Regularization of the second-order partial derivatives of the Coulomb potential of a point charge*

V Hnizdo

National Institute for Occupational Safety and Health, Morgantown, West Virginia 26505, USA

Abstract

The second-order partial derivatives of the Coulomb potential of a point charge can be regularized using the Coulomb potential of a charge of the oblate spheroidal shape that a moving rest-frame-spherical charge acquires by the Lorentz contraction. This 'physical' regularization is shown to be fully equivalent to the standard delta-function identity involving these derivatives.

Quantities with a singularity of the type $1/r^3$ at the origin $r = 0$ occur in classical electrodynamics in connection with the idealization of a point charge distribution. For example, the straightforward calculation of the second-order partial derivatives of the Coulomb potential $1/r$ of a unit point charge yields $\partial^2 r^{-1}/\partial x_i \partial x_j = (3x_i x_j - r^2\delta_{ij})/r^5$, or the field of a point electric or magnetic dipole, obtained as the straightforward gradient of a potential with radial dependence $1/r^2$, has the radial dependence $1/r^3$. Because of the $1/r^3$ singularity at the origin, the integral of such a quantity over any three-dimensional region that includes the origin $r = 0$ does not exist even in the improper-integral sense: the value of the integral obtained by excluding from the integration a region $\mathcal{V}_0$ around the origin and taking the limit of the size of $\mathcal{V}_0$ tending to zero depends on the shape and orientation of $\mathcal{V}_0$. Integrals involving derivatives of $1/r^2$ or second-order derivatives of $1/r$ therefore have to be suitably regularized. A formal way of doing that is to use the delta-function identity [1]

$$\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{r} = \frac{3x_i x_j - r^2\delta_{ij}}{r^5} - \frac{4\pi}{3}\delta_{ij}\delta(\boldsymbol{r}) \tag{1}$$

where δ_{ij} is the Kronecker delta symbol and $\delta(\boldsymbol{r}) = \delta(x_1)\delta(x_2)\delta(x_3)$ is the three-dimensional delta function. The validity of the identity (1) can be justified most easily by the use of the straightforward regularization $1/(r^2 + a^2)^{1/2}$ of the singular potential $1/r$:

$$\lim_{a\to 0}\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{\sqrt{r^2 + a^2}} = \lim_{a\to 0}\frac{3x_i x_j - r^2\delta_{ij}}{(r^2 + a^2)^{5/2}} - \lim_{a\to 0}\frac{a^2\delta_{ij}}{(r^2 + a^2)^{5/2}} \tag{2}$$

since here the second term on the right-hand side is a well-known representation of $-\frac{4}{3}\pi\delta_{ij}\delta(\boldsymbol{r})$ (e.g., see [2]). The first term on the right-hand side of the identity (1) is as such still non-integrable at the origin $r = 0$, but the regularization (2) also includes a specification of the regularization of this term; of course, the limits $a \to 0$ are understood to be

* This paper is written by V Hnizdo in his private capacity. No official support or endorsement by the Centers for Disease Control and Prevention is intended or should be inferred.

taken only after a three-dimensional integration with a well-behaved 'test' function. Regularizing the term $(3x_ix_j - r^2\delta_{ij})/r^5$ in a different, but equivalent, way, the identity (1) may be written as

$$\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{r} = \lim_{a\to 0+} \frac{3x_ix_j - r^2\delta_{ij}}{r^5}\,\Theta(r-a) - \frac{4\pi}{3}\delta_{ij}\delta(\boldsymbol{r}) \tag{3}$$

where $\Theta(\cdot)$ is the Heaviside step function. This formulation is equivalent to the stipulation that the spherical coordinates are to be used in the integration with a test function and that the angular integration is to be done first; the identity (1) has been derived in reference [1] effectively in the form (3).

