Thus, in view of (1), if $[A]_\alpha$ denotes the matrix representation of A with respect to the basis α, then $[A]_\alpha = TD$, where T is the upper triangular matrix with $T_{ij} = \binom{j}{i}$ for $i \leqq j$ and D is the diagonal matrix with $D_{jj} = \lambda_j$. Since $T_{jj} = 1$, the eigenvalues of $[A]_\alpha$, and hence of A, indeed are equal to $\lambda_j = \binom{2m+1}{m-j}$. To prove that $Af_p(x) = \lambda_p f_p(x)$, it suffices to show that

$$\sum_{j=q}^{p} \binom{j}{q} \lambda_j C(p,j) = \lambda_p C(p,q),$$

or, in view of (2),

$$\sum_{j=q}^{p} (-1)^{j-q} \frac{(p-q)!(p+j)!(m-p)!}{(p-j)!(j-q)!(m+j+1)!} = \frac{(m-q)!(p+q)!}{(m+p+1)!}.$$

This is easily proved using the beta integral. Thus

$$\begin{aligned}\sum_{j=q}^{p} (-1)^{j-q} \frac{(p-q)!(p+j)!(m-p)!}{(p-j)!(j-q)!(m+j+1)!} &= \sum_{j=q}^{p} (-1)^{j-q} \binom{p-q}{j-q} \int_0^1 t^{p+j}(1-t)^{m-p}\, dt \\ &= \int_0^1 t^{p+q}(1-t)^{m-q}\, dt \\ &= \frac{(m-q)!(p+q)!}{(m+p+1)!},\end{aligned}$$

as required.

REFERENCES

[1] M. ABRAMOWITZ AND I. STEGUN, EDS., *Handbook of Mathematical Functions*, Dover, New York, 1965.
[2] W. FELLER, *An Introduction to Probability Theory and Its Applications*, John Wiley, New York, 1967.
[3] F. B. HILDEBRAND, *Introduction to Numerical Analysis*, McGraw-Hill, New York, 1974.

Radiation from a System of Uniformly Circling Charges

Problem 89-5, *by* V. HNIZDO (University of the Witwatersrand, Johannesburg, South Africa).

Recently it has been shown by numerical calculations [1] that the power radiated by a classical system of two particles of equal (like or unlike) charge and mass, which orbit uniformly and diametrically opposite each other on a circle, equals the rate at which work is done on the particles against their retarded electromagnetic interactions and the Lorentz–Dirac radiation reaction forces. Thus, it has been demonstrated *numerically* that, contrary to a previous claim [2], the Lorentz–Dirac equation and the retarded Lienard–Wiechert potentials of classical electrodynamics satisfy energy conservation in such systems.

The total power P radiated by such a system can be calculated using Fourier series methods, and is given exactly [3] by

$$(1)\quad P = 4\omega^4 \sum_{\substack{n=1,3,5,\cdots \\ (n=2,4,6,\cdots)}} n^2 \int_0^\pi \left[\left(\frac{dJ_n(n\omega \sin\theta)}{d(n\omega\sin\theta)} \right)^2 + n^2 \cos^2\theta \left(\frac{J_n(n\omega\sin\theta)}{n\omega\sin\theta} \right)^2 \right] \sin\theta\, d\theta,$$

where $J_n(x)$ are the Bessel functions and

$$\omega = \frac{\phi}{\cos\phi}$$

is the angular frequency of the orbital motion, with ϕ being the angle subtended at the position of each particle by the retarded and current positions of the other particle (units such that the speed of light, the radius of the circle, and the magnitudes of each charge are all unity are used). The sum in (1) runs over only n odd or only n even, depending on whether the system is that of unlike or like charges, respectively. On the other hand, the rate W of work done against the Lorentz–Dirac radiation reaction forces and the retarded Lienard–Wiechert fields is given by

$$W=\frac{4\omega^4}{3(1-\omega^2)^2}\pm\frac{\omega[(\omega\cos 2\phi-\sin\phi)(1-\omega^2)+2\phi^2(\sin\phi+\omega)]}{2(1+\omega\sin\phi)^3\cos^2\phi},$$

where the plus and minus signs are for the systems of unlike and like charges, respectively. Prove for the systems of both unlike and like charges that

$$P=W \tag{2}$$

for real ω such that $|\omega|<1$. For the system of unlike charges, (2) has been shown to hold up to order ω^6 in [1], and up to order ω^{10} in [4], by expanding P and W in powers of ω.

