Hidden mechanical momentum and the field momentum in stationary electromagnetic and gravitational systems

V. Hnizdo
Department of Physics, Schonland Research Centre for Nuclear Sciences, and Centre for Nonlinear Studies, University of the Witwatersrand, Johannesburg 2050, South Africa

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The existence of hidden momentum, i.e., the mechanical momentum of a body the constituents of which move in a stationary manner but the center of mass of which is at rest in an external static field of force, is shown to follow from the general requirements of relativistic mechanics of continuous media, independently of whether the external forces are electromagnetic or gravitational. The hidden mechanical momentum is compensated by the momentum of the static fields in the system, necessitating, in the gravitational case, the existence of a gravinetic quasistatic gravitational field, which, in analogy to the magnetostatic field, is generated by a quasistationary current of mass and acts on a moving mass.

I. INTRODUCTION

A body the center of mass of which is at rest can carry a nonzero mechanical momentum, called hidden momentum. Since the discovery of this surprising result some 30 years ago,[1] in investigations of the forces that an electrically neutral current loop can experience in an electric field, the discussion of hidden momentum has focused on electrodynamic aspects of the phenomenon, such as the implications of hidden momentum for the force on a magnetic dipole in an electromagnetic field,[2] or the role of hidden momentum in the electromagnetic mass of a body carrying both charge and current.[3] Hidden momentum can arise, however, not only when electric currents are located in electric fields[4]—it is shown in the present paper that the existence of hidden momentum follows from the general requirements of special relativity, independently of whether the forces in question are of electromagnetic or gravitational nature.

In relativistic mechanics of continuous media, macroscopic bodies such as fluids and solids are described by an energy-momentum four-tensor $T^{\mu\nu}$. When the body is an ideal fluid,[5] the components of the four-tensor $T^{\mu\nu}$ give the body's energy density u, momentum density $\mathbf{g}$ and stress tensor σ_{ij} as[6]

$$u = T^{00} = \gamma^2(u_0 + pv^2/c^2), \tag{1}$$

$$g_i = T^{0i}/c = \gamma^2(u_0 + p)v_i/c^2, \tag{2}$$

$$\sigma_{ij} = -T^{ij} = -\gamma^2(u_0 + p)v_i v_j/c^2 - p\,\delta_{ij}, \tag{3}$$

where u_0 is the local-rest-frame energy density, p is the pressure in the body,[7] v_i are the components of the velocity $\mathbf{v}$ of the macroscopic motion of the volume element of the body at a given point in space, and $\gamma = (1 - v^2/c^2)^{-1/2}$ with $v = |\mathbf{v}|$. Equations (1)–(3) also give the energy-momentum tensor $T^{\mu\nu}$ of a solid body in an approximation that neglects possible violations of Pascal's law in the local rest frame of the solid.[8]

The energy-momentum tensor $T^{\mu\nu}$ satisfies the equation of motion

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = f^\mu, \tag{4}$$

where $f^\mu = (\mathbf{f}\cdot\mathbf{v}/c, \mathbf{f})$ is the four-vector of an external force density acting on the body.[9] The time ($\mu = 0$) component of Eq. (4) gives an equation of continuity for the energy density u,

$$\frac{\partial u}{\partial t} + \nabla\cdot(c^2\mathbf{g}) = \mathbf{f}\cdot\mathbf{v}. \tag{5}$$

The energy flux density is thus c^2 times the momentum density $\mathbf{g}$, which is the relation between these quantities that is demanded by special relativity, independently of the particular form [as, e.g., that of Eqs. (1)–(3) for an ideal fluid] that the components of the energy-momentum tensor $T^{\mu\nu}$ take. The space components of Eq. (4) with the energy-momentum tensor of Eqs. (1)–(3) lead to the following relativistic generalization of Euler's equation:

$$\frac{\gamma}{c^2}(u_0 + p)\left[\frac{\partial(\gamma\mathbf{v})}{\partial t} + (\mathbf{v}\cdot\nabla)(\gamma\mathbf{v})\right] + \frac{\gamma^2}{c^2}\left[\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p\right]\mathbf{v}$$
$$= -\nabla p + \mathbf{f}. \tag{6}$$