In a recent paper on the Coulomb-gauge vector potential of a uniformly moving point charge [3], an occasion has arisen of using the delta-function identity (3) for the regularization of an integral of the type $\int \mathrm{d}^3r\, f(\boldsymbol{r})\partial^2 r^{-1}/\partial x_i^2$ in terms of which the difference between the Coulomb- and Lorenz-gauge vector potentials in that problem can be obtained as the solution to a Poisson equation. Since the relation (3) is an *identity*, there should be no doubt as to the correctness of such a formal regularization. However, in a problem that concerns a moving charge, it would be reassuring if one could show that a more 'physical' regularization procedure will yield the same results. Physically, it is natural to regularize the Coulomb potential $1/r$ of a moving point charge by the Coulomb potential of a charge that has the oblate spheroidal shape that a moving rest-frame-spherical 'elementary' charge of finite extension a acquires by the Lorentz contraction, and then to take the limit $a \to 0$ of any integral involving second-order derivatives of this potential. Such regularization involves an 'ellipsoidal' approach toward the singularity at the origin because of the spheroidal shape of the moving elementary charge—in contrast to the first term on the right-hand side of identity (3) that stipulates a strictly 'spherical' approach toward this singularity. In this note, we demonstrate that a physical regularization along the above lines is indeed fully equivalent to the delta-function identity (3). We believe that the reward for carrying out the calculation that this demonstration requires will be a physical insight into and ensuing confidence in use of a useful formal relation.

As a preliminary, we note that a 'physical' justification of the well-known delta-function identity $\nabla^2(1/r) = -4\pi\delta(\boldsymbol{r})$ can be provided very simply. Let $\varphi_a(\boldsymbol{r})$ be the Coulomb potential of a unit elementary charge described by the density $\rho_a(\boldsymbol{r}) = (1/V_a)\Theta(a^2 - \gamma^2x_1^2 - x_2^2 - x_3^2)$, where $V_a = \frac{4}{3}\pi a^3/\gamma$, which is the density of a uniformly charged spheroid with semiaxes a/γ, a, a, centred at the origin. For $\gamma = (1 - v^2/c^2)^{-1/2}$, this is the density of a charge that is moving with a velocity v (c is the speed of light) along the x_1-axis and that is a uniform ball of radius a in its rest frame. Then, for any well-behaved test function $f(\boldsymbol{r})$, we have by the fact that the potential $\varphi_a(\boldsymbol{r})$ satisfies the Poisson equation $\nabla^2\varphi_a(\boldsymbol{r}) = -4\pi\rho_a(\boldsymbol{r})$:

$$\int \mathrm{d}^3r\, f(\boldsymbol{r})\nabla^2\varphi_a(\boldsymbol{r}) = -4\pi \int \mathrm{d}^3r\, f(\boldsymbol{r})\rho_a(\boldsymbol{r}) = -\frac{4\pi}{V_a}\int_{\mathcal{V}_a} \mathrm{d}^3r\, f(\boldsymbol{r}) = -\frac{4\pi}{V_a}V_a f(\boldsymbol{r}_0) \tag{4}$$

where the mean-value theorem is used on the right-hand side, with $\boldsymbol{r}_0$ being a point inside the region $\mathcal{V}_a$ occupied by the spheroid. Taking now the limit $a \to 0$, we obtain

$$\lim_{a\to 0}\int \mathrm{d}^3r\, f(\boldsymbol{r})\nabla^2\varphi_a(\boldsymbol{r}) = -4\pi \lim_{a\to 0} f(\boldsymbol{r}_0) = -4\pi f(0) = -4\pi \int \mathrm{d}^3r f(\boldsymbol{r})\delta(\boldsymbol{r}) \tag{5}$$

because the point $\boldsymbol{r}_0 \in \mathcal{V}_a$ has to converge on the origin $\boldsymbol{r} = 0$ as $a \to 0$, and thus we can write

$$\lim_{a\to 0}\nabla^2\varphi_a(\boldsymbol{r}) = -4\pi\delta(\boldsymbol{r}). \tag{6}$$

