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[18]Changing the phase angle is equivalent to changing the drop time—that is, instead of releasing the mass at $t_0=0$ one can think of the mass as being released at $t_0=T(\delta/2\pi)$, where T is the period of the driving force.

[19]The behavior of a system as it evolves from an impacting to a nonimpacting state is an area of active research. For an extensive study of such "grazing bifurcations" in a system similar to the Bender bouncer, see W. Chin, E. Ott, H. E. Nusse, and C. Grebogi, "Grazing bifurcations in impact oscillators," Phys. Rev. E 50, 4427–4444 (1994).

Alternate derivation of the Liénard–Wiechert fields of a point charge

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The Liénard–Wiechert fields for a point charge are derived from a suitable integral representation for the electric and magnetic fields. In this derivation one does not calculate derivatives of retarded functions. Retardation is introduced into the fields at the end of the calculations. The free-space retarded Green function turns out to be an essential ingredient in this derivation. A heuristic argument is given for introducing this Green function. The derivation presented here might be used in undergraduate courses on electromagnetic theory.

I. INTRODUCTION

In undergraduate-level textbooks on electromagnetic theory,[1–5] the Liénard–Wiechert fields of a point charge are derived by differentiating the *retarded* electric and magnetic potentials in the Lorentz gauge. Since the retarded time appears explicitly in these potentials, the differential operations must be performed very carefully. In particular, the space derivatives on retarded quantities are troublesome because of their subtle action on both field and source coordinates implicit in the retarded time. However, the space derivatives may be avoided by recasting them into time derivatives. For example, if f is a function of space and time then the chain rule can be used to show the property[6]

$$\nabla[f]=-\frac{\hat{\mathbf{R}}}{c}\frac{\partial[f]}{\partial t}, \tag{1}$$

where c the speed of light and $[f]$ is a retarded quantity defined by $[f]=f(\mathbf{r}',t')=f(\mathbf{r}',t-R/c)$ with $R=|\mathbf{r}-\mathbf{r}'|$ the magnitude of $\mathbf{R}=(\mathbf{r}-\mathbf{r}')$ and with $\mathbf{r}$ and $\mathbf{r}'$ the field and source points respectively. Thus $\hat{\mathbf{R}}$ is a unit vector defined as $\hat{\mathbf{R}}=\mathbf{R}/R$. By making use of properties such as Eq. (1), Jefimenko[7] has obtained time-dependent generalizations of the Biot–Savart and Coulomb laws from retarded integrals satisfying Maxwell's equations. Since the method of Jefimenko does not make use of potentials, one does not have to discuss the physical (or nonphysical) meaning of these intermediary functions. Consequently the problem of *fixing a gauge* is avoided.

The generalized Biot–Savart and Coulomb laws have been recently used by Griffiths and Heald,[8] Ton,[9] and Heras[10] to derive the Liénard–Wiechert fields of a point charge. The derivation of Griffiths and Heald[8] appears already in the *third edition* of the book of Marion and Heald.[11] We emphasize here that these derivations of the fields, though less complicated than the traditional derivation based on retarded potentials, are not as simple as one would like since all of them involve retardation from the beginning of the calculations.

The fact is that the retardation is, in one way or another, the source of cumbersome complications, and the generalized Biot–Savart and Coulomb laws involve time differentiation on the retarded charge and current densities. *But since we cannot dispose of the awkward retardation—because, among other things, our world is causal—we can nevertheless move it to the end of our calculations.* We have adopted this point of view in a recent paper[12] in which we derived the following formulas for electric and magnetic fields vanishing at infinity:

$$\mathbf{E}=\frac{1}{4\pi\epsilon_0}\int dt'\int dV'\left(\rho\nabla' G-\frac{1}{c^2}G\frac{\partial\mathbf{J}}{\partial t'}\right), \tag{2a}$$

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int dt'\int dV'\ \mathbf{J}\times\nabla' G. \tag{2b}$$

