< 3 °K; the slope of this region is the critical exponent γ. Within the temperature region $0.14\ ^\circ\mathrm{K} < T - T_c < 0.7\ ^\circ\mathrm{K}$, the data can be observed to be systematically lower than the straight line drawn through the points above $T - T_c = 0.2\ ^\circ\mathrm{K}$. Convection in the fluids became obvious over this temperature region, and it is thought that the transmitted light intensity was increased as a result, and hence, the calculated turbidities were smaller.

Data run 2 displays a plateau region close to, but above T_c ($0.01\ ^\circ\mathrm{K} < T - T_c < 0.05\ ^\circ\mathrm{K}$). The horizontal error bars are large close to T_c due to the 0.05 °K uncertainty in T_c. Far above T_c, we observe large errors on the turbidities due to the small differences between I and I_0. The effects of multiple scattering were small when far above T_c; however, close to T_c, the effect was important though negligible compared to other errors.

All of the data collected within the temperature range $0.1\ ^\circ\mathrm{K} < T - T_c < 3\ ^\circ\mathrm{K}$ were fit using a ln–ln, weighted, linear least squares routine.[14] The slope γ was determined to be 1.34 ± 0.13. This value agrees within experimental error to other experimental results and to the predicted value; for example, the binary fluid mixture, polystyrene–diethyl malonate, was found to have γ equal 1.23 ± 0.03 using a similar technique.[7]

VIII. CONCLUSION

A simple experiment on the turbidity of a binary fluid mixture near its critical consolute point has been described. Quantitative results are consistent with theoretical predictions of universal critical exponents. Much more precise data may be obtained by using a photomultiplier tube for light intensity measurements, and by using a two stage thermostat to reduce temperature gradients across the cell and hence eliminate convection currents when close to T_c. It is felt that this is an appropriate experiment for junior or senior undergraduates and provides them with an opportunity to experience the exciting field of critical phenomena.

ACKNOWLEDGMENTS

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New method for calculating electric and magnetic fields and forces

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It is shown that for the purpose of electric field calculations, polarized dielectric bodies and real electric charge distributions can be replaced by fictitious "polarization current" distributions. Electric fields can then be calculated by any of the techniques available for calculating magnetic fields, and vice versa. Also electric forces acting on electric charges and dielectrics can be calculated by using the techniques available for calculating magnetic forces on currents. The method is illustrated by examples on calculating electric fields produced by electrets and by examples on calculating forces on dielectric bodies.

I. INTRODUCTION

An important method for calculating electrostatic fields in the presence of dielectric media, sometimes known as the "Poisson transformation," is the representation of such media by "equivalent" space and surface charge distributions. This method allows one to reduce an electrostatic system in the presence of dielectric media to a system of real and fictitious charges in a vacuum yielding everywhere exactly the same electrostatic field as that of the actual system.

If one considers this method on the basis of the Poisson theorem of vector analysis, one notices that dielectric media can also be represented by "equivalent" space and surface current distributions, so that the contribution of dielectric media to the total electrostatic field of a system under consideration can be calculated by means of the same equations and techniques as those used for calculating magnetic fields produced by electric currents in a vacuum. This means that for the purpose of electrostatic field calculations an electrostatic system with dielectrics can be reduced not only to a system of real and fictitious charges but also to a system of real charges and fictitious currents also yielding exactly the same electrostatic field as that of the actual system. The latter method offers a new insight into the structure of electrostatic fields, particularly those in the presence of dielectric media, and is especially useful for determining and analyzing electrostatic fields produced by electrets (permanently charged or permanently polarized dielectric bodies).

An important consequence of this method is that for calculating electrostatic fields one can also replace real charge distributions by equivalent fictitious current distributions, and that for calculating stationary magnetic fields one can replace real electric current distributions by equivalent fictitious electrostatic charge distributions. This makes it possible to use all the techniques usually reserved for calculating magnetic fields and magnetic forces also for calculating electric fields and electric forces, and to use all the techniques available for calculating electric fields and forces also for calculating magnetic fields and forces.

II. THEORY

According to the Poisson theorem of vector analysis a vector field $\mathbf{V}$ regular at infinity can be found from its curl and its divergence by means of the integral

$$\mathbf{V}(x,y,z) = -\frac{1}{4\pi}\int_{\text{all space}} \frac{\nabla'(\nabla'\cdot\mathbf{V}) - \nabla'\times(\nabla'\times\mathbf{V})}{r}\,dv', \quad (1)$$

where r is the distance from the volume element dv' located at the source point x', y', z' to the field point x, y, z, and where the primed operator ∇' operates upon the primed coordinates only.[1]

Let us apply Eq. (1) for finding the electrostatic field $\mathbf{E}$ of a charge distribution ρ in the presence of a dielectric medium having a polarization $\mathbf{P}$. By the basic electrostatic laws we have

$$\nabla\cdot\mathbf{D} = \rho, \quad (2a)$$

$$\nabla\times\mathbf{E} = 0. \quad (2b)$$

By the definition of $\mathbf{P}$ we also have

$$\mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E}. \quad (3)$$

From Eq. (3) it follows that

$$\nabla\cdot\mathbf{E} = (1/\epsilon_0)\nabla\cdot\mathbf{D} - (1/\epsilon_0)\nabla\cdot\mathbf{P}. \quad (4)$$