A uniformly charged ball was not the most popular model of an elementary charge employed in the classical electron theory—this was a uniformly charged spherical shell, or, equivalently, a charged spherical conductor (see, e.g., [4]). The Coulomb potential of a uniformly moving charged conductor that is spherical with radius a in its rest frame is the most convenient one to use for our purpose because it equals the Coulomb potential of a charged conducting oblate spheroid of semiaxes a/γ, a, a (see [5]; an interesting historical background to the problem of a moving charged sphere can be found in [6]), and this potential can be expressed in terms of an elementary function [7]:

$$\varphi_a(\boldsymbol{r}) = \begin{cases} (1/\beta a)\arctan[\beta a/\sqrt{\frac{1}{4}(r_+ + r_-)^2 - (\beta a)^2}] & \text{for } \gamma^2 x_1^2 + \rho^2 \geq a^2 \\ (1/\beta a)\arctan(\beta\gamma) & \text{for } \gamma^2 x_1^2 + \rho^2 < a^2 \end{cases} \tag{7}$$

where

$$r_\pm = \sqrt{x_1^2 + (\rho \pm \beta a)^2} \qquad \rho = \sqrt{x_2^2 + x_3^2} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c}. \tag{8}$$

The partial derivatives $\partial\varphi_a(\boldsymbol{r})/\partial x_i$ of the potential $\varphi_a(\boldsymbol{r})$ for $\gamma^2 x_1^2 + \rho^2 \geq a^2$ are

$$\left.\frac{\partial\varphi_a(\boldsymbol{r})}{\partial x_1}\right|_{\gamma^2 x_1^2+\rho^2\geq a^2} = -\frac{x_1}{r_+ r_- \sqrt{\frac{1}{4}(r_+ + r_-)^2 - (\beta a)^2}} \tag{9}$$

$$\left.\frac{\partial\varphi_a(\boldsymbol{r})}{\partial x_{2,3}}\right|_{\gamma^2 x_1^2+\rho^2\geq a^2} = -\frac{(r_- - r_+)\beta a + (r_+ + r_-)\rho}{r_+ r_- \sqrt{\frac{1}{4}(r_+ + r_-)^2 - (\beta a)^2}}\,\frac{x_{2,3}}{(r_+ + r_-)\rho}. \tag{10}$$

Since the potential $\varphi_a(\boldsymbol{r})$ is constant inside the spheroid $\gamma^2 x_1^2 + \rho^2 = a^2$, the partial derivatives $\partial\varphi_a(\boldsymbol{r})/\partial x_i$ can be written as $\Theta[(\gamma^2 x_1^2+\rho^2)^{1/2} - a]\partial\varphi_a(\boldsymbol{r})/\partial x_i$, and thus the second-order partial derivatives $\partial^2\varphi_a(\boldsymbol{r})/\partial x_i\partial x_j$ can be written as

$$\begin{aligned} \frac{\partial^2\varphi_a(\boldsymbol{r})}{\partial x_i\partial x_j} &= \frac{\partial^2\varphi_a(\boldsymbol{r})}{\partial x_i\partial x_j}\,\Theta(\sqrt{\gamma^2 x_1^2+\rho^2}-a) + \frac{\partial\varphi_a(\boldsymbol{r})}{\partial x_i}\frac{\partial}{\partial x_j}\Theta(\sqrt{\gamma^2 x_1^2+\rho^2}-a) \\ &= \frac{\partial^2\varphi_a(\boldsymbol{r})}{\partial x_i\partial x_j}\,\Theta(\sqrt{\gamma^2 x_1^2+\rho^2}-a) + \frac{\partial\varphi_a(\boldsymbol{r})}{\partial x_i}\,\frac{[1+(\gamma^2-1)\delta_{1j}]x_j}{\sqrt{\gamma^2 x_1^2+\rho^2}}\,\delta(\sqrt{\gamma^2 x_1^2+\rho^2}-a) \end{aligned} \tag{11}$$

where the derivatives of the potential on the right-hand side are understood as those of the expression for the potential exterior to the spheroid.