II. HIDDEN MOMENTUM

The momentum $\mathbf{P}$ of a body of finite dimensions can be obtained by integrating the momentum density $\mathbf{g}$ as

$$P_i = \int g_i d^3r = \int \nabla\cdot(x_i\mathbf{g})d^3r - \int x_i\nabla\cdot\mathbf{g}d^3r$$
$$= -\int x_i\nabla\cdot\mathbf{g}d^3r, \tag{7}$$

where the integral over $\nabla\cdot(x_i\mathbf{g})$ is zero as it can be transformed into the surface integral of an integrand that vanishes outside the body. Using Eq. (5) for the divergence of the momentum density, the body's momentum (7) is expressed as

$$\mathbf{P} = -\frac{1}{c^2}\int \mathbf{r}\left(\mathbf{f}\cdot\mathbf{v} - \frac{\partial u}{\partial t}\right)d^3r. \tag{8}$$

This momentum turns out to be not necessarily zero even when the body's elements move in a stationary manner, and its center of mass is at rest. When the motion of the body's elements is stationary, $\partial u/\partial t = 0$, and, furthermore, the external force density $\mathbf{f}$ is derivable from a potential ϕ by $\mathbf{f} = -\rho\nabla\phi$, where ρ is the density of the body's ''charge'' that is acted on by the external force, the momentum (8) is given as

$$\mathbf{P} = -\frac{1}{c^2}\int \mathbf{r}(\mathbf{v}\cdot\mathbf{f})d^3r = \frac{1}{c^2}\int \mathbf{r}(\rho\mathbf{v}\cdot\nabla\phi)d^3r$$
$$= -\frac{1}{c^2}\int \phi\mathbf{j}d^3r. \tag{9}$$

Here, the localized current density $\mathbf{j} = \rho\mathbf{v}$ is assumed divergenceless, and the second integral is transformed using the integral identity

$$\int a\mathbf{h}d^3r = -\int \mathbf{r}(\mathbf{h}\cdot\nabla a)d^3r, \tag{10}$$

which holds for any well-behaved a and localized divergenceless $\mathbf{h}$.[10] Equation (9) gives the hidden momentum of the body.[11]

The body's ''charge'' referred to above is the electric charge when the external force in question is due to an electric field,[12] or it is the rest mass when the external force is due to a gravitational field. Thus, for example, according to Eq. (9), a disk of radius r and uniformly distributed electric charge q, spinning with an angular velocity $\boldsymbol{\omega}$ about its symmetry axis but the center of mass of which is at rest in a uniform electric field $\mathbf{E}$, carries a hidden momentum

$$\mathbf{P} = \frac{qr^2}{4c^2}\,\boldsymbol{\omega}\times\mathbf{E}, \tag{11}$$

while a similar disk, of uniformly distributed rest mass m and spinning in a uniform gravitational field $\mathbf{e}$, has a hidden momentum

$$\mathbf{P} = \frac{mr^2}{4c^2}\,\boldsymbol{\omega}\times\mathbf{e}. \tag{12}$$

These results are obtained most easily by using the formula $\mathbf{P} = (1/c^2)\mathbf{m}\times\mathbf{E}$, with $\mathbf{m} = (qr^2/4)\boldsymbol{\omega}$ the magnetic dipole moment; this formula, expressed in terms of electromagnetic quantities, is equivalent to Eq. (9) when the external field in question is uniform.[13]

III. TWO LIMITING CASES

The expression (9) for hidden momentum was obtained from the relativistic connection of Eq. (5) between the energy flux density and momentum density, which does not provide an insight into the physical origin of hidden momentum. There are two limiting cases in which the hidden momentum of a body can be calculated easily by integrating directly the momentum density (2): (i) when the body is a sufficiently rarefied gas the pressure of which can be neglected; and (ii) when the linear, or angular, velocity of the body's elements is constant, which occurs in the flow of a fluid of negligible compressibility constrained by a straight,[14] or circular, tube of a constant cross section, and in the rotation with a constant angular velocity of a solid body of negligible compressibility.