These formulas represent the solution of Maxwell's equations for confined sources ρ and $\mathbf{J}$. The spatial integration in Eqs. (2) is over all space while the temporal integration is from $-\infty$ to $+\infty$. The free-space retarded Green function $G=G(\mathbf{r},t;\mathbf{r}',t')$ in Eqs. (2) is given by[13]

$$G(\mathbf{r},t;\mathbf{r}',t')=\frac{\delta\left(t'+\frac{R}{c}-t\right)}{R}, \tag{3}$$

where δ is the Dirac delta function. It should be mentioned that in the derivation of Eqs. (2) the retardation was handled implicitly in the Green function G.[12] The time integration of Eqs. (2) yields the time-dependent generalizations of the Biot–Savart and Coulomb laws as given by Jefimenko.[7]

In this paper, we present in Sec. II an alternate derivation of the Liénard–Wiechert fields of a point charge based on the formulas (2). The advantage of our method consists in that it does not involve *unfamiliar* operations on retarded quantities. Retardation is introduced explicitly at the end, when all the required standard vector operations are finished, by performing the corresponding time integration. Let us make here a simile: *If retardation is a species of monster with which necessarily we have to drive by the highways of the electromagnetic theory, then with the standard method of retarded potentials we put the monster to our side, while with Jefimenko's approach we send the monster to the back seat. Well, with the approach developed here we put the monster into the trunk, together with the extra tire, to use it when necessary*. This pedagogical *strategy* has been used to a certain extent in other alternate derivations of the Liénard–Wiechert fields appearing in usual graduate textbooks.[14–16] In these derivations one works either with a covariant representation for the electric and magnetic potentials[14,15] or with a noncovariant representation of them.[16] In both cases one uses potentials in the Lorentz gauge. Instead, in the derivation presented here one arrives at the Liénard–Wiechert fields by a route in which one does not make use of potentials nor gauges and the retardation appears at the end.

It can be argued, however, that our approach for deriving the Liénard–Wiechert fields is not suitable for an undergraduate course of electromagnetic theory because of the use of the Green function defined in Eq. (3). As is well known, this Green function is regularly introduced later in a typical graduate course on the subject. In Sec. III we give a heuristic argument by which one may introduce the Green function defined by Eq. (3). This argument relies on reasonable physical assumptions which should be accessible to undergraduates. In any case we believe that the derivation of the Liénard–Wiechert fields presented here is suitable for an advanced undergraduate course on electromagnetic theory at the level of Vanderlinde's book.[3]

II. LIÉNARD–WIECHERT FIELDS OF A POINT CHARGE

In order to apply Eqs. (2) we need the gradient of the Green function G,[17]

$$\nabla' G=\frac{\hat{\mathbf{R}}}{R^2}\,\delta\left(t'+\frac{R}{c}-t\right)-\frac{\hat{\mathbf{R}}}{cR}\,\delta'\left(t'+\frac{R}{c}-t\right), \tag{4}$$

where a prime on the δ function means differentiation with respect to its argument. Consider now a point charge e whose position in an inertial system is $\mathbf{s}(t')$. Thus, the charge and current densities of this moving charge are

$$\rho(\mathbf{r}',t')=e\,\delta(\mathbf{r}'-\mathbf{s}(t')), \tag{5a}$$

$$\mathbf{J}(\mathbf{r}',t')=e\mathbf{v}(t')\,\delta(\mathbf{r}'-s(t')), \tag{5b}$$

where $\mathbf{v}(t')=d\mathbf{s}(t')/dt$ is the velocity of the charge (*notice that t' is still not identified with the retarded time*). When Eqs. (3)–(5) are substituted into Eq. (2a) it becomes

$$\mathbf{E}=\frac{e}{4\pi\epsilon_0}\int dt'\int dV'\,\frac{\hat{\mathbf{R}}}{R^2}\,\delta(\mathbf{r}'-\mathbf{s}(t'))\delta\left(t'+\frac{R}{c}-t\right)-\frac{e}{4\pi\epsilon_0}\int dt'\int dV'\,\frac{\hat{\mathbf{R}}}{cR}\,\delta(\mathbf{r}'-\mathbf{s}(t'))\delta'\left(t'+\frac{R}{c}-t\right)$$
$$-\frac{e}{4\pi\epsilon_0}\int dt'\int dV'\,\frac{\delta\left(t'+\frac{R}{c}-t\right)}{c^2R}\,\frac{\partial}{\partial t'}\,(\mathbf{v}(t')\,\delta[\mathbf{r}'-\mathbf{s}(t')]). \tag{6}$$