Once (2) is proved, the following simpler equality results immediately.

$$\omega^4\sum_{n=1}^{\infty}\int_0^{\pi}\left[\left(\frac{dJ_n(n\omega\sin\theta)}{d(n\omega\sin\theta)}\right)^2+n2\cos^2\theta\left(\frac{J_n(n\omega\sin\theta)}{n\omega\sin\theta}\right)^2\right]\sin\theta\,d\theta=\frac{2\omega^4}{3(1-\omega^2)^2},$$

where the right-hand side gives the well-known radiation rate of a single unit charge circling uniformly with an angular frequency ω along a unit radius circle.

REFERENCES

[1] V. Hnizdo, *The Lorentz-Dirac equation, Lienard-Wiechert potentials, and radiation by a system of uniformly circling charges*, Phys. Lett. A., 129 (1988), pp. 426–428.

[2] E. Comay, *A test of Lorentz-Dirac and Lienard-Wiechert equations*, Phys. Lett. A, 126 (1987), pp. 155–158; *Erratum*, Phys. Lett. A., 129 (1988), pp. 424–425.

[3] J. D. Jackson, *Classical Electrodynamics*, John Wiley, New York, 1975, pp. 695–697.

[4] K. Briggs, *On the radiation from a rotating dipole*, preprint ADP-88-75-CP7, University of Adelaide, Adelaide, Australia; Australian J. Phys., 41 (1988), p. 629.

Solution by W. B. Jordan (Scotia, NY).

Let $g(z)=J_n^2(z)$. Then $g'(z)=2J_n(z)J_n'(z)$ and $g''(z)=2(J_n'(z))^2+2J_n(z)J_n''(z)$. Using Bessel's differential equation to eliminate $J_n''(z)$,

$$2(J_n'(z))^2=g''(z)+\frac{g'(z)}{z}+2\left(1-\frac{n^2}{z^2}\right)g(z),$$

and so

$$P_{O,E}=2\omega^4\int_0^{\pi}\sum_{O,E}\left\{\frac{d^2g(nz)}{dz^2}+\frac{1}{z}\frac{dg(nz)}{dz}+2\left(1-\frac{1}{\omega^2}\right)n^2g(nz)\right\}_{z=\omega\sin\theta}\sin\theta\,d\theta,$$

the summation being over only odd (O) or only even (E) values of n. Let P_F denote the expression above with the full sum (over all positive integers) and let P_A denote the corresponding alternating sum in whch the nth term has $(-1)^{n-1}$ as a factor. Then $P_O=(P_F+P_A)/2$ and $P_E=(P_F-P_A)/2$.

To compute P_F, we sum g and n^2g following the Kepler–Bessel analysis of elliptic orbits as developed in Chapter 17 of [1]. With $M = E - z\sin E$ and $r/a = 1 - z\cos E$, the expansion

$$\frac{a}{r} = 1 + 2\sum_{n=1}^{\infty} J_n(nz)\cos(nM)$$

holds. Putting $M = E = 0$ and $M = E = \tau$ in turn, we get

$$\frac{z}{1-z} = 2\sum_{n=1}^{\infty} J_n(nz) \quad \text{and} \quad \frac{z}{1+z} = \sum_{n=1}^{\infty}(-1)^{n-1}J_n(nz),$$

whose difference is

$$\frac{z^2}{1-z^2} = 2\sum_{n=1}^{\infty} J_{2n}(2nz).$$

Now

$$-2\sum_{n=1}^{\infty} n^2 J_n(nz)\cos(nM) = \frac{d^2(a/r)}{dM^2} = \frac{3z^2\sin^2 E}{(1-z\cos E)^5} - \frac{z\cos E}{(1-z\cos E)^4},$$

and again setting $M = 0$ and $M = \pi$, we obtain

$$\frac{z}{(1-z)^4} = 2\sum_{n=1}^{\infty} n^2 J_n(nz) \quad \text{and} \quad \frac{z}{(1+z)^4} = 2\sum_{n=1}^{\infty}(-1)^{n-1}n^2 J_n(nz),$$

so

$$\frac{z}{(1-z)^4} - \frac{z}{(1+z)^4} = 16\sum_{n=1}^{\infty} n^2 J_{2n}(2nz).$$

Replacing z with $nx\sin\phi$ in these results and using the formula

$$J_n^2(t) = \frac{2}{\pi}\int_0^{\pi/2} J_{2n}(2t\sin\phi)\,d\phi, \qquad (1)$$

we get

$$(1-x^2)^{-1/2} - 1 = 2\sum_{n=1}^{\infty} g(nx),$$

$$x^2(4+x^2)(1-x^2)^{-7/2} = 16\sum_{n=1}^{\infty} n^2 g(nx).$$

(The exponent $-\frac{7}{2}$ is misprinted in [1].) Thus

$$2\sum_{n=1}^{\infty}\frac{dg(nx)}{dx} = x(1-x^2)^{-3/2},$$

$$2\sum_{n=1}^{\infty}\frac{d^2g(nx)}{dx^2} = (1-x^2)^{-3/2} + 3x^2(1-x^2)^{-5/2}.$$

