative and descriptive only, we cannot enter into mathematical details further than to say that if **H** be the vector magnetic force, and **E** the vector electric force, not counting impressed forces, the energy-current, as above defined, is $\mathbf{VEH}/4\pi$ (see equation (3) for definition of V). This is true universally, irrespective of the nature of the medium as to conductivity, capacity, and permeability, or as to eolotropy or isotropy, and true in transient as well as in steady states. A line of energy-current is perpendicular to the electric and the magnetic force, and is a line of pressure. We here give a few general notions.

Return to our wire from London to Edinburgh with a steady current from the battery in London. The energy is poured out of the battery *sideways* into the dielectric at a steady rate. Divide into tubes bounded by lines of energy-current. They pursue in general solenoidal paths in the dielectric, and terminate in the conductor. The amount of energy entering a given length of the conductor is the same wherever that length may be situated. The lines of energy-current are the intersections of the magnetic and electric equipotential surfaces. Most of the energy is transmitted parallel to the wire nearly, with a slight slant towards the wire in the direction of propagation; thus the lines of energy-current meet the wire very obliquely. But some of the outer tubes go out into space to an immense distance, especially those which terminate on the further end of the wire. Others pass between the wire and the earth, but none in the earth itself from London to Edinburgh, or *vice versâ*, although there is a small amount of energy entering the earth straight downwards, especially at the earth "plates." If there is an instrument in circuit at Edinburgh, it is worked by energy that has travelled wholly through the dielectric, then finding its way into the instrument, where it enters the coil and is there dissipated, or else used up by the visible motions it effects in moving parts of the instrument; which, however, is a different kind of affair from dissipation, as it involves impressed force.

Now, go into the line-wire. A tube of energy-current arriving at the surface of the wire by a long slant, at once turns round and goes straight to the axis. In passing from the battery to the wire through the dielectric the energy-current is continuous, the state being steady (or the ether machinery frictionless); but directly it reaches the conducting matter of the wire dissipation commences and the current begins to fall in strength, and on reaching the axis has fallen to nothing. Not a fraction of an erg is transmitted along the wire. Some small part of the energy leaving the battery may enter it again, but most of the dissipation in the battery itself is accounted for by the weakening of strength in tubes which are on their way to leave the battery.

Put the battery in the middle of the line; earth at both ends. Now, one half of the energy-current tubes leaving the battery sideways turn round to one section of the line, the other half to the other section. Otherwise the case is similar to the last.

When we have a double wire looped without earth, and battery at one end, most of the energy is transmitted between the wires.

In a circular circuit, with the battery at one end of a diameter, its other end is the neutral point; the lines of energy-current are distributed symmetrically with respect to the diameter.

On closing the battery circuit there is an immediate rush of energy into the dielectric, and, at the first moment, into all bodies in the neighbourhood of the battery, and wasted there in induced currents according to their conductivity. In the variable state the tubes of energy-current are themselves in motion. It takes some time to set the electric machinery going steadily. Also the energy-current is not continuous in the dielectric, for the potential energy of displacement and the magnetic energy have to be supplied at every place. But, in the end, the energy-current becomes continuous in the dielectric, goes round an external conductor instead of entering it, as it would do in the transient state, and finally reaches the conductor to which the battery is connected, penetrating which it terminates.

If we neglect the magnetic energy, as in Sir W. Thomson's original telegraph theory, against the energy of electric displacement, we can easily get a general idea of the setting up of the permanent state in a long suspended wire; a submarine cable is more complex on account of the sheath. The energy reaches the beginning of the wire first, and only reaches the end, save insignificantly, later on. But the theory indicates instantaneous setting up of current at the far end, though not in recognisable amount. This result follows from the neglect of the magnetic energy. In a dielectric medium the velocity of undisturbed propagation is $(c\mu)^{-\frac{1}{2}}$; where c is the capacity, and μ the permeability; that the magnetic energy $=0$ is equivalent to assuming $\mu=0$ everywhere, whence instantaneous transmission. The "retardation," however, arises from the setting up of the potential energy of displacement. But, strictly speaking, we must not neglect μ. It is, then, not so easy to follow the transient state without simplifications. There is an oscillatory phenomenon in the dielectric, a to-and-fro transmission of energy and pressure parallel to the wire all round it with a velocity whose possible maximum is that of undisturbed transmission. This is modified as it progresses by dissipation in the wire, and so gets wiped out. This usually occurs so rapidly that the waves are of importance only at the battery end of a long wire. The electric machinery must have mass, as well as elasticity, by reason of this phenomenon, since there is reason to believe (from Maxwell's theory of light) that it is not the air, but something between the air molecules that is the electromagnetic medium, the air merely modifying the phenomena somewhat.

In the state of steady current through a submarine cable, with an iron sheath outside the dielectric, the energy is transmitted wholly through the gutta percha or other suitable insulator (neglecting the small amount going to earth), thus going nearly parallel to the wire, practically quite parallel, except as regards the lines near the wire itself, as they all eventually meet the wire. There is no transmission in the sheath lengthwise, though there is dissipation there if it should contain, as it does sometimes, part of the return current. In the transient state there is, of course, always dissipation in the sheath more or less, besides the loss of energy to magnetise it.