The equivalence of the delta-function identity (3) and the regularization that uses the Coulomb potential $\varphi_a(\boldsymbol{r})$ of a charged conducting spheroid demands that, for any well-behaved test function $f(\boldsymbol{r})$,

$$\lim_{a\to 0}\int \mathrm{d}^3 r\, f(\boldsymbol{r})\frac{\partial^2\varphi_a(\boldsymbol{r})}{\partial x_i\partial x_j} = \lim_{a\to 0+}\int_{r>a}\mathrm{d}^3 r\, f(\boldsymbol{r})\frac{3x_i x_j - r^2\delta_{ij}}{r^5} - \frac{4\pi}{3}\delta_{ij} f(0). \tag{12}$$

Using (11), we write the left-hand side of (12) as

$$\begin{aligned} \lim_{a\to 0}\int \mathrm{d}^3 r\, f(\boldsymbol{r})\frac{\partial^2\varphi_a(\boldsymbol{r})}{\partial x_i\partial x_j} &= \lim_{a\to 0}\int_{\gamma^2 x_1^2+\rho^2>a^2}\mathrm{d}^3 r\, f(\boldsymbol{r})\,\frac{\partial^2\varphi_a(\boldsymbol{r})}{\partial x_i\partial x_j} \\ &\quad + [1+(\gamma^2-1)\delta_{1j}]\lim_{a\to 0}\int \mathrm{d}^3 r\, f(\boldsymbol{r})\,\frac{\partial\varphi_a(\boldsymbol{r})}{\partial x_i}\,\frac{x_j\delta(\sqrt{\gamma^2 x_1^2+\rho^2}-a)}{\sqrt{\gamma^2 x_1^2+\rho^2}}. \end{aligned} \tag{13}$$

Using the expressions (9) and (10) for the derivatives $\partial\varphi_a(\boldsymbol{r})/\partial x_i$, the second term on the right-hand side of (13) can be evaluated in closed form. Let us first assume that i=j=1. Transforming as $\gamma x_1 \to x_1$ and then to the spherical coordinates r, θ, ϕ, with x_1 as the polar axis and $\cos\theta = \xi$, we obtain

$$\begin{aligned}
&\gamma^2 \lim_{a\to 0} \int \mathrm{d}^3 r\, f(\boldsymbol{r})\, \frac{\partial\varphi_a(\boldsymbol{r})}{\partial x_1}\, \frac{x_1 \delta(\sqrt{\gamma^2 x_1^2 + \rho^2} - a)}{\sqrt{\gamma^2 x_1^2 + \rho^2}} \\
&= \lim_{a\to 0} \int \mathrm{d}^3 r\, f(x_1/\gamma, x_2, x_3)\, \frac{\partial\varphi_a(\boldsymbol{r})}{\partial x_1}\bigg|_{x_1\to x_1/\gamma} \frac{x_1 \delta(r-a)}{r} \\
&= -\lim_{a\to 0} \int_{-1}^{1} \mathrm{d}\xi \int_0^{2\pi} \mathrm{d}\phi\, f(x_1/\gamma, x_2, x_3)|_{r=a}\, \frac{\xi^2}{1-\beta^2(1-\xi^2)} \\
&= -2\pi f(0) \int_{-1}^{1} \frac{\mathrm{d}\xi\, \xi^2}{1-\beta^2(1-\xi^2)} = -2\pi \left(\frac{2}{\beta^2} - \frac{2 \arcsin\beta}{\gamma\beta^3} \right) f(0) \qquad (14)
\end{aligned}$$

where $\partial\varphi_a(\boldsymbol{r})/\partial x_1|_{x_1\to x_1/\gamma}$ denotes the exterior partial derivative (9) after the transformation $\gamma x_1 \to x_1$. Here, the delta function $\delta(r-a)$ led to an immediate radial integration, which enabled a considerable simplification of the integrand; the limit $a \to 0$ then could be taken inside the remaining integral, yielding

$$\begin{aligned}
\lim_{a\to 0} f(x_1/\gamma, x_2, x_3)|_{r=a} &= \lim_{a\to 0} f(a\cos\theta/\gamma, a\cos\phi\sin\theta, a\sin\phi\sin\theta) \\
&= f(0,0,0) \equiv f(0). \qquad (15)
\end{aligned}$$