Introducing the notation

$$\rho_P = -\nabla\cdot\mathbf{P} \quad (5)$$

and using Eq. (2a), we can rewrite Eq. (4) as

$$\nabla\cdot\mathbf{E} = (1/\epsilon_0)\rho + (1/\epsilon_0)\rho_P. \quad (6)$$

Substituting now Eqs. (6) and (2b) into Eq. (1), we obtain

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\int_{\text{all space}} \frac{\nabla'\rho}{r}\,dv' - \frac{1}{4\pi\epsilon_0}\int_{\text{all space}} \frac{\nabla'\rho_P}{r}\,dv'. \quad (7)$$

Using the vector identity

$$\frac{\nabla'\rho}{r} = \nabla'\frac{\rho}{r} - \frac{\rho}{r^2}\hat{r}$$

together with Gauss's theorem of vector analysis, we can transform Eq. (7) to the more familiar expresssion

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{\text{all space}} \frac{\rho\hat{r}}{r^2}\,dv' + \frac{1}{4\pi\epsilon_0}\int_{\text{all space}} \frac{\rho_P\hat{r}}{r^2}\,dv', \quad (8)$$

where $\hat{r}$ is a unit vector in the direction of r (from x', y', z' to x, y, z).

As can be seen from Eq. (8), the field $\mathbf{E}$ in the presence of a dielectric medium may be regarded as the sum

$$\mathbf{E} = \mathbf{E}_V + \mathbf{E}_P \quad (9)$$

of two partial fields: the ordinary "vacuum" field

$$\mathbf{E}_V = \frac{1}{4\pi\epsilon_0}\int_{\text{all space}} \frac{\rho\hat{r}}{r^2}\,dv' \quad (10)$$

identical with the field produced by the charge distribution ρ in the absence of the medium, and the "polarization" field

$$\mathbf{E}_P = \frac{1}{4\pi\epsilon_0}\int_{\text{all space}} \frac{\rho_P\hat{r}}{r^2}\,dv' \quad (11)$$

associated with the medium and attributable to a fictitious charge distribution ρ_P defined by Eq. (5). For practical applications the integral of Eq. (11) can be transformed into a volume integral over the interior of the medium and a surface integral over the surface of the medium, which gives

$$\mathbf{E}_P = \frac{1}{4\pi\epsilon_0}\int_{\text{interior}} \frac{\rho_P\hat{r}}{r^2}\,dv' + \frac{1}{4\pi\epsilon_0}\int_{\text{surface}} \frac{\sigma_P\hat{r}}{r^2}\,ds', \quad (12)$$

where

$$\sigma_P = -\hat{n}_{\text{in}}\cdot\mathbf{P}, \quad (13)$$

$\hat{n}_{\text{in}}$ being a unit vector normal to the surface and pointing into the medium.[2]

Equation (12) together with Eqs. (5) and (13) constitute the well-known Poisson transformation for dielectric media (ρ_P and σ_P are the fictitious "equivalent" space and surface charge densities). The particular method used here for obtaining this transformation indicates, however, the possibility of a similar alternative transformation. This alternative transformation is obtained by applying Eq. (1) for finding the displacement field $\mathbf{D}$, rather than the field $\mathbf{E}$, as follows.

From Eqs. (2b) and (3) we have

$$\nabla\times\mathbf{D} = \epsilon_0\nabla\times\mathbf{E} + \nabla\times\mathbf{P} = \nabla\times\mathbf{P}, \quad (14)$$

and, introducing the notation

$$\mathbf{J}_P = (1/\epsilon_0)\nabla\times\mathbf{P}, \quad (15)$$

we can write

$$\nabla\times\mathbf{D} = \epsilon_0\,\mathbf{J}_P. \quad (16)$$

Substituting Eqs. (2a) and (16) into Eq. (1), we have for the displacement field $\mathbf{D}$

$$\mathbf{D} = -\frac{1}{4\pi}\int_{\text{all space}} \frac{\nabla'\rho}{r}\,dv' + \frac{\epsilon_0}{4\pi}\int_{\text{all space}} \frac{\nabla'\times\mathbf{J}_P}{r}\,dv'. \quad (17)$$

The first integral of this expression can be transformed as before in Eq. (7); the second integral can be transformed in a similar manner by using the vector identity

$$\frac{\nabla' \times \mathbf{J}}{r} = \nabla' \times \frac{\mathbf{J}}{r} + \frac{\mathbf{J} \times \hat{r}}{r^2}$$

together with Gauss's theorem of vector analysis. We then obtain

$$\mathbf{D} = \frac{1}{4\pi} \int_{\text{all space}} \frac{\rho \hat{r}}{r^2} dv' + \frac{\epsilon_0}{4\pi} \int_{\text{all space}} \frac{\mathbf{J}_P \times \hat{r}}{r^2} dv'. \quad (18)$$

As can be seen from this expression, the displacement field $\mathbf{D}$ in the presence of a dielectric medium can be regarded as the sum

$$\mathbf{D} = \mathbf{D}_V + \mathbf{D}_P \quad (19)$$

of two partial fields: the ordinary vacuum field

$$\mathbf{D}_V = \frac{1}{4\pi} \int_{\text{all space}} \frac{\rho \hat{r}}{r^2} dv' \quad (20)$$

identical with the field produced by the charge distribution ρ in the absence of the medium, and the "polarization" field

$$\mathbf{D}_P = \frac{\epsilon_0}{4\pi} \int_{\text{all space}} \frac{\mathbf{J}_P \times \hat{r}}{r^2} dv' \quad (21)$$

associated with the medium. A surprising result here is that $\mathbf{D}_P$ is expressed by an integral identical with the well-known integral representing the magnetic field (flux density field) of a current distribution in a vacuum

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \hat{r}}{r^2} dv'. \quad (22)$$

Hence $\mathbf{D}_P$ is attributable to a fictitious current distribution $\mathbf{J}_P$ defined by Eq. (15) in the sense that mathematically $\mathbf{D}_P$ is determined by J_P in the same way as a magnetic field is determined by the current that produces this field. What is more, $\mathbf{D}_P$ can be formally obtained from a geometrically similar $\mathbf{B}$ by replacing μ_0 with ϵ_0 and $\mathbf{J}$ with $\mathbf{J}_P$.

