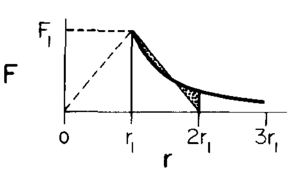

Fig. 3. Area comparison of Fig. 3 facilitated by superposing the triangle on the curve. The two shaded areas are the same.

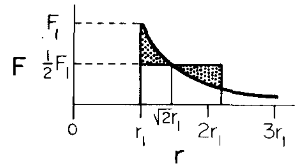

Fig. 4. The constant force $F_1/2$ does the same work as the variable force in Fig. 3.

weights of the two shapes cut from the same uniform plate.

An even easier way to compare the areas, at least for a cursory eyeball approximation, is this: Flip over the triangle so that it is superposed on the curve, as in Fig. 3. Your eye tells you at once that the two shaded areas are very closely the same. Conclusion: When the force is k/r^2, the potential energy is $-k/r$.

Finally, what constant force is equivalent, in terms of work done, to the variable inverse-square force, again over the displacement from r_1 to $2r_1$? The net work done is still $F_1 r_1/2$, so the constant force has the magnitude $F_1/2$ (see Fig. 4). It is easy to confirm that the curve crosses the horizontal line at $2^{1/2} r_1$, which is the geometric mean of r_1 and $2r_1$.

[1] J. V. Mallow, Am. J. Phys. **54**, 944 (1986).
[2] M. Iona, Am. J. Phys. **55**, 776 (1987).

Angular momentum in the field of an electron

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Normally we represent the electron as a little charged sphere in our introductory physics courses. We realize of course that this is wrong and only a conceptual approximation that we must use. Nevertheless, it is instructive to see how far down on the length scale we can go with our classical concepts of fields before there is a major breakdown. Feynman[1] shows that if we extend the classical coulomb field all the way down to $\frac{2}{3}$ of the classical electron radius ($r_0 = \alpha\hbar/mc$), the entire mass of the electron is contained in its field. However, as he points out, there are problems since the calculation ignores the mass of the "self-energy" forces which, according to the classical ideas, must bind the electron together. So we see that at about the level of r_0, our classical treatment certainly breaks down.

Another physical property of the electron is its spin angular momentum of magnitude $\sqrt{3}\hbar/2$. If we consider a "billiard ball" electron of radius r_0, it would need a maximum surface speed of about 300 times the speed of light to have this much angular momentum. However, just as some of the electron's mass is contained in its coulomb field, perhaps some of its angular momentum could be contained in the field pattern as well. In order to see how this could come about, consider the following thought experiment, also presented by Feynman[2]:

Imagine an insulating disk suspended by a silken thread along its axis. Around the periphery of the disk are several metal balls, not touching and all charged with $+Q$ coulombs. At the center of the disk and coaxial with it is a superconducting ring containing a current of I amperes so that initially there is a magnetic dipole field and an approximately radial electric field. Initially the disk is not rotating. As the temperature of the ring is raised above its critical value, the current suddenly stops, the magnetic field collapses, and, by Faraday's law, an induced electric field appears briefly in the vicinity of the charged balls tangential to the disk. The disk receives a torque and starts to rotate. Where does the angular momentum come from? If you believe it is due to the stopping of the motion of the charge carriers in the ring, reverse the sign on the charge of the metal balls to get a rotation in the opposite direction. We are still left with a problem: the final angular momentum of the disk appears to come from "nowhere." We can do one of two things at this point. We can decide that angular momentum is not conserved in electromagnetic interactions, or we can assign an angular momentum to the field pattern itself in much the same way we assign mass or inertia to an electromagnetic field. (It is well known that electromagnetic radiation exerts a "radiation pressure.") The basic problem is that the magnetic component of the interaction introduces a noncentral force to the system and thus changes its angular momentum. In order to conserve angular momentum we must assign the difference to the field pattern itself. This is most directly seen in the case of right and left circularly polarized photons that carry one unit $\hbar$ of angular momentum. This amount of angular momentum is required to balance the change in angular momentum experienced by the atomic system that created the photon (which is actually just a smaller version of the thought experiment presented here).

Having established that the initial field pattern contains angular momentum, we notice that at distances greater than the disk radius the electric and magnetic fields are "crossed"; i.e., Poynting's vector $\mathbf{N} = \mathbf{E}\times\mathbf{H}$, circulates around the disk axis. This vector gives the energy flux in the electromagnetic field pattern (energy flux = energy density $\times c$).[3] We obtain the volume density of linear momentum from this by dividing Poynting's vector by c^2 (momentum density = energy density/c):

$$\mathbf{p} = \mathbf{N}/c^2 = \mathbf{D}\times\mathbf{B}\,.$$

The fields of the electron are

$$\mathbf{D} = -e\mathbf{r}/4\pi r^3\,,$$

$$\mathbf{B}=[3(\boldsymbol{\mu}\cdot\mathbf{r})r/r^2-\boldsymbol{\mu}]/4\pi\epsilon_0c^2r^3,$$

where $\boldsymbol{\mu}$ is the magnetic dipole moment. This gives

$$\mathbf{p}=e(\mathbf{r}\times\boldsymbol{\mu})/\epsilon_0(4\pi c)^2r^6=r_0(\mathbf{r}\times\boldsymbol{\mu})/(e/m)4\pi r^6,$$

where, as above, r_0 is the classical electron radius

$$r_0=e^2/4\pi\epsilon_0mc^2.$$

The volume density of angular momentum is given by

$$\mathbf{l}=\mathbf{r}\times\mathbf{p}=(\cdots)\mathbf{r}\times(\mathbf{r}\times\boldsymbol{\mu})=(\cdots)[\mathbf{r}(\mathbf{r}\cdot\boldsymbol{\mu})-\boldsymbol{\mu}r^2].$$