Now to speak more generally. In the steady state of current due to any impressed forces, the tubes of energy-current start sideways from the places of impressed force, where energy is supplied to the electric system, and travel through definite paths, without loss in dielectric, with loss in conducting parts, to terminate finally in conducting matter; or else they may go from one place of impressed force to another with or without dissipation on the way when the current is with the impressed force at one source, and against it at the other. But with special arrangements (solenoidal) of impressed force, there is no transmission of energy in the steady state.

Since on starting a current the energy reaches the wire from the medium without, it may be expected that the electric current in the wire is first set up in the outer part, and takes time to penetrate to the middle. This I have verified by investigating some special cases.

Increase the conductivity of a wire enormously, still keeping it finite, however. Let it, for instance, take minutes to set up current at the axis. Then ordinary rapid signalling "through the wire" would be accompanied by a surface-current only, penetrating to but a small depth. The disturbance is then propagated parallel to the wire in the manner of waves, with reflection at the end, and hardly any tailing off. With infinite conductivity there can be no current set up in the wire at all. There is no dissipation; wave propagation in the medium is perfect. The wire-current is wholly superficial—an abstraction—yet it is nearly the same with very high conductivity. This illustrates the impenetrability of a perfect conductor to magnetic induction (and similarly to electric current), applied by Maxwell to the molecular theory of magnetism. Whatever state of magnetic induction and of current there may be in a perfect conductor is a fixture. If we move the conductor about in a magnetic field, superficial currents are instantaneously induced, whose only function is to ward off external induction and keep the interior state unchanged.

In a thermo-electric circuit of two metals, with one junction a little hotter than the other, there is a transmission of energy from one junction to the other through the dielectric, with a trifling amount of loss in the circuit generally. Here the source of the electric energy is heat, and the final result is heat. One junction is cooled, the other is heated, reversibly. Now, heat is the energy of molecular agitation, and at first sight the only difference is that the agitation is a little more brisk at one junction than at the other. Again, all parts of the circuit are agitating the ether. It would appear, then, that the ordinary molecular agitations set up no electric manifestations on account of their irregularity; although the electric machinery may be influenced vigorously, yet it must be done in some regularly symmetrical manner to constitute an impressed electric force. At the junctions there is a change of material, the molecules are different, and at their contact some directed quality is given to the agitations. This is very vague, no doubt, but is merely to point out that the impressed force is a symmetrical kind of radiation.

After these general remarks the temporarily interrupted mathematical treatment will be resumed.

SECTION III. RESUMPTION OF ROUGH SKETCH. EXTENSIONS.

Real transient, and suggested dissipative Magnetic Current.

As the rate of increase of the displacement in a non-conducting dielectric is the electric current, so the rate of increase of $\mathbf{B}/4\pi$ may be called the magnetic current. Let it be $\mathbf{G}$. Then

$$\mathbf{G} = \dot{\mathbf{B}}/4\pi = \mu\dot{\mathbf{H}}/4\pi. \quad \text{(Magnetic current)} \quad (12)$$

Like electric displacement currents, magnetic currents are transient only, *i.e.*, they cannot continue indefinitely in one direction, like an electric conduction current. Also, like electric currents in a dielectric, they are unaccompanied by heat generation. In ether, the electric current and the magnetic current are of equal significance.

There is probably no such thing as a magnetic conduction current, with dissipation of energy. If there be such, analogous to an electric conduction current, then let

$$\mathbf{G} = g\mathbf{H} + \mu\dot{\mathbf{H}}/4\pi. \quad \text{.........................(13)}$$

Here $g\mathbf{H}$ is the magnetic conduction current, which, added to the undoubted magnetic current as in (12), gives $\mathbf{G}$ the true magnetic current. g may be scalar, or similar to k, with rotatory ϵ. Multiply (13) by $\mathbf{H}$. Then, using (11),

$$\mathbf{HG} = \mathbf{H}g\mathbf{H} + \dot{T}. \quad \text{........ ....(14)}$$

Here $\mathbf{H}g\mathbf{H}$ is the rate of dissipation. Compare with (9).

Effect of g in a Closed Iron Ring.

The permanency of state of a steel magnet makes it improbable that

g has any existence at all, so that the conduction magnetic current is quite imaginary. But we may inquire what would happen in a closed ring of iron under magnetising force, on the supposition that g exists. Let the ring be uniformly lapped with wire, through which we pass a current from a voltaic battery.

