Relation between Interaction terms in Electromagnetic Momentum $\int d^3x\ \mathbf{E}\times\mathbf{B}/4\pi c$ and Maxwell's $e\mathbf{A}(\mathbf{r},t)/c$, and Interaction terms of the Field Lagangian $L_{em} = \int\ d^3x\ [E^2 - B^2]/8\pi$ and the Particle Interaction Lagrangian, $L_{int}\ = e\Phi - e\mathbf{v}\cdot\mathbf{A}/c$

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Electromagnetic Momenta

Consider a collection of charged particles with charges e_j, coordinates $\mathbf{r}_j(t)$, and velocities $\mathbf{v}_j(t)$ in interaction. Each particle produces electric and magnetic fields, $\mathbf{E}_j(\mathbf{r},\mathbf{r}_j,t)$, $\mathbf{B}_j(\mathbf{r},\mathbf{r}_j,t)$, at the point $(\mathbf{r},t)$. The interaction terms in the usual expression for the electromagnetic momentum in Gaussian units are

$$\mathbf{P}_{em} \ = \ \frac{1}{4\pi c}\int d^3x \ \left[\sum_{j=1}^{n}\mathbf{E}_j \times \sum_{k\neq j}^{n}\mathbf{B}_k \right] \qquad (1)$$

On the other hand, it is believed since Maxwell's time that the quantity,

$$\mathbf{p}_j^{em} = \frac{e_j}{c}\mathbf{A}(\mathbf{r}_j,t)\ , \qquad (2)$$

is the electromagnetic interaction contribution to the momentum of the j^{th} particle, where $\mathbf{A}(\mathbf{r}_j,t)$ is the vector potential caused by all the other charges, evaluated at the position of the j^{th}.

We treat the fields as quasi-static, neglecting terms of order $1/c^2$, and employ the Coulomb or radiation gauge ($\nabla\cdot\mathbf{A}=0$) in which the scalar potential is instantaneous. Consider first the interaction contribution of one term in the double sum in (1):

$$\mathbf{P}_{em}^{jk} \ = \ \frac{1}{4\pi c}\int \ d^3x\ \mathbf{E}_j\times\mathbf{B}_k \qquad (3)$$

Since $\mathbf{B}$ is itself of order $1/c$, we need only the scalar potential contribution to the j^{th} electric field, $\mathbf{E_j} = -\nabla\Phi_j$. We perform an integration by parts to obtain

$$\mathbf{P}_{em}^{jk} \ = \ \frac{1}{4\pi c}\int \ d^3x\ \Phi_j\nabla\times\mathbf{B}_k \qquad (4)$$

With $\mathbf{B}_k = \nabla \times \mathbf{A}_k$, we have $\nabla \times \mathbf{B}_k = \nabla(\nabla \cdot \mathbf{A}_k) - \nabla^2 \mathbf{A}_k$. In the Coulomb gauge the first term is absent. In the quasi-static approximation that neglects the second time derivative of $\mathbf{A}$ the wave equation for $\mathbf{A}_k$ reduces to

$$- \nabla^2 \mathbf{A}_k = \frac{4\pi}{c} \mathbf{J}_{k,trans} \tag{5}$$

Here $\mathbf{J}_{k,trans}$ is the transverse current(footnote 1) caused by the k^{th} particle. Equation (4) therefore becomes

$$\mathbf{P}_{em}^{jk} = \frac{1}{c^2} \int d^3x \ \Phi_j \mathbf{J}_{k,trans} \tag{6}$$

The scalar potential Φ_j is

$$\Phi_j(\mathbf{r}, \mathbf{r}_j, t) = \frac{e_j}{|\mathbf{r} - \mathbf{r}_j(t)|} \tag{7}$$

Hence (6) becomes

$$\mathbf{P}_{em}^{jk} = \frac{e_j}{c^2} \int d^3x \ \frac{\mathbf{J}_{k,trans}(\mathbf{r}, \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_j(t)|} \tag{8}$$

The integral in (8) is the quasi-static vector potential in the Coulomb gauge (footnote 2), evaluated at $\mathbf{r} = \mathbf{r}_j$. Thus (8) is

$$\mathbf{P}_{em}^{jk} = \frac{e_j}{c} \mathbf{A}_k(\mathbf{r}_j, \mathbf{r}_k) \tag{9}$$

where $\mathbf{A}_k$ is

$$\mathbf{A}_k(\mathbf{r}_j, \mathbf{r}_k) = \frac{e_k}{2cr_{jk}} \left[\mathbf{v}_k + \hat{\mathbf{r}}_{jk}(\hat{\mathbf{r}}_{jk} \cdot \mathbf{v}_k)\right] \tag{10}$$

Here $\mathbf{r}_{jk} = \mathbf{r}_j - \mathbf{r}_k$ and $r_{jk} = |\mathbf{r}_{jk}|$.

We define the total Coulomb gauge vector potential caused by all the particles other than the j^{th} as

$$\tilde{\mathbf{A}}_j(\mathbf{r}, t) = \sum_{k \neq j}^{n} \mathbf{A}_k(\mathbf{r}, \mathbf{r}_k) \tag{11}$$

where the individual terms are defined by (10). Then the interaction electromagnetic momentum (1) becomes

$$\mathbf{P}_{em} = \sum_{j}^{n} \frac{e_j}{c} \tilde{\mathbf{A}}_j(\mathbf{r}_j, t) \tag{12}$$

The individual terms in (12) are the Maxwell form (2) of the electromagnetic contribution to the j^{th} particle's momentum.