To compute P_F, we replace x by $\omega\sin\theta$ in the last three formulas, multiply by $\sin\theta$ in the last three formulas, multiply by $\sin\theta$ and integrate. The required integrals can all be obtained from

$$\int_0^{\pi/2}\frac{\sin^{2\mu-1}\theta\cos^{2\nu-1}\theta}{(1-\omega^2\sin^2\theta)^{\mu+\nu}}\,d\theta = \frac{B(\mu,\nu)}{(1-\omega^2)^{\mu}} \qquad (\mu,\nu>0),$$

and the result is

$$P_F = \frac{8\omega^4}{3(1-\omega^2)^2},$$

in agreement with the proposer's "simpler equality."

To compute P_A, we proceed as follows. Put $M = \pi/2$ and $E = \pi/2 + u$, so $z = u \sec u$. Write

$$h = u \sec u, \qquad p = 1 + u \tan u,$$

so $h' = p \sec u$. Then

$$\frac{u \tan u}{2p} = -\sum_{n=1}^{\infty} J_n(nz) \cos(n\pi/2) = \sum_{n=1}^{\infty} (-1)^{n-1} J_{2n}(2nz),$$

so, with $\sin t = (u/x) \sec u$, and again using (1)

$$\sum_{n=1}^{\infty} (-1)^{n-1} g(nx) = \frac{2}{\pi} \int_0^{\pi/2} \frac{u \tan u}{2p}\, dt.$$

Write $R = (x^2 - h^2)^{1/2} = x \cos t$, so $dt = h'\,du/R$, and get

$$\sum_{n=1}^{\infty} (-1)^{n-1} g(nx) = \frac{1}{\pi} \int_0^U \frac{h(u) \tan u}{R(x,u)}\, du,$$

where $U = U(x)$ satisfies $U \sec U = x$. The standard method for differentiating an integral leads to an unpleasant $\infty - \infty$, so we integrate by parts first:

$$R\,dR = -h\,dh = -hp \sec u\, du$$

and the integral is

$$\sum_{n=1}^{\infty} (-1)^{n-1} g(nx) = -\frac{1}{\pi} \int L\, dR = \frac{1}{\pi} \int_0^U R(x,u) L'(u)\, du,$$

where

$$L(u) = \frac{\sin u}{p}.$$

Thus

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{dg(nx)}{dx} = \frac{x}{\pi} \int_0^U \frac{L'(u)}{R(x,u)}\, du = \frac{x}{\pi} \int_0^{\pi/2} \frac{L'(u)}{h'(u)}\, dt,$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{d^2 g(nx)}{dx^2} = \frac{1}{x} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{dg(nx)}{dx} + \frac{1}{\pi} \int_0^U \frac{d}{du}\left(\frac{L'(u)}{h'(u)}\right) \frac{h(u)}{R(x,u)}\, du.$$

Again with $M = \pi/2$,

$$8 \sum_{n=1}^{\infty} (-1)^{n-1} n^2 J_n(nz) = 3p^{-5} u^2 + p^{-4} u \tan u,$$

where $p = p(u) = 1 + u \tan u$. Thus

$$\sum_{n=1}^{\infty} (-1)^{n-1} n^2 g(nx) = \frac{1}{4\pi} \int_0^U \frac{3u^2 + p(u) u \tan u}{p^4(u) R(x,u)} \sec u\, du.$$

A typical integral needed to evaluate P_A is

$$I = \int_0^{\pi} \sin\theta\, d\theta \int_0^U \frac{f(u)}{R(x,u)}\, du,$$

with $x = \omega \sin\theta$. Interchanging the order of integration, we get

$$I = \int_0^{\phi} f(u)\,du \int_{\psi}^{\pi-\psi} \frac{\sin\theta}{R}\,d\theta,$$

in which $R = \sqrt{\omega^2 \sin^2\theta - h^2}$, $\sin\psi = h/\omega$ and $\phi \sec\phi = \omega$. The inner integral evaluates to π/ω and so

$$I = \frac{\pi}{\omega} \int_0^{\phi} f(u)\,du.$$

Thus

$$\int_0^{\pi} \sum_{n=1}^{\infty} \left\{ \frac{1}{x} \frac{dg(nx)}{dx} \right\}_{x=\omega\sin\theta} \sin\theta\,d\theta = \frac{1}{\omega} \int_0^{\phi} L'(u)\,du$$