A similar calculation for i=j=2 yields

$$\begin{aligned}
&\lim_{a\to 0} \int \mathrm{d}^3 r\, f(\boldsymbol{r})\, \frac{\partial\varphi_a(\boldsymbol{r})}{\partial x_2}\, \frac{x_2 \delta(\sqrt{\gamma^2 x_1^2 + \rho^2} - a)}{\sqrt{\gamma^2 x_1^2 + \rho^2}} \\
&= \lim_{a\to 0} \int \frac{\mathrm{d}^3 r}{\gamma}\, f(x_1/\gamma, x_2, x_3) \frac{\partial\varphi_a(\boldsymbol{r})}{\partial x_2}\bigg|_{x_1\to x_1/\gamma} \frac{x_2 \delta(r-a)}{r} \\
&= -\frac{f(0)}{\gamma^2} \int_{-1}^{1} \mathrm{d}\xi \int_0^{2\pi} \mathrm{d}\phi\, \frac{(1-\xi^2)\cos^2\phi}{1-\beta^2(1-\xi^2)} \\
&= -\frac{\pi f(0)}{\gamma^2} \int_{-1}^{1} \mathrm{d}\xi\, \frac{1-\xi^2}{1-\beta^2(1-\xi^2)} = -2\pi \left(1 - \frac{1}{\beta^2} + \frac{\arcsin\beta}{\gamma\beta^3} \right) f(0). \qquad (16)
\end{aligned}$$

Here, similarly as in the integration in (14), even with the relatively complicated expression (10) for the exterior partial derivative $\partial\varphi_a(\boldsymbol{r})/\partial x_2$, the integrand could be simplified considerably after the radial integration. The case i=j=3 will obviously yield the same result, while for any mixed case $i \neq j$, the integration with respect to the azimuthal angle ϕ will lead to a vanishing result. Collecting these results, we have

$$[1+(\gamma^2-1)\delta_{1j}] \lim_{a\to 0} \int \mathrm{d}^3 r\, f(\boldsymbol{r})\, \frac{\partial\varphi_a(\boldsymbol{r})}{\partial x_i}\, \frac{x_j \delta(\sqrt{\gamma^2 x_1^2+\rho^2}-a)}{\sqrt{\gamma^2 x_1^2 + \rho^2}} = -2\pi g_{ij}(\beta) f(0) \qquad (17)$$

where

$$g_{ij}(\beta) = \begin{cases} 2/\beta^2 - (2/\gamma\beta^3)\arcsin\beta & \text{for } i{=}j{=}1 \\ 1 - 1/\beta^2 + (1/\gamma\beta^3)\arcsin\beta & \text{for } i{=}j{=}2,3 \\ 0 & \text{for } i \neq j. \end{cases} \qquad (18)$$

We note that $\lim_{\beta\to 0} g_{ij}(\beta) = \frac{2}{3}\delta_{ij}$. Using (13) and (17), the condition (12) of regularization equivalence can now be written as

$$\lim_{a\to 0}\int_{\gamma^2 x_1^2+\rho^2>a^2} \mathrm{d}^3 r\, f(\boldsymbol{r})\,\frac{\partial^2\varphi_a(\boldsymbol{r})}{\partial x_i\partial x_j} - \lim_{a\to 0+}\int_{r>a} \mathrm{d}^3 r\, f(\boldsymbol{r})\frac{3x_i x_j - r^2\delta_{ij}}{r^5}$$
$$= 2\pi[g_{ij}(\beta) - \tfrac{2}{3}\delta_{ij}]f(0). \tag{19}$$

To prove that the condition (19) holds true, we proceed as follows. The first limit on the left-hand side of (19) can be written more simply as

$$\lim_{a\to 0}\int_{\gamma^2 x_1^2+\rho^2>a^2} \mathrm{d}^3 r\, f(\boldsymbol{r})\,\frac{\partial^2\varphi_a(\boldsymbol{r})}{\partial x_i\partial x_j} = \lim_{a\to 0}\int_{\gamma^2 x_1^2+\rho^2>a^2} \mathrm{d}^3 r\, f(\boldsymbol{r})\frac{3x_i x_j - r^2\delta_{ij}}{r^5} \tag{20}$$

since the derivatives $\partial^2\varphi_a(\boldsymbol{r})/\partial x_i\partial x_j$ for $\gamma^2 x_1^2 + \rho^2 > a^2$ can be expanded in powers of $(\beta a)^2$ (such an expansion is obtained most easily by differentiating term by term the corresponding expansion of φ_a), where the zeroth-order term equals $(3x_i x_j - r^2\delta_{ij})/r^5$ and the integrals involving the higher-order terms vanish in the limit $a\to 0$. It suffices to show this only for the case i=j=1. Here we have