Although the integrals in Eq. (6) may appear forbidding, they are really simple to calculate. We consider first a change of notation for the unit vector $\hat{\mathbf{R}}$. This vector is defined by $\hat{\mathbf{R}}=\mathbf{R}/R=(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|$. But now we see that the volume integration over the delta terms $\delta\{\mathbf{r}'-\mathbf{s}(t')\}$ in Eqs. (6) implies $\mathbf{r}'=\mathbf{s}(t')$ so that $\hat{\mathbf{R}}$ becomes $\hat{\mathbf{R}}(t')=\{\mathbf{r}-\mathbf{s}(t')\}/|\mathbf{r}-\mathbf{s}(t')|$, i.e., after the volume integration the vector $\hat{\mathbf{R}}$ is function of t' (*notice that t' is still not identified with the retarded time*). The vector $\hat{\mathbf{R}}(t')$ will be denoted by the symbol "$\mathbf{n}$". Another ingredient we require in our integration process is the result[8]

$$\int dV'\,\delta(\mathbf{r}'-\mathbf{s}(t'))=\frac{1}{K}, \tag{7}$$

where $K\equiv\partial(\mathbf{z})/\partial(\mathbf{r}')$ is the Jacobian of the transformation from the variable $\mathbf{r}'$ to the variable $\mathbf{z}=\mathbf{r}'-s(t')$. In particular, when t' becomes the retarded time, the Jacobian is

$$K=1-\mathbf{v}\cdot\mathbf{n}/c, \tag{8}$$

where by economy we have written $\mathbf{v}\cdot\mathbf{n}$ to specify $\mathbf{v}(t')\cdot\hat{\mathbf{R}}(t')$. We claim here (by nth time) that *t' is still not identified with the retarded time*. Therefore, at this stage, K is still not defined by Eq. (8). We are now ready to carry out the volume integrations in Eqs. (6):

$$\int dt'\int dV'\,\frac{\hat{\mathbf{R}}}{R^2}\,\delta(\mathbf{r}'-\mathbf{s}(t'))\delta\left(t'+\frac{R}{c}-t\right)=\int dt'\,\delta\left(t'+\frac{R(t')}{c}-t\right)\frac{\mathbf{n}}{KR^2}, \tag{9}$$

where $R(t')$ stands for $|\mathbf{r}-\mathbf{s}(t')|$. Henceforth, after performing the volume integrations, the dependence of R on t' will not be shown explicitly, except in the argument of the delta function. Taking into account the result

$$\delta'\left(t'+\frac{R}{c}-t\right)=\frac{d}{dt'}\,\delta\left(t'+\frac{R}{c}-t\right), \tag{10}$$

we calculate the remaining volume integrals in Eq. (6):

$$\int dt' \int dV' \,\frac{\hat{\mathbf{R}}}{cR}\,\delta(\mathbf{r}'-\mathbf{s}(t'))\,\frac{d}{dt'}\,\delta\left(t'+\frac{R}{c}-t\right)=-\frac{1}{c}\int dt'\;\delta\left(t'+\frac{R(t')}{c}-t\right)\frac{d}{dt'}\left(\frac{\mathbf{n}}{KR}\right), \tag{11}$$

$$\int dt' \int dV' \,\frac{\delta\left(t'+\frac{R}{c}-t\right)}{c^2R}\,\frac{\partial}{\partial t'}\,(\mathbf{v}(t')\,\delta[\mathbf{r}'-\mathbf{s}(t')])=\frac{1}{c^2}\int dt'\,\delta\left(t'+\frac{R(t')}{c}-t\right)\frac{d}{dt'}\left(\frac{\mathbf{v}}{KR}\right). \tag{12}$$

Making use of Eqs. (9), (11), and (12), the field $\mathbf{E}$ in Eq. (6) reduces to

$$\mathbf{E}=\frac{e}{4\pi\epsilon_0}\int dt'\;\delta\left(t'+\frac{R(t')}{c}-t\right)\left[\frac{\mathbf{n}}{KR^2}+\frac{1}{c}\frac{d}{dt'}\left(\frac{\mathbf{n}}{KR}\right)-\frac{1}{c^2}\frac{d}{dt'}\left(\frac{\mathbf{v}}{KR}\right)\right]. \tag{13}$$