First, let us consider case (i). Using the momentum density (2) with $p=0$ and $u_0=n_0m_0c^2$, where n_0 and m_0 are the local-rest-frame density and rest mass of the gas particles, respectively, we get the hidden momentum **P** as

$$\mathbf{P}=\frac{1}{c^2}\int \gamma^2 n_0 m_0 c^2 \mathbf{v}\, d^3r=\int \gamma m_0 n\mathbf{v}\, d^3r, \tag{13}$$

where $n=\gamma n_0$ is the laboratory-frame density of the particles. In a steady motion of the particles in a static external potential ϕ, the total mechanical energy $\mathscr{E}$ of each particle is the same conserved quantity

$$\mathscr{E}=\gamma m_0 c^2+q\phi=\text{const}, \tag{14}$$

where q is the ''charge'' of each particle. Thus, the momentum (13) can be expressed in the same way as in Eq. (9),

$$\mathbf{P}=\frac{1}{c^2}\int (\mathscr{E}-q\phi)n\mathbf{v}\, d^3r=-\frac{1}{c^2}\int \phi\mathbf{j}\, d^3r. \tag{15}$$

Here, the first term in the first integral yields zero as the particle current density $n\mathbf{v}$ is divergenceless in a stationary situation; $\mathbf{j}=qn\mathbf{v}$.

Now let us calculate the hidden momentum for case (ii). Assuming a stationary case, in which all partial time derivatives vanish, and projecting onto the direction of the velocity **v** of the body's element at a given point, the relativistic Euler's equation (6) simplifies to

$$\gamma^2\mathbf{v}\cdot\nabla p+\frac{\gamma}{c^2}(u_0+p)\mathbf{v}\cdot[(\mathbf{v}\cdot\nabla)(\gamma\mathbf{v})]=\mathbf{f}\cdot\mathbf{v}. \tag{16}$$

When the velocity **v** is constant, or the motion is circular with a constant angular velocity (i.e., $\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}$ with $\boldsymbol{\omega}$ constant), the second term on the left-hand side of Eq. (16) vanishes,[15] resulting in[16]

$$\gamma^2\mathbf{v}\cdot\nabla p=\mathbf{f}\cdot\mathbf{v}. \tag{17}$$

Integrating the momentum density (2) and using Eq. (17) with $\mathbf{f}=-\rho\nabla\phi$, the hidden momentum **P** is calculated, again in a full agreement with Eq. (9), as

$$\begin{aligned}\mathbf{P}&=\frac{1}{c^2}\int \gamma^2 u_0\mathbf{v}\, d^3r+\frac{1}{c^2}\int \gamma^2 p\mathbf{v}\, d^3r\\ &=-\frac{1}{c^2}\int \mathbf{r}(\gamma^2\mathbf{v}\cdot\nabla p)d^3r\\ &=\frac{1}{c^2}\int \mathbf{r}(\mathbf{j}\cdot\nabla\phi)d^3r\\ &=-\frac{1}{c^2}\int \phi\mathbf{j}\, d^3r.\end{aligned} \tag{18}$$

Here, the first integral vanishes as u_0 is constant and $\nabla\cdot(\gamma^2\mathbf{v})=0$ in the motion with a constant linear or angular velocity,[17] and the second and fourth integrals are transformed using the integral identity (10).

These calculations show that hidden momentum arises in case (i) from the relativistic variation of the mass of the gas particles with their speed,[18] while, in case (ii), it can be traced to the relativistic properties of the pressure in a body of low compressibility.[19]