If the radius of the ring be large compared with its section, the core may be treated as straight, and the manner in which the current would rise in the coil and the accompanying core phenomena may be easily worked out by a slight modification of the corresponding case with $g=0$ [Art. xxvii., § 29, Example 2, p. 394]. Let a be the radius of the core, also of the coil of negligible depth surrounding it, having N windings per unit length of core. Let k and μ be the conductivity and permeability of the core, and **H** (parallel to the axis) the magnetic force at distance r from the axis. The differential equation of H will be

$$\frac{1}{r}\frac{d}{dr}r\frac{dH}{dr}=(4\pi)^2gkH+4\pi k\mu\dot{H};$$

whence $J_0(nr)\epsilon^{mt}$ is a normal system of magnetic force, if

$$-m=\frac{n^2}{4\pi k\mu}+\frac{4\pi g}{\mu}.$$

Thus the effect of g is to increase the reciprocal of the time-constant of every normal system by the same quantity $4\pi g/\mu$; in this respect resembling the effect of uniform leakage along a telegraph line, and having a similar result, viz., to accelerate the establishment of the permanent state. When this is reached, we do not have uniform strength of magnetic force in the core; but, if H_0 is the strength at the axis, that at distance r therefrom is

$$H=H_0\left(1+\frac{xr^2}{2^2}+\frac{x^2r^4}{2^24^2}+\dots\right),$$

where $x=(4\pi)^2gk$. This is accompanied by core-currents parallel to the coil-current, of density

$$-\frac{1}{4\pi}H_0\frac{xr}{2}\left(1+\tfrac{1}{2}\frac{xr^2}{2^2}+\dots\right).$$

The coil-current will be a little less strong than if $g=0$; for the work of the battery is spent not merely in supporting the coil-current, but in heating the core, both by reason of the weak electric current in the core and the supposed weak magnetic current g**H**. The back E.M.F. in the coil will be of strength

$$-F\left\{1+\frac{R}{4\pi Lg}\frac{1+\dfrac{xa^2}{2^2}+\dots}{1+\tfrac{1}{2}\dfrac{xa^2}{2^2}+\dots}\right\}^{-1},$$

where $F=$ E.M.F. of battery, R resistance of coil-circuit, and L its inductance without the core—*i.e.*, with air replacing it. Or, since g is to be small,

$$-F(1+R/4\pi Lg)^{-1}.$$

If $L/R=\cdot01$ second, $g=1/4\pi$ would make the back force $=1/101$ of the battery force, so g, if existent, must be very small.

In the following, $g=0$, so that equation (12) is the equation of the magnetic current.

First Cross Connection of Magnetic and Electric Force.

In the foregoing we have been dealing with the direct connection of the electric force and its consequences, electric conduction current and displacement, and of the magnetic force and magnetic induction. We have also brought in the displacement current in a dielectric, and the true current in a conducting dielectric. Also, to balance the displacement current, we have introduced the magnetic current. But, so far, we have no relations whatever between the electric and the magnetic quantities, which we must have, in order to make a consistent system.

The first cross connection is expressed by

$$\text{curl}\,\mathbf{H}=4\pi\Gamma, \quad\dots\dots(15)$$

H being the magnetic force and Γ the true current. Here "curl" is, like sin and cos, the symbol of an operation. It is so recurrent in electromagnetism that it might be termed *the* electromagnetic operator. It may be defined with reference to Cartesian coordinates thus: If H_1, H_2, H_3, are the three rectangular components of **H**, those of curl **H** are

$$\frac{dH_3}{dy}-\frac{dH_2}{dz},\quad \frac{dH_1}{dz}-\frac{dH_3}{dx},\quad \frac{dH_2}{dx}-\frac{dH_1}{dy}. \quad\dots\dots(16)$$

But the most useful definition is that which is virtually contained in the fundamental Theorem of Version:—The line-integral of a vector **H** round any closed curve or circuit (or the "circulation" of **H**) equals the surface-integral of another vector, viz., curl **H**, over any surface bounded by the circuit. Apply this to small squares in planes perpendicular to **x**, **y**, and **z** successively, and the three expressions given in (16) for the components of curl **H** follow at once. Apply the theorem to suitably chosen infinitely small areas in any system of coordinates and we obtain the proper expressions in, usually, a far simpler manner than by laborious transformations of differential coefficients. Whilst the expressions for the components vary according to the system of coordinates chosen as most suitable for a special problem, the theorem, on the other hand, is universal, and gives us the inner meaning of the operation. It is far the best in general investigations not to employ any system of coordinates, but to emancipate one's self from their complexity by employing symbols which only relate to the intrinsic meaning of the operations; besides which, there is a great gain in the ease of manipulation. In the present paper the meanings of all forms of expression likely to be unfamiliar are briefly stated, and we shall avoid occupying valuable space by lengthy formulæ.

The operator "curl" is connected with rotation thus: if **H** be the instantaneous velocity at a point in a moving fluid, curl **H** is a vector whose direction is that of the axis of instantaneous rotation of the fluid surrounding the point, and whose length equals twice the angular velocity of rotation.