The time derivatives in this equation can be calculated from the results[14]

$$\frac{1}{c}\frac{d\mathbf{n}}{dt'}=\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{R}, \tag{14}$$

$$\frac{1}{c}\frac{d(KR)}{dt'}=\beta^2-\boldsymbol{\beta}\cdot\mathbf{n}-\frac{R}{c}\,\mathbf{n}\cdot\dot{\boldsymbol{\beta}}, \tag{15}$$

where $\boldsymbol{\beta}=\mathbf{v}/c$. Hence, Eq. (13), after some vector manipulation, takes the form

$$\mathbf{E}=\frac{e}{4\pi\epsilon_0}\int dt'\;\delta\left(t'+\frac{R(t')}{c}-t\right)\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2K^3R^2}+\frac{\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}}{cK^3R}\right), \tag{16}$$

where $\gamma=(1-\beta^2)^{-1/2}$ and $\dot{\boldsymbol{\beta}}=d\boldsymbol{\beta}/dt$. By a similar procedure, we find from Eqs. (2b), (3), (4), and (5b) that the magnetic field is

$$\mathbf{B}=\frac{e\mu_0}{4\pi}\int dt'\int dV'\;\delta(\mathbf{r}'-\mathbf{s}(t'))\,\frac{\mathbf{v}\times\hat{\mathbf{R}}}{R^2}\,\delta\left(t'+\frac{R}{c}-t\right)-\frac{e\mu_0}{4\pi}\int dt'\int dV'\,\delta(\mathbf{r}'-\mathbf{s}(t'))\frac{\mathbf{v}\times\hat{\mathbf{R}}}{cR}\,\delta'\left(t'+\frac{R}{c}-t\right). \tag{17}$$

When the volume integrations in this equation are performed, we obtain

$$\mathbf{B}=\frac{e\mu_0}{4\pi}\int dt'\;\delta\left(t'+\frac{R(t')}{c}-t\right)\left[\frac{\mathbf{v}\times\mathbf{n}}{KR^2}+\frac{1}{c^2}\frac{d}{dt'}\left(\frac{\mathbf{v}\times\mathbf{n}}{KR}\right)\right]. \tag{18}$$

By a straightforward calculation in which Eqs. (14) and (15) are used, we can write Eq. (18) as

$$\mathbf{B}=\frac{e\mu_0}{4\pi}\int dt'\;\delta\left(t'+\frac{R(t')}{c}-t\right)\left(\frac{\mathbf{v}\times\mathbf{n}}{\gamma^2K^3R^2}+\frac{\mathbf{n}\times\{-\dot{\boldsymbol{\beta}}+\mathbf{n}\times(\dot{\boldsymbol{\beta}}\times\boldsymbol{\beta})\}}{K^3R}\right). \tag{19}$$

The formulas (16) and (19) are our main result. We have arrived at them by a route in which we have avoided to speak of retardation, potentials, and gauges. In Eqs. (16) and (19) t' *is still not identified with the retarded time* (the monster is still inside the trunk). But if we perform the time integration (i.e., we take the monster out of the trunk!) on Eqs. (16) and (19) then we obtain

$$\mathbf{E}=\frac{e}{4\pi\epsilon_0}\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2K^3R^2}+\frac{\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}}{cK^3R}\right]_{\text{ret}}, \tag{20a}$$

$$\mathbf{B}=\frac{e\mu_0}{4\pi}\left[\frac{\mathbf{v}\times\mathbf{n}}{\gamma^2K^3R^2}+\frac{\mathbf{n}\times\{-\dot{\boldsymbol{\beta}}+\mathbf{n}\times(\dot{\boldsymbol{\beta}}\times\boldsymbol{\beta})\}}{K^3R}\right]_{\text{ret}}, \tag{20b}$$

where the square brackets with the subscript "ret" denote the retardation symbol indicating that the enclosed quantity is to be evaluated at the *retarded time* $t'=t-R(t')/c$ (at last!), and then K is defined by Eq. (8). Equations (20) are the Liénard–Wiechert fields of a point charge. Equivalent forms for these fields are the so-called Heaviside–Feynman formulas.[18] We challenge the reader to derive these last formulas from our Eqs. (13) and (18).

