the momentum is $\mathbf{P}=[m_1(1+\frac{1}{2}v^2/c^2)+m_2(1+\frac{1}{2}v^2/c^2)+2e_1e_2/(c^2\ell)]\mathbf{v}$; the energy is $E=[m_1(c^2+\frac{1}{2}v^2)+m_2(c^2+\frac{1}{2}v^2)+e_1e_2/\ell]$. There is a factor-of-two discrepancy in the electromagnetic field contributions corresponding to the presence of the summation term in Eq. (4).

(d) For the classical model of the electron, as in case (c), forces of constraint are present but do no *net* work so that Eq. (4) holds. The forces are applied at points with differing displacements $\mathbf{r}_i$ in the direction of motion and so the summation term in Eq. (4) is non-vanishing. The term involving the external forces gives an additional contribution so that we find Eq. (2), which is not in agreement with the form in Eq. (1). This case is discussed in Ref. 9.

The difference between the system (mechanical and electromagnetic) momentum $\mathbf{P}$ and the term $d[(E/c^2)\mathbf{X}]/dt$ involving the system (mechanical and electromagnetic) energy E is sometimes termed "hidden momentum."[10] We see in Eq. (3) that this hidden momentum is given by $-\Sigma_i(\mathbf{F}_{\text{ext},i}\cdot\mathbf{v}_i)\mathbf{r}_i/c^2$ and involves forces that are external to the electromagnetic system. The designation of this term as hidden momentum tells us little about the character of the non-electromagnetic energy and momentum flow associated with the external forces.

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Comment on "A generalized Helmholtz theorem for time-varying vector fields," by Artice M. Davis [Am. J. Phys. 74, 72–76 (2006)]

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In a recent paper Davis formulated the following generalization of the Helmholtz theorem for a time-varying vector field:[1]

$$\boldsymbol{F}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(\nabla\phi+\frac{\partial \boldsymbol{A}}{\partial t}\right)+\nabla\times(\nabla\times\boldsymbol{A}), \tag{1}$$

where ϕ and $\boldsymbol{A}$ are the Lorenz gauge retarded potentials. The purposes of this Comment are to point out that Davis's generalization is a version of the generalization of the Helmholtz theorem formulated some years ago by McQuistan[2] and Jefimenko,[3] and more recently by the present author,[4–6] and to show that Davis's expression for the field $\boldsymbol{F}$ is also valid for potentials in gauges other than the Lorenz gauge.

The generalized Helmholtz theorem states that a retarded vector field vanishing at infinity can be written as[4]

$$\boldsymbol{F}=-\nabla\int d^3x'\frac{[\nabla'\cdot\boldsymbol{F}]}{4\pi R}+\nabla\int d^3x'\frac{[\nabla'\times\boldsymbol{F}]}{4\pi R}+\frac{1}{c^2}\frac{\partial}{\partial t}\int d^3x'\frac{[\partial\boldsymbol{F}/\partial t]}{4\pi R}, \tag{2}$$

where the square brackets denote the retardation symbol, $R=|\mathbf{x}-\mathbf{x}'|$, and the integrals are over all space. If we define the potentials Φ, $\mathbf{A}$, and $\mathbf{C}$ by

$$\Phi=\int d^3x'\frac{[\nabla'\cdot\boldsymbol{F}]}{4\pi R}, \tag{3a}$$

$$\mathbf{A}=\int d^3x'\frac{[\nabla'\times\boldsymbol{F}]}{4\pi R}, \tag{3b}$$

$$\mathbf{C} = \int d^3x' \frac{[\partial \boldsymbol{F}/\partial t]}{4\pi R}, \tag{3c}$$

then Eq. (2) can be written compactly as

$$\boldsymbol{F} = -\nabla\Phi + \nabla\times\mathbf{A} + \frac{1}{c^2}\frac{\partial\mathbf{C}}{\partial t}. \tag{4}$$

The potentials Φ and $\mathbf{A}$ in this formulation of the theorem are different from the potentials ϕ and $\boldsymbol{A}$ in Davis's formulation. It is not difficult to derive the relations $\partial\phi/\partial t = -c^2\Phi$, $\nabla\times\boldsymbol{A} = \mathbf{A}$, and $\partial\boldsymbol{A}/\partial t = \mathbf{C}$, which imply the formal equivalence between the two formulations.

The standard Helmholtz theorem is usually applied to solve the equations of electrostatics and magnetostatics. The generalization of this theorem[4] can be used to solve Maxwell's equations. The generalization proposed by Davis[1] can be used to elucidate the form of Maxwell's equations. The two versions of the generalized Helmholtz theorem are complementary.