Notice that (15) contains no physical constants. It is therefore, in a sense, a purely geometrical equation. Given a system of magnetic force **H**, mentally represented by lines or tubes of force mapping out space in one way, by the operator "curl" we find another system of lines or tubes mapping out space in another way, viz., the lines and tubes of current. Whether **H** be wholly continuous or not, the derived Γ is necessarily continuous [that is, circuital]. The curl of a vector can have no divergence anywhere, which we express by

$$\text{div}\,\Gamma = 0\,; \quad \text{or,} \quad \frac{d\Gamma_1}{dx}+\frac{d\Gamma_2}{dy}+\frac{d\Gamma_3}{dz}=0, \quad \ldots\ldots\ldots\ldots(17)$$

which defines "divergence" with reference to Cartesian coordinates. The divergence of Γ is the amount of Γ leaving a point, reckoned per unit volume. When Γ, as here, signifies electric current, it is continuous; as much current leaves as enters any volume, or the integral amount leaving it, reckoning that entering it as negative, is zero. That (17) is involved in (15) is tested by differentiating the three components in (16) to x, y, and z respectively and adding them, when (17) results.

Given **H**, we have Γ, by (15), perfectly definite. But given Γ (necessarily continuous), **H** is not definitely fixed by (15). For, on finding one function **H** satisfying (15) with Γ given, we may add to **H** any function **I** such that curl **I** = 0, without disturbing the relation (15). The nature of **I** is given by

$$\mathbf{I} = -\nabla\Omega\,; \quad \text{or,} \quad I_1 = -d\Omega/dx, \quad I_2 = -d\Omega/dy, \quad I_3 = -d\Omega/dz, \quad (18)$$

where Ω is a *scalar* function of position, a scalar potential in fact. We require some other condition than (15) to find **H** completely when Γ is given; this is, that the magnetic induction $\mathbf{B} = \mu\mathbf{H}$, (equation (10),) is continuous, or div **B** = 0. **H** is now perfectly definite. If μ = constant, or all space is equally magnetisable isotropically, then **B** is the same multiple of **H** everywhere, hence div **H** = 0, so that the proper solution of (15) is that function **H** satisfying (15) which is continuous, like Γ. But **H** is not continuous when μ varies from one part of the field to another.

Having now defined "curl," "divergence," and ∇ applied to a scalar function, consider (15) from a less abstract point of view, in the light of the Version Theorem. Let there be any closed circuit in space,—whether passing through conducting or dielectric matter is immaterial. The amount of current passing through the circuit in the positive direction (that passing the other way being counted negatively) equals the circulation of **H** round the circuit ÷ 4π. The actual distribution of Γ is got by taking the circuit infinitely small and applying it to all parts of the field. Let us, whilst considering a finite circuit, yet take it sufficiently small to make the current pass all one way through it. Then, setting up current through the circuit, we set up magnetic force round it.

But there is another way of setting up magnetic force round the circuit, viz., by motion of the circuit itself in a previously undisturbed electric field. Thus, let there be a steady field of electric force, say in air, with therefore steady electric displacement, and no electric current. Let the closed circuit be a thin wire. When at rest in the field there is no current through it, and no magnetic force round it. But if we move the circuit so that the amount of electric displacement through it varies, there is electric current through the circuit, to be measured by the rate of increase of the amount of displacement through it at any moment; or, in another form, by the number of tubes of displacement added to the circuit per second by the motion of the circuit across them. Hence there will be magnetic force round the circuit, and if it be a thin iron wire, it will become magnetised by the motion in the electric field. In general, the motion of matter in an electric field sets up magnetic force.

As an example, fix a thin circular iron ring in air. Call the line through its centre perpendicular to its plane the axis. Let there be no current or magnetic force in the first place. Now shoot a small bullet, having an electrical charge, through the ring, along its axis. The electric displacement due to the charge will be continually changing; thus, there is a system of electric current in the air accompanying the motion of the bullet. The velocity of propagation of disturbances in air is so great that, unless the velocity of the bullet be *not* a very small fraction of the velocity of propagation, we may neglect the disturbance in the field of force due to the latter velocity not being infinite, and suppose that the bullet carries with it in its motion its normal field of force (radiating straight lines) unchanged. The distribution of displacement current about the moving bullet is then the same as that of the lines of magnetic force that would come from it if it were uniformly magnetised parallel to the axis, or line of actual motion in the real case, and the lines of magnetic force accompanying the displacement currents are circles centred upon the axis, in planes perpendicular thereto, the strength of magnetic force in the air being inversely proportional to the square of the distance from the centre of the bullet, and directly proportional to the cosine of the latitude; the equator being the circle on the bullet's surface in the plane perpendicular to the axis passing through the centre of the bullet. (With very high velocity this distribution of displacement current and magnetic force is departed from.) The fixed ring coincides with the lines of magnetic force during the whole motion of the bullet, and is therefore solenoidally magnetised thereby, most strongly when the magnetic force is strongest there, *i.e.*, when the bullet has just reached the centre of the ring, and the current through the ring is a maximum. The current through the ring may be measured either by the displacement current through a surface bounded by the ring, or by the rate at which the ring cuts the lines of electric force (supposed undisturbed) of the bullet.