For practical applications the integral of Eq. (21) can be transformed into a volume integral over the interior of the medium and a surface integral over the surface of the medium, which gives[3]

$$\mathbf{D}_P = \frac{\epsilon_0}{4\pi} \int_{\text{interior}} \frac{\mathbf{J}_P \times \hat{r}}{r^2} dv' + \frac{\epsilon_0}{4\pi} \int_{\text{surface}} \frac{\mathbf{J}_P^{(s)} \times \hat{r}}{r} ds', \quad (23)$$

where

$$\mathbf{J}_P^{(s)} = (1/\epsilon_0)\, \hat{n}_{\text{in}} \times \mathbf{P} \quad (24)$$

is the fictitious surface current ("current per unit width").

Equation (23) together with Eqs. (15) and (24) constitute the desired transformation alternative to the Poisson transformation ($\mathbf{J}_P$ and $\mathbf{J}_P^{(s)}$ are the fictitious equivalent space and surface current densities).

It should be emphasized that the practical significance of this transforamtion consists not only in the possibility of using Eq. (23) directly for finding $\mathbf{D}_P$, but also in the fact that $\mathbf{D}_P$ is formally related to $\mathbf{J}_P$ and $\mathbf{J}_P^{(s)}$ just as a magnetic field is related to the electric currents producing this field. This means that $\mathbf{D}_P$ can be found from $\mathbf{J}_P$ and $\mathbf{J}_P^{(s)}$ by any of the techniques available for the calculation of magnetic fields, i.e., not only by direct integration but also indirectly through vector potentials, scalar potentials, method of harmonics, axial expansion, etc. Thus this transformation results in a substantial increase of methods that can be used for calculating and analyzing electrostatic fields in the presence of dielectric media.

Since electric fields are force fields, it is plausible that the equivalent currents can be used for direct calculation of forces on dielectric bodies. To obtain explicit formulas for such calculations we shall start with the vector identity

$$\oint (\mathbf{V}\cdot\mathbf{W}) d\mathbf{s} - \oint \mathbf{V}(\mathbf{W}\cdot d\mathbf{s}) - \oint \mathbf{W}(\mathbf{V}\cdot d\mathbf{s})$$
$$= \int \mathbf{V} \times (\nabla \times \mathbf{W}) dv + \int \mathbf{W} \times (\nabla \times \mathbf{V}) dv$$
$$- \int \mathbf{V}(\nabla \cdot \mathbf{W}) dv - \int \mathbf{W}(\nabla \cdot \mathbf{V}) dv. \quad (25)$$

Let the surface integrals be extended over a surface enclosing the dielectric body the force upon which is to be found. Let the external field at the location of this body be $\mathbf{E}'$. Substituting in Eq. (25) $\mathbf{P}$ for $\mathbf{V}$ and $\mathbf{E}'$ for $\mathbf{W}$ and taking into account that $\mathbf{P} = 0$ outside the dielectric, that $\nabla \times \mathbf{E}' = 0$, and that $\nabla \cdot \mathbf{E}' = 0$ (the sources of the external field are outside the surface of integration), we obtain

$$0 = \int \mathbf{E}' \times (\nabla \times \mathbf{P}) dv - \int \mathbf{E}' \nabla \cdot \mathbf{P}\, dv. \quad (26)$$

But the force on a dielectric in an external field is given by

$$\mathbf{F} = \int \rho_P \mathbf{E}'\, dv = -\int (\nabla \cdot \mathbf{P}) \mathbf{E}'\, dv. \quad (27)$$

Hence, by Eq. (26), the force can also be calculated as

$$\mathbf{F} = -\int \mathbf{E}' \times (\nabla \times \mathbf{P}) = \int (\nabla \times \mathbf{P}) \times \mathbf{E}'\, dv \quad (28)$$

or, with Eq. (15), as

$$\mathbf{F} = \int \mathbf{J}_P \times \mathbf{D}' dv, \quad (29)$$

where we have replaced $\epsilon_0 \mathbf{E}'$ with $\mathbf{D}'$. Except for the symbols, Eq. (29) is identical with the equation describing the force acting on a current in an magnetic field

$$\mathbf{F} = \int \mathbf{J} \times \mathbf{B}'\, dv. \quad (30)$$

Consequently, electric forces on dielectrics can be calculated by the same techniques that are used for calculating magnetic forces on currents. In particular, for a surface distribution of the equivalent current the force can be calculated from

$$\mathbf{F} = \int \mathbf{J}_P^{(s)} \times \mathbf{D}'\, ds, \quad (31)$$

and for a straight filamentary equivalent current from

$$\mathbf{F} = \mathbf{I}_P \times \mathbf{D}' l \quad (32)$$

with

$$\mathbf{I}_P = \mathbf{J}_P^{(s)} t, \quad (33)$$

where t is the width of the equivalent surface current, and l is its length.