In Ref. 1 the electric field $-\nabla\phi - \partial\boldsymbol{A}/\partial t$ and the magnetic field $\nabla\times\boldsymbol{A}$ are expressed in terms of the Lorenz gauge potentials, which were used to formulate Eq. (1) for the time-varying vector field $\boldsymbol{F}$. Equation (1) can also be formulated using potentials in other gauges. For example, it can be formulated for potentials in the velocity gauge[7] $\nabla\cdot\boldsymbol{A} + (1/v^2)\partial\phi/\partial t = 0$, a class of gauges containing the Coulomb gauge ($v = \infty$), the Lorentz gauge ($v = c$), and the Kirchhoff gauge[8] ($v = ic$). Jackson[7] recently derived the gauge function χ_v [Eq. (7.5) of Ref. 7], which transforms the Lorenz gauge potentials ϕ_L and $\boldsymbol{A}_L$ to the velocity gauge potentials ϕ_v and $\boldsymbol{A}_v$,

$$\phi_v = \phi_L - \frac{\partial\chi_v}{\partial t}, \tag{5a}$$

$$\boldsymbol{A}_v = \boldsymbol{A}_L + \nabla\chi_v. \tag{5b}$$

From Eq. (5) we obtain

$$\nabla\phi_L + \frac{\partial\boldsymbol{A}_L}{\partial t} = \nabla\phi_v + \frac{\partial\boldsymbol{A}_v}{\partial t} \tag{6a}$$

$$\nabla\times\boldsymbol{A}_L = \nabla\times\boldsymbol{A}_v. \tag{6b}$$

Equations (6) imply that Eq. (1) is also valid for potentials in the velocity gauge, which means that it is valid for the Coulomb and Kirchhoff gauges also. The application of Eq. (1) to potentials in the velocity gauge requires the identification $\boldsymbol{F} = \mu_0\mathbf{J}$, where $\mathbf{J}$ is the current density and μ_0 the permeability of free space.

Davis introduced causality in Eq. (1) when he chose retarded potentials. But we can equally choose acausal advanced potentials to obtain Eq. (1). Causality in Eq. (1) is not a necessary assumption, but it is required to identify $-\nabla\phi - \partial\boldsymbol{A}/\partial t$ and $\nabla\times\boldsymbol{A}$ with the retarded electric and magnetic fields. As pointed out by Rohrlich,[9] causality must be inserted by hand in classical field theories as a condition.

The reader might wonder why Eq. (1) can also be written in terms of the Coulomb gauge potentials when the instantaneous scalar potential ϕ_C in this gauge is clearly acausal. The explanation is that the Coulomb gauge vector potential $\mathbf{A}_C$ contains two parts, one of which is causal (retarded) and the other is acausal (instantaneous). Jackson[7] recently derived a novel expression for $\mathbf{A}_C$ [Eq. (3.10) in Ref. 7] which exhibits both parts. The fact that $\mathbf{A}_C$ carries a causality violating instantaneous component has also been recently emphasized by Yang.[10] The effect of the acausal part of $\mathbf{A}_C$ vanishes identically when we take the curl and obtain $\nabla\times\boldsymbol{A}_C = \nabla\times\boldsymbol{A}_L$. A direct calculation gives[7] $-\partial\mathbf{A}_C/\partial t = -\nabla\phi_L - \partial\boldsymbol{A}_L/\partial t + \nabla\phi_C$. The last (acausal) term cancels exactly the instantaneous electric field $-\nabla\phi_C$ generated by ϕ_C and we again obtain $-\nabla\phi_C - \partial\boldsymbol{A}_C/\partial t = -\nabla\phi_L - \partial\boldsymbol{A}_L/\partial t$. This expression has also been recently demonstrated in Ref. 11 using a different approach [see Eq. (29) in Ref. 11]. In other words, the explicit presence of an acausal term in Eq. (1), when it is written in terms of the Coulomb gauge potentials, is irrelevant because such a term is always canceled, which means that causality is never effectively lost.

Similar conclusions can be drawn when Eq. (1) is expressed in terms of the Kirchhoff gauge potentials ϕ_K and $\mathbf{A}_K$ (Ref. 8). In this case the potential ϕ_K propagates with the imaginary speed ic and generates the imaginary field $-\nabla\phi_K$. The Kirchhoff gauge vector potential $\mathbf{A}_K$ contains three parts: one is causal (retarded), one is imaginary, and the remaining one mixes imaginary and retarded contributions [see Eq. (42) in Ref. 8]. The effect of the imaginary terms in the last two parts vanishes identically when we take the curl and obtain $\nabla\times\boldsymbol{A}_K = \nabla\times\boldsymbol{A}_L$. A direct calculation gives[8] $-\partial\boldsymbol{A}_K/\partial t = -\nabla\phi_L - \partial\boldsymbol{A}_L/\partial t + \nabla\phi_K$. The last term cancels exactly the imaginary field $-\nabla\phi_K$ and we again obtain $-\nabla\phi_K - \partial\boldsymbol{A}_K/\partial t = -\nabla\phi_L - \partial\boldsymbol{A}_L/\partial t$. The explicit presence of an imaginary term in Eq. (1) when it is written in terms of the Kirchhoff gauge potentials is irrelevant because such a term is always canceled, which means that causality is never effectively lost.