IV. HIDDEN MOMENTUM AND THE FIELD MOMENTUM

An important aspect concerning hidden momentum is that it is compensated by an equal and opposite momentum of the external field combined with the field created by the current **j** in the system, so that the total linear momentum, i.e., the mechanical one plus that of the fields, vanishes in a finite system of a stationary distribution of matter and static fields.[20] This can be shown to follow immediately from the requirement of special relativity that the total-momentum density $\mathbf{g}_{\text{tot}}$ of a closed system, which is the sum of the mechanical part **g** and a field part, satisfies the continuity equation

$$\frac{\partial u_{\text{tot}}}{\partial t}+\nabla\cdot(c^2\mathbf{g}_{\text{tot}})=0, \tag{19}$$

which expresses the conservation of the total energy, with density u_{tot}, of the closed system. As the system is assumed finite, including the sources of the fields, the total-momentum density $\mathbf{g}_{\text{tot}}$ vanishes at least as $1/r^4$ at infinity. The total linear momentum $\mathbf{P}_{\text{tot}}$ of the system is then given as

$$\begin{aligned}P_{i\ \text{tot}}&=\int g_{i\ \text{tot}}\, d^3r\\ &=\int \nabla\cdot(x_i\mathbf{g}_{\text{tot}})d^3r-\int x_i\nabla\cdot\mathbf{g}_{\text{tot}}\, d^3r\\ &=0.\end{aligned} \tag{20}$$

Here, both the second and third integrals are zero: the second one because it can be transformed into the surface integral of an integrand that vanishes sufficiently fast at infinity, and the third one on the account of Eq. (19) and the stationary condition $\partial u_{\text{tot}}/\partial t=0$.

In the electromagnetic case, the result that the total linear momentum in a finite stationary system vanishes is verified immediately on transforming the expression (9) for hidden momentum into one in terms of the external electric field $\mathbf{E}=-\nabla\phi$ and the magnetic field **B** created by the current **j** (i.e., $\nabla\times\mathbf{B}=\mu_0\mathbf{j}$),

$$\begin{aligned}\mathbf{P}&=-\frac{1}{c^2}\int \phi\mathbf{j}\, d^3r\\ &=-\epsilon_0\int \phi\nabla\times\mathbf{B}\, d^3r\\ &=-\epsilon_0\int \nabla\times(\phi\mathbf{B})d^3r+\epsilon_0\int \nabla\phi\times\mathbf{B}\, d^3r\\ &=-\epsilon_0\int \mathbf{E}\times\mathbf{B}\, d^3r.\end{aligned} \tag{21}$$

Here, the third integral is zero as it can be transformed into a surface integral of a vanishing integrand; the last expression is the negative of the usual formula for the momentum of the electromagnetic field (**E**,**B**).

In the gravitational case, the requirement that the total linear momentum of a finite stationary system must vanish necessitates the existence of quasistatic ''gravinetic'' fields, which, in analogy to magnetostatic fields, are generated by quasistationary currents of mass and act on moving mass, and, when combined with ordinary, ''gravistatic'' fields, can contain linear momentum. Gravinetic effects are predicted by general relativity, but arguments based solely on special relativity only have been made for the existence of quasistatic gravinetic fields on several occasions.[21] Following Bedford and Krumm,[22] a gravinetic field $\mathbf{b}$ can be defined so that it is created by a quasistationary rest-mass current density $\mathbf{j}_m$ according to a gravitational analog of Ampère's law:

$$\nabla\times\mathbf{b}=-\frac{4\pi G}{c^2}\,\mathbf{j}_m\,, \tag{22}$$

where G is the gravitational constant; the gravinetic field $\mathbf{b}$ acts on a particle of rest mass m, moving with a velocity $\mathbf{v}$, with the force

$$\mathbf{F}=m\mathbf{v}\times\mathbf{b}. \tag{23}$$

Postulating that the quasistatic-gravitational-field momentum density $\mathbf{g}_g$ is given by[23]

$$\mathbf{g}_g=\frac{1}{4\pi G}\,\mathbf{b}\times\mathbf{e}, \tag{24}$$

where $\mathbf{e}=-\nabla\phi$ is the ordinary, gravistatic, gravitational field with a potential ϕ, the gravitational-case hidden momentum $-(1/c^2)\int\phi\mathbf{j}_m d^3r$ can be transformed into the negative of the gravitational-field momentum $(1/4\pi G)\int\mathbf{b}\times\mathbf{e}\,d^3r$ in exactly the same manner as it is done for the electromagnetic case in Eq. (21).