Although the vector $\mathbf{D}'$ appearing in Eqs. (29) and (32) is the external displacement field, it can be replaced by the total field $\mathbf{D}$, since the self-field (field due to $\mathbf{J}_P$) produces no net forces on the dielectric. In linear isotropic dielectrics $\mathbf{D}'$ (or $\mathbf{D}$) can be replaced by $\epsilon_0 \epsilon \mathbf{E}'$ (or $\epsilon_0 \epsilon \mathbf{E}$), where ϵ is the permittivity of the dielectric. It also can be replaced by $\mathbf{P} + \epsilon_0 \mathbf{E}$, in accordance with Eq. (3).

III. EXAMPLES

To illustrate some of the ways in which one can use the equivalent current transformation obtained above, we shall find with its help the electric fields of several differently shaped electrets and shall find electric forces acting on dielectrics in some typical electrostatic systems. For simplicity we shall assume that the electrets carry no real space or surface charge, and that the polarization of the electrets is not affected by depolarizing fields (the latter assumption has no effect on the calculations, but the fields which we shall find using this assumption may differ from the fields of real electrets in which a depolarization may occur).

Example 1: Find the electric field on the axis of a disk electret of thickness t, radius a, and uniform polarization $\mathbf{P}$ directed along the axis of the electret if $t \ll a$ [Fig. 1(a)].

Since $\mathbf{P}$ is constant, $\nabla \times \mathbf{P} = 0$, and hence, by Eq. (15), the space current $\mathbf{J}_P$ is zero. On the flat surfaces of the electret $\hat{n}_{\text{in}}$ is parallel to $\mathbf{P}$, and hence, by Eq. (23), the surface current density $\mathbf{J}_P^{(s)}$ is zero there. On the cylindrical surface, however, $\hat{n}_{\text{in}}$ is perpendicular to $\mathbf{P}$, so that on this surface $\mathbf{J}_P^{(s)} = (1/\epsilon_0) P \hat{\theta}$, where $\hat{\theta}$ is a unit vector tangential to this surface and right-handed with respect to $\mathbf{P}$. The electret may be represented therefore by an equivalent ring current $I_P = J_P^{(s)} t = (1/\epsilon_0) P t$ along the edge of the electret, as shown in Fig. 1(b). The magnetic field of such current is known and can be expressed at a point z of the symmetry axis with the origin at the center of the ring as (see, for example, Ref. 1, p. 347)

$$\mathbf{B} = \mu_0 \mathbf{H} = [\,\mu_0 I a^2 / 2(a^2 + z^2)^{3/2}]\hat{k},$$

where $\hat{k}$ is a unit vector along the z axis. Hence, outside the electret the fields $\mathbf{D}$ and $\mathbf{E}$ are (observing that in a vacuum $\mathbf{D} = \epsilon_0 \mathbf{E}$)

$$\mathbf{D} = \mathbf{P} t a^2 / 2(a^2 + z^2)^{3/2}$$

and

$$\mathbf{E} = \mathbf{D}/\epsilon_0 = \mathbf{P} t a^2 / 2\epsilon_0 (a^2 + z^2)^{3/2}.$$

Inside the electret (the electret is thin), $z \approx 0$ and, by Eq. (3), $\mathbf{E} = (1/\epsilon_0)(\mathbf{D} - \mathbf{P})$. The internal fields on the axis of the electret are therefore

$$\mathbf{D} = \mathbf{P} t / 2a$$

and

$$\mathbf{E} = \frac{1}{\epsilon_0}\left(\frac{\mathbf{P}t}{2a} - \mathbf{P}\right) = -\frac{\mathbf{P}}{\epsilon_0}\left(1 - \frac{t}{2a}\right) \approx -\frac{\mathbf{P}}{\epsilon_0},$$

where the last expression represents $\mathbf{E}$ in a very thin electret.

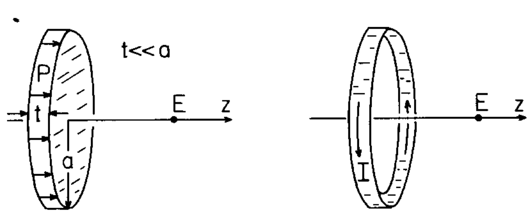

Fig. 1. To find the electric field on the axis of a disk electret (a), the electret is replaced by an equivalent current ring (b).

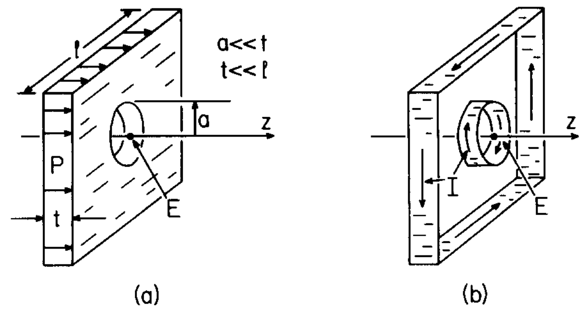

Fig. 2. To find the electric field in a central hole of a rectangular electret (a), the electret is replaced by equivalent rectangular and cylindrical currents (b). (The radius of the hole is very much exaggerated.)

Example 2: A square electret of thickness t, length l, and uniform axial polarization $\mathbf{P}$ has at its center a small cylindrical hole of radius a oriented along the symmetry axis [Fig. 2(a)]. Find the electric field at the center of the hole if $t \ll l$ and $a \ll t$.