Interaction Lagrangians

The interaction part of the electromagnetic field Lagrangian for the interacting particles is

$$L_{int}^{em} \;=\; \frac{1}{4\pi}\int\, d^3x\, \sum_{j=1}^{n}\, \sum_{k\neq j}^{n} [\, \mathbf{E}_j \cdot \mathbf{E}_k - \mathbf{B}_j \cdot \mathbf{B}_k] \tag{13}$$

Without approximation the electric field $\mathbf{E}_k = -\nabla \Phi_k - \partial \mathbf{A}_k / c\,\partial t$. Substituting this expression into (13) and integrating by parts on the first term, we transform

$$\mathbf{E}_j \cdot \mathbf{E}_k \rightarrow\; \nabla \cdot \mathbf{E}_j \Phi_k - \mathbf{E}_j \cdot \partial \mathbf{A}_k / c\,\partial t \tag{14}$$

For the point charge e_j the divergence of its electric field is

$$\nabla \cdot \mathbf{E}_j = 4\pi e_j \delta(\mathbf{r} \;-\; \mathbf{r}_j)$$

Similarly, use of $\mathbf{B}_k = \nabla \times \mathbf{A}_k$ followed by an integration by parts yields

$$\mathbf{B}_j \cdot \mathbf{B}_k \rightarrow \nabla \times \mathbf{B}_j \cdot \mathbf{A}_k = \left[\frac{4\pi e_j}{c}\mathbf{v}_j \delta(\mathbf{r} \;-\; \mathbf{r}_j)] + \frac{\partial \mathbf{E}_j}{c\,\partial t}\right] \cdot \mathbf{A}_k \tag{15}$$

If the results in (14) and (15) are inserted into (13), we have

$$\begin{aligned} L_{int}^{em} \;=\; & \int\, d^3x\, \sum_{j=1}^{n}\, \sum_{k\neq j}^{n}\, e_j \left[\, \Phi_k(\mathbf{r},\mathbf{r}_k) - \frac{\mathbf{v}_j}{c} \cdot \mathbf{A}_k(\mathbf{r},\mathbf{r}_k)\right] \delta(\mathbf{r} - \mathbf{r}_j) \\ & -\frac{1}{4\pi c}\int\, d^3x\, \sum_{j=1}^{n}\, \sum_{k\neq j}^{n} \frac{\partial}{\partial t}(\mathbf{E}_j \cdot \mathbf{A}_k) \end{aligned} \tag{16}$$

The pieces of the integrand in the square brackets can be integrated immediately to give

$$\begin{aligned} L_{int}^{em} \;=\; & \sum_{j=1}^{n}\, e_j\, \sum_{k\neq j}^{n}\, \left[\Phi_k(\mathbf{r}_j,\mathbf{r}_k) - \frac{\mathbf{v}_j}{c} \cdot \mathbf{A}_k(\mathbf{r}_j,\mathbf{r}_k)\right. \\ & \left. -\;\frac{1}{4\pi e_j}\frac{d}{c\,dt}\int d^3x(\mathbf{E}_j \cdot \mathbf{A}_k)\;\right] \end{aligned} \tag{17}$$

The remaining integral is a perfect differential with respect to time and so can be dropped from the Lagrangian. If we define the sums of the scalar and vector potentials of all but the j^{th} particle as $\tilde{\Phi}_j(\mathbf{r},t)$ and $\tilde{\mathbf{A}}_j(\mathbf{r},t$, that is,

$$\tilde{\Phi}_j(\mathbf{r},t) \;=\; \sum_{k\neq j}^{n}\, \Phi_k(\mathbf{r},\mathbf{r}_k)\ ; \quad \tilde{\mathbf{A}}_j(\mathbf{r},t) \;=\; \sum_{k\neq j}^{n}\, (A_k(\mathbf{r},\mathbf{r}_k)\ , \tag{18}$$

then the electromagnetic interaction Lagrangian can be written

$$L_{int}^{em} \;=\; \sum_{j=1}^{n}\, e_j \left[\, \tilde{\Phi}_j(\mathbf{r}_j,t) - \frac{\mathbf{v}_j}{c} \cdot \tilde{\mathbf{A}}_j(\mathbf{r}_j,t)\right] \tag{19}$$

For each particle we have the familiar relativistic interaction for a charged particle with "external" electromagnetic fields described by scalar and vector potentials. Inspection of the development shows that for each particle it is immaterial whether the potentials are thought to be caused by a set of charged particles or by macroscopic external sources.

It is worth observing that the above argument on the Lagrangians is exact and is independent of a choice of gauge. A gauge transformation of the potentials in the interaction form is of no consequence:

$$e(\Phi - \mathbf{v}\cdot\mathbf{A}/c) \rightarrow e(\Phi' - \frac{\partial\chi}{c\,\partial t} - \mathbf{v}\cdot\mathbf{A}'/c - \mathbf{v}\cdot\nabla\chi/c) = e(\phi' - \mathbf{v}\cdot\mathbf{A}'/c) - \ ed\chi/c\,dt \quad (20)$$

The total time derivative of the gauge function χ does not contribute to the equation of motion.

Footnotes

1. See Jackson, p. 242
2. See Jackson, Sect. 12.6, p.596ff.