Next, fix the charged bullet and move the ring instead, so that their relative motion shall be as before. There is exactly the same amount of electric displacement through the circuit added per second as before, in corresponding positions of the bullet and ring, with, therefore, the same magnetic force in the ring and the same magnetisation. Otherwise, however, there is a great difference in the two experiments. In the first case, changing electric displacement or electric current all through the dielectric, the greatest strength of current being at the poles of the bullet; whilst in the latter case the field is practically undisturbed except near the moving ring itself. Compare with the induction of electric force in a ring in a magnetic field, first when the field is moving, and next when the ring is moved in the field.

The induced magnetic force per unit length in a wire moved perpendicularly across the lines of force in an electric field equals the amount $\times 4\pi$ of electric displacement of the field crossed by the unit length of wire per second, and is perpendicular to the electric displacement and to the direction of motion. In general,

$$\mathbf{h} = \mathrm{V}\mathbf{D}\mathbf{v} \times 4\pi, \quad \text{.........................(18}a\text{)}$$

where $\mathbf{D}$ is the displacement of the field, $\mathbf{v}$ the velocity, $\mathbf{h}$ the induced magnetic force, and V is as in equation (3). There are, of course, corrections due to the reactions set up, due to the wire not being infinitely thin, and to finite length.*

In electromagnetic units, c in air $=(v_1)^{-2}$, if v_1 = velocity of propagation $= 3 \times 10^{10}$ cm. per sec. Therefore, in the case of motion of a thin wire perpendicularly across the lines of force in a uniform electric field of strength E,

$$h = Ev(v_1)^{-2} = Ev/(9 \times 10^{20}).$$

Let $E = 10^{12}$ c.g.s., or 10^4 volts per cm., which is less than the disruption force in air in its ordinary state, then

$$h = v/(9 \times 10^8).$$

To get magnetic force of strength 10^{-5} c.g.s., v must equal 90 metres or 300 feet per second.

Magnetic Energy of Moving Charged Spheres.

In passing, I may remark that J. J. Thomson (*Phil. Mag.*, April, 1881) found the magnetic energy ΣT due to a sphere of radius a with an electric charge q moving with velocity v in a medium of permeability μ to be

$$\Sigma T = \tfrac{2}{15}\mu q^2 v^2/a.$$

I find that the $\frac{2}{15}$ should be $\frac{1}{3}$. Also, he found the mutual magnetic energy ΣT_{12} of two infinitely small spheres at distance r with charges q_1 and q_2, moving with velocities defined by the rectangular components u_1, u_2, u_3, and v_1, v_2, v_3, with u_1 and v_1 the velocities parallel to the line joining the spheres, to be

$$\Sigma T_{12} = (\mu q_1 q_2/3r)\,(u_1 v_1 + u_2 v_2 + u_3 v_3),$$

against which I find it to be

$$\Sigma T_{12} = (\mu q_1 q_2/2r)\,(2u_1 v_1 + u_2 v_2 + u_3 v_3).$$

I do not know what corrections, if any, have been published, and should be glad to receive information on the point, whether in corroboration of my results or otherwise.

* [The force defined by (18a) I now term the motional magnetic force, and its companion (21a) below, the motional electric force. Examples of their use will occur in later papers.]

SECTION IV. COMPLETION OF ROUGH SKETCH.

Second Connection between Electric Force and Magnetic Force.

The equation (15), $\text{curl}\,\mathbf{H} = 4\pi\Gamma$, expressing a relation, independent of physical constants, between the magnetic force and the electric current, is an extension of Ampère's results for linear circuits. By Γ must be understood Maxwell's true current—that is, the sum of the conduction current and the displacement current if the body considered be both a dielectric and a conductor, or the conduction current alone or the displacement current alone if the body have no dielectric capacity or no conductivity respectively. All bodies are either conducting or dielectric, or both, and ether is dielectric, so that electric current may exist everywhere. Putting Γ in terms of $\mathbf{E}$ by the equation of true current (8), [p. 443], we get

$$\text{curl}\,\mathbf{H} = 4\pi k\mathbf{E} + c\dot{\mathbf{E}}, \quad \text{.........................(19)}$$

which is one connection between $\mathbf{E}$ and $\mathbf{H}$.

The second connection may be obtained by translating Faraday's law of induced electric force in a linear circuit into a mathematical form. It is remarkable that the ideas of Faraday, who was no mathematician, should admit of immediate translation into mathematical language; a fact due to his dispensing with the direct action-at-a-distance hypothesis, and employing the intermediate mechanism of lines or tubes of force. In popular language, the total E.M.F. of induction round a linear circuit is measured by the number of lines of force taken out of the circuit per second. Here the conventional connection between the assumed positive direction of translation through a circuit, and the assumed positive direction of motion in the circuit must be remembered. Selecting either direction through a circuit as the positive direction of translation, look through the circuit in this direction. Then the positive direction of rotation is right-handed, or with the hands of a watch whose front faces the spectator. Thus, increasing the number of lines of force through a circuit sets up negative E.M.F. round it.