Just as in the preceding example the equivalent currents by which the electret can be replaced are [Fig. 2(b)] a current along the edge of the electret and a current along the walls of the hole (directed oppositely to the current along the edge). As before, the magnitude of these currents is $I_P = (1/\epsilon_0) P t$. A real rectangular current along the edge would produce the magnetic field (see, for example, Ref. 1, p. 359)

$$\mathbf{H} = (2\sqrt{2} I / \pi l)\hat{k}.$$

The cylindrical current along the walls of the cavity is similar to the current in a long solenoid of length t, where the magnetic field is

$$\mathbf{H} = -(I/t)\hat{k}.$$

The total magnetic field due to these currents is

$$\mathbf{H} = \left(\frac{2\sqrt{2}I}{\pi l} - \frac{I}{t}\right)\hat{k} \approx -\frac{I}{t}\hat{k}.$$

The electrostatic fields $\mathbf{D}$ and $\mathbf{E}$ in the cavity are therefore

$$\mathbf{D} \approx -(Pt/t)\hat{k} = -\mathbf{P}$$

and

$$\mathbf{E} = \mathbf{D}/\epsilon_0 \approx -\mathbf{P}/\epsilon_0.$$

Example 3: An electret is made in the shape of a thin hemispherical shell of radius a, thickness t, and uniform radial polarization $\mathbf{P}$ [Fig. 3(a)]. Find the electric field at the center of the shell.

Since $\mathbf{P}$ is radial, the only equivalent current is a ring current along the flat surface of the shell [Fig. 3(b)], and its magnitude is, as before, $I_P = (1/\epsilon_0) P t$. By Example 1, at the

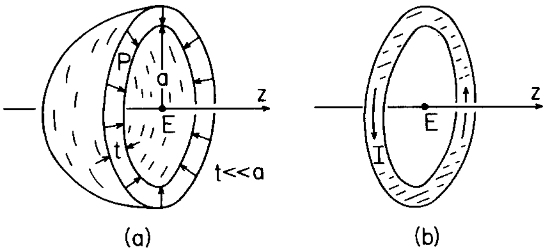

Fig. 3. To find the electric field at the center of a hemispherical electret (a), the electret is replaced by an equivalent current ring (b).

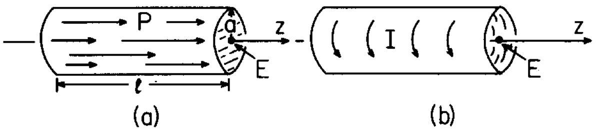

Fig. 4. To find the electric field at the end surface of a cylindrical electret (a), the electret is replaced by an equivalent current (b) on the cylindrical surface of the electret.

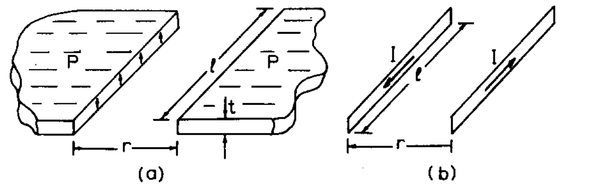

Fig. 6. To find the force between two electrets (a), the electrets are replaced by two equivalent current ribbons (b).

center of the ring a real circular current produces the magnetic field

$$\mathbf{H} = (I/2a)\hat{k}.$$

Hence the fields **D** and **E** at the center of the electret are

$$\mathbf{D} = \mathbf{P}t/2a$$

and

$$\mathbf{E} = \mathbf{P}t/2a\epsilon_0.$$

Example 4: A cylindrical electret of radius a and length l has a uniform axial polarization **P** [Fig. 4(a)]. Find the external electric field at the center of a flat surface of the electret.

The only equivalent current corresponding to this electret is a circular current along the cylindrical surface [Fig. 4(b)] of magnitude $I = (1/\epsilon_0)Pl$. The magnetic field produced by such a current at the center of an end surface is (see, for example, Ref. 1, p. 349)

$$\mathbf{H} = [I/2(l^2 + a^2)^{1/2}]\,\hat{k}.$$

The fields **D** and **E** at the point under consideration are therefore

$$\mathbf{D} = \mathbf{P}l/2(l^2 + a^2)^{1/2}$$

and

$$\mathbf{E} = \mathbf{D}/\epsilon_0 = \mathbf{P}l/2\epsilon_0(l^2 + a^2)^{1/2}.$$

Example 5: Find the magnitude of the external electric field near an edge of a large thin electret of thickness t and uniform polarization **P** normal to the electret [Fig. 5(a)].

The only equivalent current is along the edge of the electret and is $I = (1/\epsilon_0)Pt$. Since the edge is straight and long, the current I is just a straight filamentary current. As is known, such a current produces a circular magnetic field

$$H = I/2\pi r.$$

Hence the fields D and E near an edge of the electret are

$$D = Pt/2\pi r$$

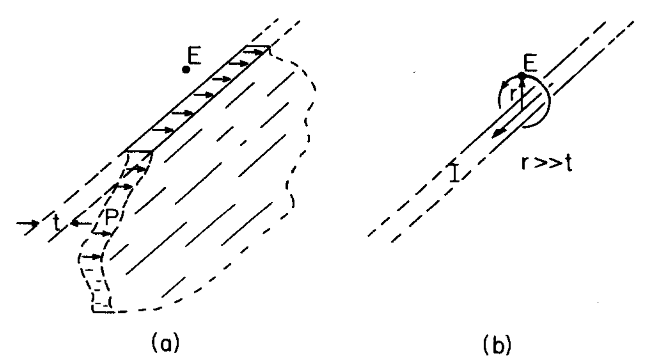

Fig. 5. To find the electric field outside an edge of a plane electret (a), the electret is replaced by an equivalent current ribbon (b).

and

$$E = Pt/2\pi\epsilon_0 r,$$

where r is the distance from the edge.