The approach developed here to find Eqs. (16) and (19) and hence Eqs. (20) can be extended to find the Liénard–Wiechert fields of a point dipole in arbitrary motion. The radiative part of these fields have been recently derived[19] by using a reformulation of Jefimenko's formulas with material sources.

III. DISCUSSION

Is the preceding derivation of the Liénard–Wiechert fields suitable for an undergraduate course of electromagnetic theory? At first sight the answer would be negative since the specific Green function used in the derivation is commonly introduced in a graduate course on the subject. However, this Green function may be alternatively introduced following a heuristic argument rather than a formal procedure.

Let us begin with the free-space Green function for the Poisson equation:

$$G_p(\mathbf{r},\mathbf{r}')=\frac{1}{R}. \tag{21}$$

As is well known this function satisfies

$$\nabla'^2\left(\frac{1}{R}\right)=-4\pi\delta(\mathbf{r}-\mathbf{r}'). \tag{22}$$

If we multiply both sides of this equation by the factor $\delta(t-t')$ we obtain

$$\nabla'^2\left(\frac{\delta(t-t')}{R}\right)=-4\pi\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{23}$$

Now, this equation may be integrated with respect to time from $-\infty$ to $+\infty$,

$$\int dt'\ \nabla'^2\left(\frac{\delta(t-t')}{R}\right)=-4\pi\int dt'\ \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{24}$$

It is easy to see that Eq. (22) is recovered if the condition of *simultaneity*,

$$\tau-\tau'=0, \tag{25}$$

is assumed in Eq. (24). This argument suggests defining the *instantaneous* Green function G_p^i as

$$G_p^i(\mathbf{r},t;\mathbf{r}',t')=\frac{\delta(t-t')}{R}. \tag{26}$$

It is evident that this Green function allows us to obtain the free-space solution of the *instantaneous* Poisson equation

$$\nabla^2\Phi=-4\pi\rho(\mathbf{r},t). \tag{27}$$

This solution is

$$\Phi(\mathbf{r},t)=\int dV'\int dt'\ G_p^i(\mathbf{r},t;\mathbf{r}',t')\rho(\mathbf{r}',t'). \tag{28}$$

Accordingly, we can say that G_p^i is an instantaneous propagator or equivalently G_p^i propagates with infinite speed. In fact, if the "signal" of propagation between the source and field points "travels" in a straight line with constant speed v then we have

$$v=\frac{|\mathbf{r}-\mathbf{r}'|}{t-t'}, \tag{29}$$

and we see that v becomes infinite when the condition of simultaneity (25) is assumed.

At this stage we invoke the *sacrosanct* principle of causality according to which the cause precedes its effect. If at the time t' we have the cause and at the time t the effect, then causality demands that $t>t'$, i.e.,

$$t-t'>0. \tag{30}$$

This condition implies that the speed of propagation of the signal is finite. From the definition of the speed $v=dS/dt$, we have

$$t-t'=\int_{\mathbf{r}'}^{\mathbf{r}}\frac{dS}{v}. \tag{31}$$

We will make here two crucial assumptions: The speed v is a constant equal to the speed of light c and the path of integration in Eq. (31) is the straight line joining the source and field points. Under these assumptions Eq. (31) becomes

$$t-t'=\frac{|\mathbf{r}-\mathbf{r}'|}{c}, \tag{32}$$

or with the employed notation $t'=t-R/c$, that is, the retarded time. This argument suggests introducing the causal (not instantaneous) Green function G by generalizing Eq. (26) as follows:

$$G(\mathbf{r},t;\mathbf{r}',t')=\frac{\delta\left(t-t'-\frac{R}{c}\right)}{R}. \tag{33}$$

Since $\delta(u)=\delta(-u)$, we see that Eq. (33) becomes Eq. (3). Evidently the propagator in Eq. (33) cannot be solution of an instantaneous equation like Eq. (23) [together with Eq. (25)]. Instead, the Green function G satisfies the wave equation:

$$\left(\nabla'^2-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right)G(\mathbf{r},t;\mathbf{r}',t')=-4\pi\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{34}$$

In this context, we can say that G is a hyperbolic finite-speed propagator or equivalently G propagates hyperbolically with speed c. The propagator G allows us to solve the free-space inhomogeneous wave equation.[13] Thus we have deduced our basic Green function following a heuristic argument which we think is suitable for undergraduate students. Hence, our derivation of the Liénard–Wiechert fields would be suitable for an undergraduate course of electromagnetism.