In the same sense that the Helmholtz theorem is considered as the mathematical foundation of electrostatics and magnetostatics, the generalized Helmholtz theorem can be considered as the mathematical foundation of electromagnetism. I advocate the use of both formulations of the generalized Helmholtz theorem[12] in courses of electromagnetism and invite instructors to decide which formulation they find more useful.

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Comments on the tethered galaxy problem

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In a recent paper Davis, Lineweaver, and Webb make the counterintuitive assertion that a galaxy held "tethered" at a fixed distance from our own could emit blueshifted light. This effect was derived from the simplest Friedmann-Robertson-Walker (FRW) spacetimes and the $\Omega_M=0.3$, $\Omega_\Lambda=0.7$ case, which is believed to be a good late time model of our universe. In this paper, we recover their results in a more transparent way, revise their calculations, and propose a formulation of the tethered galaxy problem based on radar distance rather than comoving "proper" distance. This formulation helps to remove the coordinate-dependent nature of the tethered galaxy problem and establishes consistency between the empty FRW model and special relativity. In the general case, we see that, although the radar distance tethering reduces the redshift of a receding object, it does not do so sufficiently to cause the blueshift as found by Davis, Lineweaver, and Webb. We also discuss some important issues raised by this approach relating to the interpretation of the redshift, velocity, and distance in relativistic cosmology.
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I. INTRODUCTION

The homogeneous and isotropic expansion of the universe is described by Friedmann-Robertson-Walker (FRW) spacetimes.[1] In these spacetimes, we may construct a co-moving frame in which the spacetime manifold of general relativity is treated as expanding, while on average matter is at rest. If we wish to study independent dynamical objects within an expanding universe, we would like to quantify the effect of imposing such a cosmological background. We follow Davis *et al.*[2] by considering a galaxy endowed with a large peculiar velocity that characterizes the velocity deviation from the universal expansion or Hubble flow. The tethered galaxy problem considers the physics of the extreme case, where a galaxy is endowed with sufficient peculiar velocity so as to cancel the Hubble flow and remain, in some sense, at a fixed distance from a co-moving observer who follows the Hubble flow. We study how light from such a galaxy would be redshifted and propose modifications to the previous calculations[2] that suggest that a receding source could be significantly blueshifted.

By recasting the problem in a coordinate independent form, with the radar distance as our measure, we achieve results that are more intuitive than previous work.[2] With this construction, tethered galaxies in the empty FRW universe have zero redshift and the problem may be reconciled with the notion of constant spatial separation in special relativity.

We find that the reduction in redshift is much lower than that obtained by the previous analysis.[2] The effect of imposing the tether is not sufficient to cause the blueshift of an object in an expanding universe and is only significant at very large distances, at which we would not expect proportionally large peculiar velocities.

Throughout this paper, we shall refer to the unevaluated quantity $1+z$ as the redshift of a light source, but note that a value of $z<1$ will actually correspond to a blueshift. Similarly, the condition for light to be received with zero redshift implies that it is observed at its emission frequency, that is, is neither redshifted nor blueshifted.

In Sec. II, we review the tethered galaxy problem as posed in Ref. 2. We remove the explicit redshift dependence from their calculations and recover their results as a combination of cosmic redshift and the special relativistic Doppler shift. We are thus able to demonstrate why the peculiar velocity required to cancel the cosmic redshift does not correspond to that proposed to tether a galaxy against the Hubble flow.

We then discuss two problems with this approach. First, we note that the peculiar velocities do not correspond to a quantity we might regard as a worldline velocity except in the special relativistic limit. In Sec. III, we construct a general relativistic condition on the 4 velocity of a luminous particle in order that light is received at the fixed spatial origin without redshift. In Sec. IV, we discuss the limitations of the distance scale used in the original formulation. Using this measure, we show that for the Milne model under a coordinate transformation, the problem does not—as we might expect—agree with the analogous system in special relativity. More details of this example appear in Appendix A.

Motivated by this example, we propose recasting the tethered galaxy problem in terms of a theoretically observable quantity, the radar distance. We construct this new system of observers in Sec. V and propose a method for solving the system in terms of light signals. In Sec. VI, we compare the phenomenological results with a physical model. We see that the effects of the tethered galaxy problem persist although they are comparatively small below scales of 10^4-10^5 megaparsec (Mpc).