So far in a medium of unit permeability. But when we make allowances for differences of magnetic permeability, it is not the variation of the magnetic force $\mathbf{H}$, but of the magnetic induction $\mathbf{B} = \mu\mathbf{H}$, which determines the induced E.M.F. The amended statement is that the total E.M.F. of induction round a circuit equals the rate of decrease of the amount of magnetic induction through the circuit. Now, since we have here a line-integral, viz., of the electric force of induction round a circuit, and a surface-integral, viz., of $-\mu\dot{\mathbf{H}}$ or $-\dot{\mathbf{B}}$ over any surface bounded by the circuit, we may at once apply the Version Theorem before referred to [p. 443] and deduce

$$\text{curl}\,\mathbf{E} = -\dot{\mathbf{B}} = -\mu\dot{\mathbf{H}}, \quad \text{......................(20)}$$

which is one form of the second relation between $\mathbf{E}$ and $\mathbf{H}$.

The following method is also instructive. Since the rate of increase of the magnetic induction at a point equals $4\pi\mathbf{G}$, where $\mathbf{G}$ is the magnetic current, as defined by equation (12), we may state the law of induced electric force thus:—The total E.M.F. of induction round a circuit in the negative direction equals 4π times the total magnetic current through the circuit in the positive direction. Now compare this statement with the statement regarding equation (15) [p. 443], viz., that the total magnetic force round a circuit equals 4π times the total electric current through the circuit, and change this so as to produce the statement in the last sentence. We must change magnetic force to electric force taken negatively, and electric current to magnetic current. Hence

$$\operatorname{curl} \mathbf{H} = 4\pi\Gamma \quad \text{.........................(15) } bis$$

becomes

$$-\operatorname{curl} \mathbf{E} = 4\pi\mathbf{G}, \quad \text{.........................(21)}$$

which is equivalent to (20).

We have, in order to simplify the establishment of (20) or (21), avoided mentioning the E.M.F. induced in a linear circuit by its motion in the field, which may or may not be varying independently. The amount of induction added to or taken out of a circuit from this cause may be obviously represented by a line-integral, as it depends upon the rate at which the different elements of the circuit cross the lines of induction. If the induction were of the same strength at all the moving parts of the circuit, and they all moved at right angles to their lengths and also perpendicularly across the lines of induction in the same sense, the total E.M.F. would be of strength $= B \times$ rate of increase of area of circuit. But when B varies, and likewise the velocity of the different elements across the lines of B, each element must be considered separately. The amount contributed to the total E.M.F. by an element of unit length equals the component parallel to its length of

$$\mathrm{V}\mathbf{v}\mathbf{B}, \quad \text{.............................(21}a\text{)}$$

if $\mathbf{v}$ be the vector velocity. But, if there be current induced, this brings in working mechanical forces, and should therefore be separately considered. At present we return to the case of c, k, and μ constant with respect to the time, and no parts moveable.

In equation (21), $\mathbf{E}$ is the electric force of induction only, not the actual electric force. There may be in addition electrostatic force, and also impressed electric force. But the electrostatic force is polar; it is derived from a scalar potential. If this be P, the force is $-\nabla P$. But $\operatorname{curl} \nabla P = 0$, as was before remarked [p. 444] with reference to Ω, consequently the polar force may be included in $\mathbf{E}$ in equation (21). Similarly any polar force may be included in $\mathbf{H}$ in the previous equation (15). Now in all our equations, from (1) up to (14), not containing any relations between $\mathbf{E}$ and $\mathbf{H}$, those symbols mean the actual resultant electric and magnetic force from all causes. Hence, in order that the two equations (15) and (21) may harmonise with the preliminary equations (1) to (14), not only in space where there is no impressed force, but at the places where such exist as well, we must, whilst still using $\mathbf{E}$ and $\mathbf{H}$ to denote the actual forces, deduct from them the impressed forces in using the relations (15) and (21). So, let $\mathbf{e}$ be the impressed electric force, and $\mathbf{h}$ the impressed magnetic force. Our two connections between $\mathbf{E}$ and $\mathbf{H}$ are then

$$\operatorname{curl}(\mathbf{H} - \mathbf{h}) = 4\pi\Gamma = 4\pi k\mathbf{E} + c\dot{\mathbf{E}}, \quad \text{................(22)}$$

$$\operatorname{curl}(\mathbf{e} - \mathbf{E}) = 4\pi\mathbf{G} = 4\pi g\mathbf{H} + \mu\dot{\mathbf{H}}, \quad \text{................(23)}$$

where the coefficient g of magnetic conductivity is introduced to show the symmetry, and may be put $=0$. We have now a dynamically complete system.

The subject of impressed force will be considered in a following section more fully, especially as regards impressed magnetic force, and its interpretation in terms of magnetisation. In the meantime we may define impressed electric force thus. If $\mathbf{e}$ be the impressed electric force at a point, and Γ the electric current there, $\mathbf{e}\Gamma$ of energy is taken into the electromagnetic system there per unit volume per second. Similarly, we may define the impressed magnetic force $\mathbf{h}$ at a point, by saying that if there be a magnetic current $\mathbf{G}$ there, $\mathbf{hG}$ of energy is taken in per second per unit volume by the electromagnetic system there. In general, $\mathbf{e}\Gamma$ and $\mathbf{hG}$ are scalar products [*see* equation (5)], having the ordinary signification when $\mathbf{e}$ is parallel to Γ, or $\mathbf{h}$ to $\mathbf{G}$; in other cases to be multiplied by the cosine of the angle between $\mathbf{e}$ and Γ or between $\mathbf{h}$ and $\mathbf{G}$.