$$\frac{\partial^2\varphi_a(\boldsymbol{r})}{\partial x_1^2}\Big|_{\gamma^2 x_1^2+\rho^2>a^2} = \sum_{n=0}^{\infty}(-1)^n(2n+2)P_{2n+2}(x_1/r)\frac{(\beta a)^{2n}}{r^{2n+3}}, \tag{21}$$

where $P_m(\cdot)$ are the Legendre polynomials. Using (21) and the multipole expansion of the function $f(\boldsymbol{r})$ after the transformation $x_1 \to x_1/\gamma$,

$$f(r\cos\theta/\gamma, r\cos\phi\sin\theta, r\sin\phi\sin\theta) = \sum_{lm} f_{lm}(r,\gamma)Y_{lm}(\theta,\phi) \tag{22}$$

we obtain

$$\int_{\gamma^2 x_1^2+\rho^2>a^2} \mathrm{d}^3 r\, f(\boldsymbol{r})\frac{\partial^2\varphi_a(\boldsymbol{r})}{\partial x_1^2} = \sum_{n=0}^{\infty} c_n(a,\gamma)(\beta a)^{2n} \tag{23}$$

where

$$c_n(a,\gamma) = \sum_{lm} C_{nlm}(\gamma)\int_a^{\infty}\mathrm{d}r\,\frac{f_{lm}(r,\gamma)}{r^{2n+1}} \tag{24}$$

$$C_{nlm}(\gamma) = (-1)^n(2n+2)\int \mathrm{d}\Omega\, Y_{lm}(\theta,\phi)\frac{P_{2n+2}(\cos\theta/\gamma u)}{\gamma u^{2n+3}} \qquad u = \sqrt{1-\beta^2\cos^2\theta}. \tag{25}$$

We note that $C_{n00}(\gamma) = 0$ for any $n \geq 0$. As $\lim_{r\to 0} f_{lm}(r,\gamma) = 0$ for any $l > 0$, we have that $\lim_{a\to 0}[a^{2n}\int_a^{\infty}\mathrm{d}r\, f_{lm}(r,\gamma)/r^{2n+1}] = 0$ when both $n > 0$ and $l > 0$, and thus $\lim_{a\to 0}[c_n(a,\gamma)(\beta a)^{2n}] = 0$ for any $n > 0$. Therefore, we indeed obtain

$$\lim_{a\to 0}\int_{\gamma^2 x_1^2+\rho^2>a^2} \mathrm{d}^3 r\, f(\boldsymbol{r})\frac{\partial^2\varphi_a(\boldsymbol{r})}{\partial x_1^2} = \lim_{a\to 0} c_0(a,\gamma) = \lim_{a\to 0}\int_{\gamma^2 x_1^2+\rho^2>a^2} \mathrm{d}^3 r\, f(\boldsymbol{r})\frac{3x_1^2 - r^2}{r^5}. \tag{26}$$

Using (20), the condition (19) can be expressed as

$$\lim_{a\to 0}\int_{\mathcal{U}_a} \mathrm{d}^3 r\, f(\boldsymbol{r})\frac{3x_i x_j - r^2\delta_{ij}}{r^5} = 2\pi[g_{ij}(\beta) - \tfrac{2}{3}\delta_{ij}]f(0) \tag{27}$$

where the integration region $\mathcal{U}_a$ is the region between the surfaces of the oblate spheroid $\gamma^2 x_1^2 + \rho^2 = a^2$ and the sphere $x_1^2 + \rho^2 = a^2$:

$$\mathcal{U}_a = \{(x_1, x_2, x_3);\ \gamma^2 x_1^2 + x_2^2 + x_3^2 > a^2,\ x_1^2 + x_2^2 + x_3^2 < a^2\}. \tag{28}$$