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The rise of a liquid in a capillary tube revisited: A hydrodynamical approach

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We use the time-dependent Navier–Stokes equation to study the dynamics of a liquid rising in a capillary tube. We show that the evolution consists of an initial stage of inertial motion, followed by a dynamical-balance stage in which gravity and viscosity compensate the capillarity force. This approach is compared with other formulations, based on the study of variable mass systems or on some simplified hydrodynamical formulas. Our attention is then focused on the crossover between the regimes of monotonic and oscillatory motion, and we find that numerical solutions to the relevant equations are in good agreement with an analytical linearized approach. © *1996 American Association of Physics Teachers.*

I. INTRODUCTION

The motion of a liquid in a capillary tube presents several interesting features, both in its static and in its dynamical properties. From a pedagogical viewpoint, this phenomenon provides an appropriate framework for introducing basic ideas of surface tension effects—such as adhesion of liquids to solids, contact angles, and wetting[1,2]—as well as some of the fundamental concepts of liquid mechanics.[2,3] This has motivated the publication of several papers on the subject that have appeared in this journal in the last two decades.[4–9]

In particular, Menon and Agrawal[6] have described the liquid column inside the tube as a variable-mass mechanical system, writing the related Newton equation and solving it under some simplifying assumptions. Within a different approach, Peiris and Tennakone[4] have treated the same problem by means of the Poiseuille formula for the rate of flow of a viscous fluid in a cylindrical pipe. Although the validity of such a formulation is not discussed in their paper, they have found excellent agreement with experiments.

In this paper, we aim at describing the motion of a viscous fluid in a capillary tube starting from the basic laws of hydrodynamics. The proper description is provided, in this case, by the Navier–Stokes equation (NSE). By means of a suitable average of the NSE, we obtain an evolution equation for the height of the liquid column in the tube. This nonlinear ordinary differential equation is similar to the one obtained by means of the variable-mass technique and reduces to the Poiseuille formulation for long times, when the inertial effects of the initial stages of the rise have disappeared. Our formulation then reconciles both approaches, and clarifies the range of applicability of the Poiseuille formula to this particular problem.

By means of numerical techniques, we study the evolution of the fluid column toward its equilibrium state. Depending on the value of the relevant parameters, the liquid rise is monotonic or exhibits oscillations, damped by viscosity and friction against the tube walls. In order to study in an analytical way the crossover between these two regimes, we use an approximate perturbation method and compare the results with those obtained from the numerical solution of the evolution equation.

II. HYDRODYNAMICAL APPROACH

A. The Navier–Stokes equation

We consider a capillary tube of inner radius R immersed vertically into an incompressible viscous liquid of density ρ, viscosity η, and liquid–vapor surface tension γ. The lower end of the tube is at a depth a below the free surface of the liquid. Inside the tube, the interface between the fluid and its vapor is considered to form a constant angle θ with respect to the wall (Fig. 1). The surface tension in that interface produces a net force directed upward and, as a consequence, the liquid rises inside the tube. Its motion is governed by the time-dependent Navier–Stokes equation (NSE), which describes the dynamical momentum balance in the fluid,[3]

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\frac{1}{\rho}\nabla p+\mathbf{f}+\frac{\eta}{\rho}\nabla^2\mathbf{v}, \quad (1)$$

along with the incompressibility condition

$$\nabla\cdot\mathbf{v}=0. \quad (2)$$

Here, $\mathbf{v}(\mathbf{r},t)$ and $p(\mathbf{r},t)$ are the velocity and pressure fields, respectively. In our case, the force per unit fluid mass $\mathbf{f}$ is exclusively due to gravity, $\mathbf{f}=-g\hat{\mathbf{z}}$. As we shall see later, the