The Equation of Energy and its Transfer.

We must find the rate of working of the impressed forces, and compare with the dissipativity and with the changes taking place in the energy of displacement and the magnetic energy. Multiply (22) by $(\mathbf{e} - \mathbf{E})$, and (23) by $(\mathbf{h} - \mathbf{H})$, and add the results. We get

$$4\pi\{(\mathbf{e} - \mathbf{E})\Gamma + (\mathbf{h} - \mathbf{H})\mathbf{G}\} = (\mathbf{e} - \mathbf{E}) \operatorname{curl}(\mathbf{H} - \mathbf{h}) + (\mathbf{h} - \mathbf{H}) \operatorname{curl}(\mathbf{e} - \mathbf{E}).$$

Or

$$\mathbf{e}\Gamma + \mathbf{hG} = \mathbf{E}\Gamma + \mathbf{HG} + \{(\mathbf{H} - \mathbf{h}) \operatorname{curl}(\mathbf{E} - \mathbf{e}) - (\mathbf{E} - \mathbf{e}) \operatorname{curl}(\mathbf{H} - \mathbf{h})\}/4\pi, \quad (24)$$

by rearrangement. $\mathbf{E}\Gamma$ and $\mathbf{HG}$ here occurring have been already expressed in terms of the dissipativity Q, the electric energy of displacement U, and the magnetic energy T; *see* equations (9) and (14). Thus

$$\mathbf{E}\Gamma + \mathbf{HG} = Q + \dot{U} + \dot{T}. \quad \text{......................(25)}$$

On the left side we have the rate of working per unit volume of the actual forces $\mathbf{E}$ and $\mathbf{H}$ on the currents Γ and $\mathbf{G}$; on the right side the dissipativity, or rate at which energy is being lost from the system irreversibly, producing heat according to Joule's law, and the rate of increase of the electric and magnetic energies, all per unit volume.

Now looking at (24), the left side expresses the rate at which energy is being taking in (reversibly) per unit volume, in virtue of the impressed forces $\mathbf{e}$ and $\mathbf{h}$. Therefore the excess of $(\mathbf{e}\Gamma + \mathbf{hG})$ over $(\mathbf{E}\Gamma + \mathbf{HG})$ must be the energy leaving the unit volume per second through its sides. Now, $\mathbf{X}$ and $\mathbf{Y}$ being any two vectors,

$$\mathbf{Y} \operatorname{curl} \mathbf{X} - \mathbf{X} \operatorname{curl} \mathbf{Y} = \operatorname{div} \mathrm{V}\mathbf{XY}; \quad \text{................(26)}$$

or in full, by (5), (16), (17), and (3),

$$\begin{aligned} &Y_1(dX_3/dy - dX_2/dz) + Y_2(dX_1/dz - dX_3/dx) + Y_3(dX_2/dx - dX_1/dy) \\ &- X_1(dY_3/dy - dY_2/dz) - X_2(dY_1/dz - dY_3/dx) - X_3(dY_2/dx - dY_1/dy) \\ &= (d/dx)(X_2Y_3 - X_3Y_2) + (d/dy)(X_3Y_1 - X_1Y_3) + (d/dz)(X_1Y_2 - Y_1X_2); \end{aligned}$$

using the numbers 1, 2, 3, to denote the x, y, and z components. Let

then $$\mathbf{W} = \mathrm{V}(\mathbf{E} - \mathbf{e})(\mathbf{H} - \mathbf{h})/4\pi, \qquad (27)$$

then by (26), with $\mathbf{X} = \mathbf{E} - \mathbf{e}$, and $\mathbf{Y} = \mathbf{H} - \mathbf{h}$, equation (24) becomes

$$\left.\begin{aligned} \mathbf{e}\Gamma + \mathbf{hG} &= \mathbf{E}\Gamma + \mathbf{HG} + \operatorname{div} \mathbf{W} \\ &= Q + \dot{U} + \dot{T} + \operatorname{div} \mathbf{W}, \end{aligned}\right\} \qquad (28)$$

the equation of energy put in its most significant form. Summing up through all space, $\mathbf{W}$ goes out; or the total work per second of the impressed forces equals the total dissipativity plus the rate of increase of the total electric and magnetic energy.

$\mathbf{W}$ is the vector rate of transfer of energy, or what we before [p. 438] termed the energy-current, a vector whose direction is that of the transfer of energy, and whose magnitude equals the amount transferred per second across unit area of a plane perpendicular to that of transfer. Note [p. 438] that impressed forces were said to be not counted; hence as $\mathbf{E}$ and $\mathbf{H}$ are the actual forces now, the impressed forces are deducted, as shown in (27). The magnitude of $\mathbf{W}$ is the product of the strengths of the two forces and the sine of the angle between their directions, and the direction of $\mathbf{W}$ is perpendicular to both forces, with the before-stated convention regarding positive directions.