Example 6: A large electret is broken along a straight line of length l in two pieces, located at a distance $d \ll l$ from each other. Find the force between the two pieces if the polarization is constant and perpendicular to the flat surface of the electret [Fig. 6(a)].

According to Example 5, the system is equivalent to a system of two parallel currents of magnitude $I = (1/\epsilon_0)Pt$, shown in Fig. 6(b). (The current corresponding to the remaining portions of the electret edge may be neglected if d is sufficiently small.) The magnetic force (repulsion) between such currents is (see, for example, Ref. 1, p. 451, Problem 13.15)

$$F = (\mu_0 I_1 I_2/2\pi d)l.$$

Hence the two electrets repel each other with the force

$$F = \epsilon_0 \left(\frac{Pt}{\epsilon_0}\right)^2 \frac{l}{2\pi d} = \frac{P^2 t^2 l}{2\pi\epsilon_0 d}.$$

Example 7. A voltage V is applied to a parallel-plate capacitor [Fig. 7(a)] consisting of two square plates of length a separated by a distance d. A large dielectric slab of thickness d and dielectric constant ϵ is inserted between the plates. Neglecting end effects, find the force acting on the slab.

The field in the capacitor is everywhere $\mathbf{E} = -(V/d)\hat{j}$ where $\hat{j}$ is a unit vector along the y axis. The displacement in the slab is $\mathbf{D} = \epsilon_0\epsilon\mathbf{E}$. The polarization of the slab is $\mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E} = -\epsilon_0(\epsilon - 1)(V/d)\hat{j}$ and is constant throughout the part located inside the capacitor. Hence $\mathbf{J}_P$ is zero everywhere except near the edge of the capacitor, where **P** abruptly changes to zero (because the edge effects are neglected). The abrupt change in **P** creates $\nabla \times \mathbf{P}$. If the change is assumed to take place over a small distance Δx, the corresponding equivalent space current density is, by

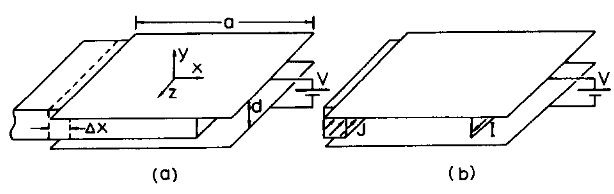

Fig. 7. To find the force with which a dielectric is pulled into the parallel-plate capacitor (a), the dielectric is replaced by an equivalent current ribbon I_P and space current J_P (b).

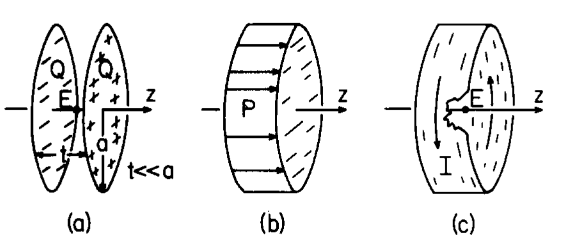

Fig. 8. To find the electric field between two charged disks (a), the disks are first replaced by an equivalent polarized dielectric (b), and then by an equivalent current ring (c).

Eq. (15),[4]

$$\mathbf{J}_P = \frac{1}{\epsilon_0}\nabla\times\mathbf{P} = -\frac{(\epsilon-1)}{\Delta x}\frac{V}{d}\hat{k}.$$

The only surface current that needs to be taken into account is on the forward edge of the slab, where it produces a current ribbon $\mathbf{I}_P = (\epsilon - 1)V\hat{k}$ (the effects of the currents along the front and back edges cancel by symmetry). The equivalent current system is shown in Fig. 7(b). The external displacement field $\mathbf{D}'$ (that is, the field due to capacitor alone) at the location of this ribbon is $-\epsilon_0(V/d)\hat{j}$. At the location of $\mathbf{J}_P$ the average external field $\mathbf{D}'$ is $-\frac{1}{2}\epsilon_0(V/d)\hat{j}$ (because in the capacitor the electric field is $-V/d$, and just outside it is zero). Hence the force on the slab is, by Eqs. (29) and (32),

$$\mathbf{F} = \int_{\text{edge}} \mathbf{J}_P\times\mathbf{D}'dv + \mathbf{I}_P\times\mathbf{D}'a. \quad (34)$$

Since Δx is small, the integral may be replaced by the product $\mathbf{J}_P\times\mathbf{D}'_{\text{average}}\,\Delta x\,da$. We then obtain

$$\mathbf{F} = -\frac{(\epsilon-1)}{\Delta x}\frac{V}{d}\hat{k}\times\left(-\frac{\epsilon_0 V}{2d}\hat{j}\right)\Delta x\,da + (\epsilon-1)V\hat{k}\times\frac{\epsilon_0 V_a}{d}\hat{j}$$
$$= -\epsilon_0(\epsilon-1)\frac{V^2a}{2d}\hat{i} + \epsilon_0(\epsilon-1)\frac{V^2a}{d}\hat{i}$$

or[5]

$$\mathbf{F} = \epsilon_0(\epsilon-1)\frac{V^2a}{2d}\hat{i}. \quad (35)$$

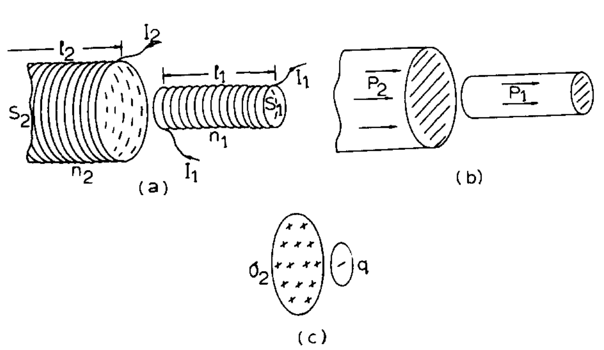

Fig. 9. To find the force between two solenoids (a), the solenoids are first replaced by two polarized cylinders (b), and then by two charged disks (c).