We now evaluate the left-hand side of (27). When the size parameter a tends to zero, the integration region $\mathcal{U}_a$ gets progressively smaller and closer to the origin $r = 0$, and thus, for $\boldsymbol{r} \in \mathcal{U}_a$, $f(\boldsymbol{r}) \to f(0)$ as $a \to 0$. We can therefore write the left-hand side of (27) as

$$\lim_{a\to 0} \int_{\mathcal{U}_a} \mathrm{d}^3 r\, f(\boldsymbol{r}) \frac{3x_i x_j - r^2\delta_{ij}}{r^5} = f(0) \lim_{a\to 0} \int_{\mathcal{U}_a} \mathrm{d}^3 r\, \frac{3x_i x_j - r^2\delta_{ij}}{r^5}. \tag{29}$$

Transforming here the integral on the right-hand side to the spherical coordinates, with x_1 as the polar axis and $\cos\theta = \xi$, we obtain for i=j=1:

$$\begin{aligned}
\int_{\mathcal{U}_a} \mathrm{d}^3 r\, \frac{3x_1^2 - r^2}{r^5} &= 2\pi \int_{-1}^{1} \mathrm{d}\xi\, (3\xi^2 - 1) \int_{a/\sqrt{1+(\gamma^2-1)\xi^2}}^{a} \frac{\mathrm{d}r}{r} \\
&= \pi \int_{-1}^{1} \mathrm{d}\xi\, (3\xi^2 - 1) \ln[1 + (\gamma^2 - 1)\xi^2] \\
&= 2\pi \left(\frac{2}{\beta^2} - \frac{2 \arcsin\beta}{\gamma\beta^3} - \frac{2}{3} \right).
\end{aligned} \tag{30}$$

The case i=j=2 gives

$$\begin{aligned}
\int_{\mathcal{U}_a} \mathrm{d}^3 r\, \frac{3x_2^2 - r^2}{r^5} &= \frac{1}{2} \int_{-1}^{1} \mathrm{d}\xi \int_{0}^{2\pi} \mathrm{d}\phi\, [3(1-\xi^2)\cos^2\phi - 1] \ln[1 + (\gamma^2 - 1)\xi^2] \\
&= \frac{\pi}{2} \int_{-1}^{1} \mathrm{d}\xi\, (1 - 3\xi^2) \ln[1 + (\gamma^2 - 1)\xi^2] \\
&= 2\pi \left(\frac{1}{3} - \frac{1}{\beta^2} + \frac{\arcsin\beta}{\gamma\beta^3} \right)
\end{aligned} \tag{31}$$

and the same result will obviously be obtained for i=j=3. The mixed cases $i \neq j$ will all yield zero on account of the integration with respect to ϕ. The values of the integrals (30) and (31) are independent of a, and using these results and (29), we obtain (27). This completes the proof of the regularization equivalence (12).

In closing, we would like to stress that the regularization equivalence (12) is bound to hold also when the potential $\varphi_a(\boldsymbol{r})$ is the Coulomb potential of a uniformly charged spheroid, or of any other Lorentz-contracted charge distribution that is, in its rest frame, spherically symmetric and characterized by a finite size parameter a. (Explicit expressions for the potential of a uniformly charged spheroid can be found in the literature [8, 9, 10], but they are rather more complicated than the conducting-spheroid expression (7).) In fact, using the powerful results of the theory of generalized functions and derivatives, one can very easily give a formal proof that (12) holds for the Coulomb potential $\varphi_a(\boldsymbol{r})$ of a charge distribution $\rho_a(\boldsymbol{r})$ of any shape, subject only to the condition $\lim_{a\to 0} \rho_a(\boldsymbol{r}) = \delta(\boldsymbol{r})$; this proof is given in Appendix.