When we align $\boldsymbol{\mu}$ with the z axis and use spherical polar coordinates, the z component of $\mathbf{l}$ is

$$l_z=-(\cdots)\mu r^2\sin^2\theta.$$

The total angular momentum in the field extending from $r=a$ to infinity is given by the integral

$$S=\int l_z\,dV=-(2/3)r_0\mu/a(e/m).$$

For $a=(2/3)r_0$ we have

$$S=-\mu/(e/m),$$

which just happens to be the entire spin angular momentum of the electron! Recall that it was for this radius that Feynman found the entire mass of the electron in its field. This can only be regarded as an amazing coincidence, albeit an interesting one. Notice also that we get the correct "g factor" of $g_s=2$:

$$\boldsymbol{\mu}=(-e/2m)g_s\mathbf{S}.$$

This is the value Dirac obtained from his relativistic quantum theory and since relativity is just an extension of classical electromagnetism, perhaps this calculation helps us appreciate a little better that the electron g factor is primarily a property of the electron's field pattern rather than some incomprehensible feature of Dirac's theory.

Feynman stresses the fact that no classical model of the electron is possible, and these calculations emphasize the idea that quantum theory is needed below a certain limit. However, they can be used as an interesting application of the energy and momentum contained in an electromagnetic field pattern.

[1]R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1964), Vol. II, Chap. 28.

[2]Reference 1. The problem is posed as a paradox in Sec. 17-4 and resolved in Sec. 27-6.

[3]W. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, MA, 1962), 2nd ed., Secs. 10-5 and 10-6. Notice particularly the last paragraph of Chap. 10.

The energy eigenvalues of the Dirac hydrogen atom

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The solution of the Dirac equation for the hydrogen atom is well known and can be found in any good textbook on relativistic quantum mechanics. However, we shall see[1] in the following that we can obtain the eigenvalues and bypass the complication of eigenfunctions for the relativistic case. In fact, we shall cast the relativistic hydrogen equation in a form that is of exactly the same type as the second-order differential equation of the nonrelativistic case. Thus by comparison we obtain the energy eigenvalues, thus bypassing the calculation of the eigenfunctions.

The Dirac equation of the hydrogenlike atom is

$$[c(\alpha_x p_x+\alpha_y p_y+\alpha_z p_z)+\beta mc^2-ze^2/r]\psi=E\psi. \tag{1}$$

Multiplying both sides of Eq. (1) from the left by $\alpha r/r$, we get (see Appendix)

$$c\left(\frac{rp}{r}\right)\psi+\frac{ic}{r}(\sigma L)\psi-mc^2\left(\beta\frac{\alpha r}{r}\right)\psi-\frac{ze^2}{r}\left(\frac{\alpha r}{r}\right)\psi=E\left(\frac{\alpha r}{r}\right)\psi. \tag{2}$$

We now have a lemma,[2]

$$\begin{aligned}(\sigma L+\hbar)^2&=L^2+\hbar^2-\hbar\sigma L+2\hbar\sigma L\\&=[L+(\hbar/2)\sigma]^2+\tfrac14 h^2=j(j+1)\hbar^2+\tfrac14\hbar^2\\&=(j+\tfrac12)^2\hbar^2=k^2\hbar^2.\end{aligned}$$

However, in Dirac's book it has been shown that $(\sigma L+\hbar)$ anticomments with (αp) and therefore we have taken the $\beta(\sigma L=\hbar)$ as the required quantity. This $\beta(\sigma L+\hbar)$ also has the same property with the additional feature that it commutes with the Hamiltonian. Thus Eq. (2) turns out to be

$$\frac{c\hbar}{i}\frac{\partial\psi}{\partial r}+\frac{ic}{r}(\beta k\hbar-\hbar)\psi-mc^2\left(\beta\frac{\alpha r}{r}\right)\psi-\frac{ze^2}{r}\left(\frac{\alpha r}{r}\right)\psi=E\left(\frac{\alpha r}{r}\right)\psi,$$

i.e.,

$$\frac{c\hbar}{i}\left(\frac{\partial}{\partial r}+\frac1r\right)\psi=\left(E+\frac{ze^2}{r}\right)\left(\frac{\alpha r}{r}\right)\psi+mc^2\left(\beta\frac{\alpha r}{r}\right)\psi-\frac{ckh}{r}\beta\psi. \tag{3}$$

Multiplying from the left by $(\hbar c/i)\ (\partial/\partial r+1/r)$, we get

$$\begin{aligned}-\hbar^2c^n\left(\frac{\partial}{\partial r}+\frac1r\right)\left(\frac{\partial}{\partial r}+\frac1r\right)&=\frac{c\hbar}{i}\left(\frac{\partial}{\partial r}+\frac1r\right)\left[E+\frac{ze^2}{r}\left(\frac{\alpha r}{r}\right)+mc^2\left(\beta\frac{\alpha r}{r}\right)-\frac{ick\hbar}{r}\beta\right]\psi\\&=\left[\left(E+\frac{ze^2}{r}\right)\left(\frac{\alpha r}{r}\right)+mc^2\left(\beta\frac{\alpha r}{r}\right)-\frac{ick\hbar}{r}\beta\right]^2\psi\\&\quad+\frac{c^2\hbar^2}{r^2}k\beta\psi-\frac{c\hbar}{i}\frac{ze^2}{r^2}\left(\frac{\alpha r}{r}\right)\psi,\end{aligned} \tag{4}$$