The general nature of the energy-current was described in Section II. "On the transmission of energy through wires by the electric current" [p. 434], where, however, only impressed electric force was considered. The same general results apply to impressed magnetic force; energy proceeding from places where such exists, to be dissipated as heat in conducting matter, or to increase the electric and magnetic energies, or to go to other places of impressed magnetic force. But there are great practical differences between impressed electric and magnetic force, owing to the transient nature of magnetic currents and other causes.

Differential Equations of E *and* H.

By eliminating $\mathbf{E}$ or $\mathbf{H}$ between (22) and (23) we obtain the characteristic equation of $\mathbf{E}$ or of $\mathbf{H}$. Put $g=0$, and eliminate $\mathbf{H}$. Then, $$\operatorname{curl} \mu^{-1} \operatorname{curl}(\mathbf{e} - \mathbf{E}) = \operatorname{curl} \dot{\mathbf{h}} + 4\pi k\dot{\mathbf{E}} + c\ddot{\mathbf{E}}, \qquad (29)$$

which is the equation of $\mathbf{E}$. Here $\mathbf{e}$ and $\mathbf{h}$, being impressed, must be supposed to be given. μ^{-1} is the operator inverse to μ, that is, in the general case of eolotropy μ^{-1} is defined by the three principal axes and the values $1/\mu_1$, $1/\mu_2$, $1/\mu_3$, along them, as was explained [p. 430] in speaking of k. Similar remarks apply to k^{-1} and c^{-1} should they occur.

In space where there is no impressed electric force, and no, or else constant, impressed magnetic force,

$$\operatorname{curl} \mu^{-1} \operatorname{curl} \mathbf{E} + 4\pi k\dot{\mathbf{E}} + c\ddot{\mathbf{E}} = 0. \qquad (30)$$

In a non-dielectric conductor,

$$\left.\begin{aligned} &\operatorname{curl} \mu^{-1} \operatorname{curl} \mathbf{E} + 4\pi k\dot{\mathbf{E}} = 0, \\ \text{and} \quad &\operatorname{curl} k^{-1} \operatorname{curl} \mathbf{H} + 4\pi\mu\dot{\mathbf{H}} = 0. \end{aligned}\right\} \qquad (31)$$

Here propagation of $\mathbf{E}$ and of $\mathbf{H}$ is by diffusion. And in a non-conducting dielectric,

$$\left.\begin{aligned} &\operatorname{curl} \mu^{-1} \operatorname{curl} \mathbf{E} + c\ddot{\mathbf{E}} = 0, \\ &\operatorname{curl} c^{-1} \operatorname{curl} \mathbf{H} + \mu\ddot{\mathbf{H}} = 0. \end{aligned}\right\} \qquad (32)$$

Here propagation is by waves, *i.e.*, propagation of $\mathbf{E}$ or $\mathbf{H}$, not of energy.

c and μ self-conjugate; k not necessarily so.

We should note that

$$\dot{T} = (d/dt)(\mathbf{H}\mu\mathbf{H}/8\pi) = \mathbf{H}\mu\dot{\mathbf{H}}/8\pi + \dot{\mathbf{H}}\mu\mathbf{H}/8\pi,$$

whilst $$\mathbf{HG} = \mathbf{H}\mu\dot{\mathbf{H}}/4\pi;$$

so for $\mathbf{HG}$ to equal $\dot{T}$ we require

$$\mathbf{H}\mu\dot{\mathbf{H}} = \dot{\mathbf{H}}\mu\mathbf{H},$$

i.e., μ must be self-conjugate, or contain no rotatory ϵ [equation (3), p. 431]. Similarly, for $\dot{U}$ to equal $\mathbf{E}\dot{\mathbf{D}}$, c must be self-conjugate. But there is no such limitation thrown upon k the electric conductivity operator, nor would there be upon g the magnetic conductivity operator, did such exist. There are other proofs of these conclusions, but the above are very short. There is, however, an objection to be raised against the rotatory conductivity vector ϵ, which want of space does not permit to be mentioned at present.

SECTION V. IMPRESSED MAGNETIC FORCE. INTRINSIC MAGNETISATION.

The energy definition of impressed electric force, due originally, it not explicitly, at least substantially, to Sir W. Thomson, has long been well recognised by most writers on electrical subjects, especially since the practical introduction of dynamo machines, accumulators, etc., which raised the energy transformations concerned in electrical phenomena from being matters of almost purely scientific interest to matters of the extremest practical commercial importance.

But in our last we gave an energy definition of impressed magnetic force, precisely similar to that of impressed electric force. Thus, if $\mathbf{h}$ be the impressed magnetic force at a point, and $\mathbf{G}$ the magnetic current there, the rate of working is $\mathbf{hG}$ per unit volume, and this amount of energy is taken in per second by the electromagnetic system at the