Appendix

Lemma. Let

$$\varphi_a(\boldsymbol{r}) = \int \mathrm{d}^3 r' \, \frac{\rho_a(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}, \tag{32}$$

where the integration extends over all space and $\rho_a(\boldsymbol{r})$ is a localized (not necessarily spherically symmetric) function of $\boldsymbol{r} = (x_1, x_2, x_3)$ that depends on a parameter a so that

$$\operatorname*{wlim}_{a \to 0} \rho_a(\boldsymbol{r}) = \delta(\boldsymbol{r}). \tag{33}$$

Then

$$\operatorname*{wlim}_{a \to 0} \frac{\partial^2 \varphi_a(\boldsymbol{r})}{\partial x_i \partial x_j} = -\frac{4\pi}{3} \delta_{ij} \delta(\boldsymbol{r}) + \mathrm{w} \lim_{\varepsilon \to 0+} \frac{3 x_i x_j - r^2 \delta_{ij}}{r^5} \, \Theta(r - \varepsilon), \tag{34}$$

where $r = |\boldsymbol{r}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ and $\Theta(\cdot)$ is the Heaviside step function. The symbol wlim denotes the weak limit: $\mathrm{wlim}_{a \to a_0} f_a(\boldsymbol{r}) = g(\boldsymbol{r})$ iff $\lim_{a \to a_0} \int \mathrm{d}^3 r \, t(\boldsymbol{r}) f_a(\boldsymbol{r}) = \int \mathrm{d}^3 r \, t(\boldsymbol{r}) g(\boldsymbol{r})$ for any 'well-behaved' test function $t(\boldsymbol{r})$.
Proof. To prove (34), we need to show that, for any 'well-behaved' test function $f(\boldsymbol{r})$,

$$\lim_{a \to 0} \int \mathrm{d}^3 r \, f(\boldsymbol{r}) \frac{\partial^2 \varphi_a(\boldsymbol{r})}{\partial x_i \partial x_j} = -\frac{4\pi}{3} f(0) \delta_{ij} + \lim_{\varepsilon \to 0+} \int_{r > \varepsilon} \mathrm{d}^3 r \, f(\boldsymbol{r}) \, \frac{3 x_i x_j - r^2 \delta_{ij}}{r^5}. \tag{35}$$

To evaluate the left-hand side of (35), we note that we can replace the derivative $\partial^2/\partial x_i \partial x_j$ by the generalized (distributional) derivative $\bar{\partial}^2/\partial x_i \partial x_j$. This allows us to to exchange the order of the limit $a \to 0$ and the differentiation [11, p 12] since the space of the generalized functions is complete [12]:

$$\begin{aligned}
\lim_{a \to 0} \int \mathrm{d}^3 r \, f(\boldsymbol{r}) \frac{\partial^2 \varphi_a(\boldsymbol{r})}{\partial x_i \partial x_j} &= \lim_{a \to 0} \int \mathrm{d}^3 r \, f(\boldsymbol{r}) \frac{\bar{\partial}^2 \varphi_a(\boldsymbol{r})}{\partial x_i \partial x_j} \\
&= \int \mathrm{d}^3 r \, f(\boldsymbol{r}) \frac{\bar{\partial}^2}{\partial x_i \partial x_j} \lim_{a \to 0} \varphi_a(\boldsymbol{r}) \\
&= \int \mathrm{d}^3 r \, f(\boldsymbol{r}) \frac{\bar{\partial}^2}{\partial x_i \partial x_j} \frac{1}{r}.
\end{aligned} \tag{36}$$

Here, the 3rd line was obtained using (32) and (33). But

$$\frac{\bar{\partial}^2}{\partial x_i \partial x_j} \frac{1}{r} = -\frac{4\pi}{3} \delta_{ij} \delta(\boldsymbol{r}) + \mathrm{w} \lim_{\varepsilon \to 0+} \frac{3 x_i x_j - r^2 \delta_{ij}}{r^5} \, \Theta(r - \varepsilon) \tag{37}$$

(see [11], p 28, but note that the signs of the right-hand sides of (3.129) and (3.130) are there misprinted; see also [13,14]), and thus

$$\int \mathrm{d}^3 r \, f(\boldsymbol{r}) \frac{\bar{\partial}^2}{\partial x_i \partial x_j} \frac{1}{r} = -\frac{4\pi}{3} f(0) \delta_{ij} + \lim_{\varepsilon \to 0+} \int_{r > \varepsilon} \mathrm{d}^3 r \, f(\boldsymbol{r}) \, \frac{3 x_i x_j - r^2 \delta_{ij}}{r^5}. \tag{38}$$

Using (38) in (36) results in (35). *QED*

The author is grateful to F Farassat for a useful discussion on the matter of this appendix.

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