$$= \frac{\sin\theta}{\omega q},$$

where $q = 1 + \omega\phi$ and

$$\int_0^{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \left\{ \frac{d^2 g(nx)}{dx^2} - \frac{1}{x} \frac{dg(nx)}{dx} \right\}_{x=\omega\sin\theta} \sin\theta\,d\theta = \frac{1}{\omega} \int_0^{\phi} h(u) \frac{d}{du}\left(\frac{L'(u)}{h'(u)} \right) du$$

$$= \frac{(\cos 2\phi - \omega \sin^3\phi)}{q^3} - \frac{\sin\theta}{\omega q},$$

on integrating by parts. Also

$$4\omega \int_0^{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} n^2 g(n\omega\sin\theta) \sin\theta\,d\theta = \int_0^{\phi} \left(\tan u + \frac{3u}{p(u)} \right) \frac{h(u)}{p^3(u)}\,du$$

$$= \int_0^{\phi} \left(\frac{3p'(u)}{p(u)} - 2\tan u \right) \frac{h(u)}{p^3(u)}\,du$$

$$= -\omega q^{-3} + \int_0^{\phi} (1 - u\tan u) \frac{\sec}{p^3(u)}\,du$$

$$= -\omega q^{-3} + q^{-2} \sec\phi \tan\phi$$

$$= (\sin\phi - \omega\cos 2\phi) q^{-3} \sec^2\phi.$$

Combining these results, we obtain

$$P_A = 4\omega^2 \left[\frac{\omega + \sin\phi}{2\omega q^3} + \left(1 - \frac{1}{\omega^2}\right) \frac{\sin\phi - \omega\cos 2\phi}{4\omega q^3 \cos^2\phi} \right]$$

$$= \frac{\omega}{q^3 \cos^2\phi} [2\omega^2 \cos^2\phi(\omega + \sin\phi) + (1 - \omega^2)(\omega\cos 2\phi - \sin\phi)].$$

Since $P_O = (P_F + P_A)/2$ and $P_E = (P_F - P_A)/2$, the fact that $P = W$ is verified.

REFERENCE

[1] G. N. WATSON, *Theory of Bessel Functions*, Cambridge University Press, Cambridge, U.K., 1944.

Also solved by C. C. GROSJEAN (State University of Ghent, Belgium), O. P. LOSSERS (Eindhoven University of Technology, Eindhoven, the Netherlands), MARK STAMP (Texas Tech University), and the proposer.

A Definite Integral Arising in Ohmic Dissipation

Problem 92-4 (Quickie).

Let $-\pi < x < \pi$ and $0 \le \rho \le 1$. From

$$\ln(1+2\rho\cos x+\rho^2) = \ln(1+\rho e^{ix}) + \ln(1+\rho e^{-ix}),$$
$$\ln(1+z) = -\sum_{n=1}^{\infty}(-1)^n \frac{z^n}{n}, \qquad |z| \le 1, \quad z \ne -1,$$

we get the Fourier cosine expansion

$$\ln(1+2\rho\cos x+\rho^2) = -2\sum_{n=1}^{\infty}(-1)^n\left(\frac{\rho^n}{n}\right)\cos nx.$$

Then by Parseval's relation,

$$\int_{-\pi}^{\pi} \ln^2(1+2\rho\cos x+\rho^2)dx = 2\pi\sum_{n-1}^{\infty}\frac{\rho^{2n}}{n^2}, \qquad 0 \le \rho \le 1.$$

For $\rho \ge 1$, we write

$$\ln(1+2\rho\cos x+\rho^2) = 2\ln\rho + \ln(1+2\rho^{-1}\cos x+\rho^{-2})$$

and apply the previous results. Then by Parseval's relation again,

$$\int_{-\pi}^{\pi} \ln^2(1+2\rho\cos x+\rho^2)dx = 8\pi\ln^2\rho + 2\pi\sum_{n=1}^{\infty}\frac{\rho^{-2n}}{n^2}.$$

Errata[2]

Problem 89-5, *Radiation from a System of Uniformly Circling Charges*, March, 1990.

p. 150, line 16. Insert an n^2 after the summation sign; change $n2$ in the integrand to n^2.
p. 151, line 5. Change τ to π.
line 6. Insert a 2 before the second summation sign.
line -12. Change $nx\sin\varphi$ to $x\sin\varphi$.
line -3. Delete "in the last three formulas, multiply by $\sin\theta$."
line -1. Insert a 2 before the integrand.

p. 153, line 7. Insert $(-1)^{n-1}$ after the summation sign.
line 9. Change $\omega\varphi$ to $\omega\sin\varphi$.
line 11. Change $(\sin\theta)/\omega q$ to $(\sin\varphi)/\omega q$.
line -9. Change sec to $\sec u$.
line -5. Change $4\omega^2$ to $4\omega^4$.

[2] We are grateful to Vladimir Hnizdo for noting these typographical corrections.