IV. TRANSFORMATIONS FOR REAL CHARGE DISTRIBUTIONS

An interesting consequence of the Poisson transformation and of the "equivalent current" transformation is that by using them in succession one can reduce also real charge distributions to equivalent fictitious current distributions yielding the same electrostatic field as that of the actual charge distribution. Indeed, by reversing the Poisson transformations, a system of real charges can be replaced by an equivalent dielectric medium of polarization $\mathbf{P}$. Next, this medium can be replaced by an equivalent current system $\mathbf{J}_P$. From this current system the displacement field $\mathbf{D}_P$ can be found. Finally, the electric field of the original system can be obtained from $\mathbf{E} = \epsilon_0(\mathbf{D}_P - \mathbf{P})$.

Consider, for example, the electric field at the center of the system of two charged layers $\pm Q$ (surface charge densities $\pm\sigma$) of radius a and separation t shown in Fig. 8(a) ($t \ll a$). By inspection and by Eq. (13) this system is equivalent to a uniformly polarized dielectric disk of polarization $P=\sigma$ [Fig. 8(b)]. The transformations for such a disk have been already demonstrated in Example 1: the equivalent current system is a ring of current shown in Fig. 8(c). Hence, by Example 1, the electric field midway between the two charged layers is

$$E = (\sigma/\epsilon_0)(1 - t/2a),$$

which, incidentally, is a more accurate expression than the usual $E = \sigma/\epsilon_0$ given for this system in most textbooks.

It is easy to see that by reversing the equivalent current transformation one can also reduce real current systems to equivalent charge systems for the purpose of calculating magnetic fields or magnetic forces.

Consider, for example, a system of two long coaxial solenoids placed close to each other, as shown in Fig. 9(a). Let the thin solenoid be of cross sectional area S_1 and length l_1; let it have n_1 turns, and let it carry a current I_1. Let the other solenoid be of area $S_2 \gg S_1$ and length l_2; let it have n_2 turns and let it carry a current I_2. Suppose that we want to find the force exerted by one solenoid on the other. Reversing the steps of the preceding example, we can replace the two solenoids by equivalent polarized rods [Fig. 9(b)], whose polarizations are $P_1 = \epsilon_0 n_1 I_1/l_1$ and $P_2 = \epsilon_0 n_2 I_2/l_2$. The two rods can in turn be replaced by two charged disks of surface charge density $\sigma_1 = \epsilon_0 n_1 I_1/l_1$ and $\sigma_2 = \epsilon_0 n_2 I_2/l_2$. Since the diameter of the first disk is much smaller than that of the second, it can be regarded as a point charge $q = \sigma_1 S_1 = \epsilon_0 n_1 I_1 S_1/l_1$. The entire system thus reduces to a point charge q in front of a large disk of surface charge density σ_2 [Fig. 9(c)]. The force on q in this system is

$$F = qE' = q\frac{D'}{\epsilon_0} = q\frac{\sigma_2}{2\epsilon_0} = \epsilon_0\frac{n_1 I_1 n_2 I_2}{2l_1 l_2}S_1.$$

Hence the force between the two solenoids is (replacing ϵ_0 with μ_0, as explained in Sec. II)

$$F = \frac{\mu_0 n_1 n_2 I_1 I_2}{2l_1 l_2}S_1.$$

V. CONCLUSION

The equivalent current transformation presented here allows one to supplement the usual methods for the calculation of the electrostatic fields in the presence of dielectrics by the methods developed for the calculation of the stationary magnetic fields produced by electric currents. When used together with the Poisson transformation it allows one to calculate *all* electrostatic fields in terms of equivalent stationary magnetic fields produced by electric currents, and vice versa. The possibility of such inversions can be stated in the form of the following two propositions:

Proposition I: To every electrostatic field there corresponds a geometrically equivalent magnetic field that can differ from it only in a limited region of space, and to every magnetostatic field there corresponds a geometrically equivalent electric field that can differ from it only in a limited region of space.

Proposition II: Within the limitations of Proposition I, electric forces can be calculated in terms of the equivalent electric currents, and magnetic forces can be calculated in terms of the equivalent electric charges.

The existence of such reciprocal relations between electric and magnetic systems contributes to the internal unity and harmony of the electromagnetic theory and expands the range of techniques available for calculating electric and magnetic fields and forces.

[1]Details on using this theorem for calculating electric and magnetic fields can be found in O. D. Jefimenko, *Electricity and Magnetism* [Appleton-Century-Crofts (now Plenum), New York, 1966], pp. 42, 93, 248, 249, 343, 344, 480, 481.

[2]One obtains Eq. (12) from Eqs. (11) and (5) by taking into account that $\nabla\cdot\mathbf{P}$ vanishes outside the medium and reduces to "surface divergence" $\hat{n}_{in}\cdot\mathbf{P}$ at the surface of the medium.