In conclusion, it is emphasized that the close analogy, exploited in this paper, between electromagnetic and gravitational fields holds only when the fields are quasistatic. Moreover, it is obvious that, in the framework of the general theory of relativity, the concept of the quasistatic gravinetic field $\mathbf{b}$ in a flat space-time can be introduced meaningfully only in a low-field approximation.

[1]W. Shockley and R. P. James, ''\`Try simplest cases' discovery of \`hidden momentum' forces on \`magnetic currents,' '' Phys. Rev. Lett. **18**, 876–879 (1967); H. A. Haus and P. Penfield, ''Force on a current loop,'' Phys. Lett. A **26**, 412–413 (1968).

[2]L. Vaidman, ''Torque and force on a magnetic dipole,'' Am. J. Phys. **58**, 978–983 (1990); V. Hnizdo, ''Comment on \`Torque and force on a magnetic dipole,' '' *ibid*. **60**, 279–280 (1992).

[3]V. Hnizdo, ''Hidden momentum and the electromagnetic mass of a charge and current carrying body,'' Am. J. Phys. **65**, 55–65 (1997).

[4]L. Vaidman, Ref. 2, discusses briefly the hidden momentum in gravitational systems.

[5]An ideal fluid has a negligible viscosity and thermal conductivity, and it cannot support any shear stresses.

[6]L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1987), 2nd ed., Sec. 133; *The Classical Theory of Fields* (Pergamon, Oxford, 1975), 4th ed., Sec. 35. The mechanical stress tensor σ_{ij}, as usually defined, equals the negative of the momentum flux density T^{ij}.

[7]The pressure in an ideal fluid is an invariant scalar, see C. Møller, *The Theory of Relativity* (Clarendon, Oxford, 1972), 2nd ed., pp. 191–192.

[8]A solid can support nonzero shear stresses, which may result in a local-rest-frame stress tensor $\sigma_{ij}\neq-p\,\delta_{ij}$, in a violation of Pascal's law (see the first footnote in Sec. 35 of L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Ref. 6).

[9]Strictly speaking, the force density $\mathbf{f}$ acting on a given element of the body should include the contributions from the ''body'' (as opposed to ''surface'') forces due to all the other elements of the body, but we shall assume that such contributions can be neglected in comparison with the external forces (see Ref. 12, however, for a case where such an assumption cannot be made).

[10]This can be proved separately for each component, using Eq. (7) with $g_i=ah_i$ and the fact that $\nabla\cdot(a\mathbf{h})=\mathbf{h}\cdot\nabla a$ for a divergenceless $\mathbf{h}$.

[11]Following the already accepted terminology, by ''hidden momentum'' is thus meant a nonzero mechanical momentum of a body the center of mass of which is at rest; however, in a recent paper of E. Comay, ''Exposing \`hidden momentum,' '' Am. J. Phys. **64**, 1028–1034 (1996), the term ''hidden momentum'' is used in a rather less restricted way.

[12]The potential ϕ appearing in Eq. (9) is then, strictly speaking, that of the total electric field acting on the body, including that which originates from the charge distribution within the body itself. When the body that carries the electric currents is a conductor, the external electric field induces a surface charge distribution on the conductor such that the net potential ϕ in the body is constant and thus, according to Eq. (9), the hidden momentum vanishes, as $\int\mathbf{j}d^3r=0$ for a divergenceless $\mathbf{j}$.

[13]With the quantities $\mathbf{j}=\rho\mathbf{v}$ and $\phi=-\mathbf{E}\cdot\mathbf{r}$ an electric current density and the potential of a uniform electric field $\mathbf{E}$, respectively, Eq. (9) can be rewritten as $c^2\mathbf{P}=-\int\mathbf{r}(\mathbf{j}\cdot\mathbf{E})d^3r=-\mathbf{E}\times\int\mathbf{r}\times\mathbf{j}d^3r-\int\mathbf{j}(\mathbf{E}\cdot\mathbf{r})d^3r=2\mathbf{m}\times\mathbf{E}+\int\phi\mathbf{j}d^3r=2\mathbf{m}\times\mathbf{E}-c^2\mathbf{P}$, where $\mathbf{m}=\frac{1}{2}\int\mathbf{r}\times\mathbf{j}d^3r$ is the magnetic dipole moment, resulting in $\mathbf{P}=(1/c^2)\mathbf{m}\times\mathbf{E}$.