[3]One obtains Eq. (23) from Eqs. (21) and (15) by taking into account that $\nabla\times\mathbf{P}$ vanishes outside the medium and reduces to "surface curl" $\hat{n}_{in}\times\mathbf{P}$ at the surface of the medium.

[4]To find $\nabla\times\mathbf{P}$, we use the relation $\nabla\times\mathbf{P}=\hat{i}(\Delta P_z/\Delta y-\Delta P_y/\Delta z)+\hat{j}(\Delta P_x/\Delta z-\Delta P_z/\Delta x)+\hat{k}(\Delta P_y/\Delta x-\Delta P_x/\Delta y)$. In the present case $P_x=P_z=0$, while $P_y=-\epsilon_0(\epsilon-1)V/d$ and changes along the x axis only. Since $\mathbf{P}=0$ outside the capacitor, $\Delta P_y/\Delta x=P_y/\Delta x$, and hence $\nabla\times P=-\hat{k}\epsilon_0(\epsilon-1)V/d\Delta x$.

[5]The same result is obtained if the total field $\mathbf{D}$ rather than the external field $\mathbf{D}'$ is used in Eq. (34). Inside the slab $\mathbf{D}=-\epsilon_0\epsilon(V/d)\hat{j}$. Outside the capacitor $\mathbf{D}=0$. Inside the capacitor but outside the slab $\mathbf{D}=-\epsilon_0(V/d)\hat{j}$. The average fields are therefore $\mathbf{D}_{average}=-\frac{1}{2}\epsilon_0\epsilon(V/d)\hat{j}$ at the location of $\mathbf{J}_P$ and $\mathbf{D}_{average}=\frac{1}{2}\epsilon_0(\epsilon+1)(V/d)\hat{j}$ at the location of $\mathbf{I}_P$. Replacing $\mathbf{D}'$ in Eq. (34) by these expressions yields Eq. (35).

Lise Meitner and the beta-ray energy controversy: An historical perspective

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Much has been written about the early history of beta-ray research and the controversy it provoked. Many prominent physicists lent their talents to the solution of this puzzling problem. One of the foremost of these was Lise Meitner, conspicuous in the fray not only because of her creative experimental work, but also because of her attachment to a physical principle—the simplicity of nature. This article reviews the sequence of events as they preceded, coincided with, and followed her work on the subject.

Lise Meitner began her graduate studies at the University of Vienna in 1901. She received the Ph.D. in physics in 1905, but remained on at the University until 1907 to clear up some unfinished work. (One study was that of the deflection of alpha-rays. At that time, there was some uncertainty as to whether they were deflected in passing through matter. She designed and carried out one of the first experiments to show some deflection does, indeed, occur.) In 1907, she went to Berlin and soon began studies on radioactivity with a young chemist, Otto Hahn.

By 1907, several significant discoveries had been made in the field of beta rays. Chronologically presented, they are

1899: Rutherford had shown that the nucleus emits two kinds of radiation, one about 100 times as penetrating as the other. He named them alpha and beta rays.[1]

1899: Becquerel showed that beta rays can be deflected by a magnetic field.[2]

1900: Pierre Curie found that beta rays have ranges of the order of several tens of centimeters of air. He showed that they produce much less ionization than alpha rays, and are markedly deviated by magnetic fields.[3]

1900: Becquerel[4] and the Curies[5] showed that beta rays are negative. They established this by studying the direction of deflection in a magnetic field. Curiously, however, the photographic image was diffuse and extended, even though the entering beam was well collimated. This seemed to suggest that either (a) beta rays from the same source were not all identical or (b) they were emitted from the source with a range of energies.

Becquerel had also applied an electric field to the beam of beta rays. Combining the measurements thus obtained with the magnetic field observations, he was able to calculate a rough value of e/m. This was close enough to that of the electron to make it fairly clear that beta rays are streams of electrons.

1902: Kaufmann used simultaneous parallel electric and magnetic fields to measure the ratio of charge to mass and the velocity of beta rays.[6] (The electric field causes a

For field points outside the region, the previous integral expression given by Eq. (6) reduces to

$$U = \frac{1}{r} \int_0^R f(r')(r')^2 dr' = \frac{K}{r},$$

with K the constant integral factor. The inverse square field follows immediately, and we have obtained an alternative direct proof to the previously stated theorem.

In conclusion, it is evident that Helmholtz' theorem is an extremely significant theorem of vector calculus which to a large extent has been neglected in mathematical physics.

[1]G. Arfken, *Mathematical Methods for Physicists* (Academic, New York, 1966), pp. 58–62.
[2]W. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, MA, 1962), pp. 2–7.
[3]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), p. 140.
[4]L. Eyges, *The Classical Electromagnetic Field* (Dover, New York, 1962), pp. 123, 387–8.
[5]P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Chap. 13.

Erratum: "New method for calculating electric and magnetic fields and forces" [Am. J. Phys. 51, 545–551 (1983)]

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The external displacement field $\mathbf{D}'$ in Eq. (34) is incorrectly described. The actual external field is the sum of the original field of the capacitor and the field due to charges induced on the capacitor plates by the dielectric. However, due to the symmetry of the system under consideration, the latter field has no effect on the force in this particular case.

Erratum: "Correct use of Maxwell stress equations for electric and magnetic fields" [Am. J. Phys. 51, 988–996 (1983)]

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The external field $\mathbf{E}'$ used in Eq. (83) is incorrectly described. The external field is actually the sum of the original field V/d of the capacitor and the field due to charges induced on the capacitor plates by the dielectric. Due to the symmetry of the system under consideration the latter field has no effect on the force in this particular case.