[14]A strictly straight tube cannot confine the fluid to a finite region of space; the tube has to have bends that fashion it into a closed loop. One can assume, however, that the bends are negligible in size compared to the straight sections of the tube.

[15]That this is true for $\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}$ with $\boldsymbol{\omega}$ constant (''constant angular velocity'') is seen most easily by using cylindrical coordinates ρ,φ: $\mathbf{v}\cdot[(\mathbf{v}\cdot\nabla)(\gamma\mathbf{v})]=\mathbf{v}\cdot[v\,\partial(\gamma\mathbf{v})/\rho\partial\varphi]=0$, as putting the z axis along $\boldsymbol{\omega}$, $\mathbf{v}=(-v\sin\varphi,v\cos\varphi,0)$ with $v=|\mathbf{v}|=\omega\rho$.

[16]This equation can be obtained also by transforming Euler's equation from the local rest frame of a constant-velocity fluid to the laboratory frame, see, in the electromagnetic setting, V. Hnizdo, ''Hidden momentum of a relativistic fluid carrying current in an external electric field,'' Am. J. Phys. **65**, 92–94 (1997).

[17]For $\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}$ with $\boldsymbol{\omega}$ constant (''constant angular velocity''), $\nabla\cdot[f(v^2)\mathbf{v}]=f(v^2)\nabla\cdot\mathbf{v}$ as $\mathbf{v}\cdot\nabla v^2=0$, while the divergence of the velocity itself $\nabla\cdot\mathbf{v}=0$.

[18]When the relativistic mass γm_0 of a gas particle is replaced by the nonrelativistic constant mass m_0, the hidden momentum of Eq. (13) vanishes as the particle current density $n\mathbf{v}$ is divergenceless in the stationary situation.

[19]Vaidman, Ref. 2.

[20]S. Coleman and J. H. Van Vleck, ''Origin of \`hidden momentum forces' on magnets,'' Phys. Rev. **171**, 1370–1375 (1968); M. G. Calkin, ''Linear momentum of the source of a static electromagnetic field,'' Am. J. Phys. **39**, 513–516 (1971); Y. Aharanov, P. Pearle, and L. Vaidman, ''Comment on \`Proposed Aharanov–Casher effect: Another example of Aharanov-Bohm effect arising from a classical lag,' '' Phys. Rev. A **37**, 4052–4055 (1988); L. Vaidman, Ref. 2.

[21]P. Lorrain and D. R. Corson, *Electromagnetic Fields and Waves* (Freeman, San Francisco, 1970), 2nd ed., p. 251; D. Bedford and P. Krumm, ''On relativistic gravitation,'' Am. J. Phys. **53**, 889–890 (1985); P. Krumm and D. Bedford, ''The gravitational Poynting vector and energy transfer,'' Am. J. Phys. **55**, 362–363 (1987).

[22]D. Bedford and P. Krumm, Ref. 21.

[23]It should be noted that the formula of Eq. (24) cannot give the momentum density of gravitational waves, for which the quasistatic fields $\mathbf{e}$ and $\mathbf{b}$ have no meaning. Note also the reverse order, in comparison with the electromagnetic case, of the factors in the vector product $\mathbf{b}\times\mathbf{e}$—this is due to the negative sign in the ''Ampère's law'' of Eq. (22), which, in turn, comes about because the like gravitational ''charges'' attract (see D. Bedford and P. Krumm, Ref. 21). The quasistatic gravitational Poynting vector is then $c^2\mathbf{g}_g=(c^2/4\pi G)\mathbf{b}\times\mathbf{e}$, as proposed by P. Krumm and D. Bedford, Ref. 21.