Correct use of Maxwell stress equations for electric and magnetic fields

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By neglecting edge or end effects in electric or magnetic systems and by using certain other conventional approximation techniques one may inadvertently create conditions that contradict some of the assumptions upon which Maxwell stress tensors and stress integrals are derived. Correction terms must then be added to the stress equations to make them usable. The nature of these correction terms is discussed and several illustrative examples are given.

I. INTRODUCTION

The Maxwell stress tensor and Maxwell stress integral for time-independent electric systems are derived under the assumption that $\nabla \times \mathbf{E} = 0$. For time-independent magnetic systems they are derived under the assumption that $\nabla \cdot \mathbf{B} = 0$. Although these relations are always satisfied in real electric and magnetic systems, they do not always hold in systems that are conventionally simplified for the purpose of calculations. Thus, for example, when edge or end effects are neglected, as is frequently the case with systems that otherwise would be too complicated for an exact solution, regions of space with $\nabla \times \mathbf{E} \neq 0$ or $\nabla \cdot \mathbf{B} \neq 0$ are implicitly created. For instance, if one neglects edge effects in a parallel-plate capacitor, one creates $\nabla \times \mathbf{E} \neq 0$ at the edges. This is because the electric field then experiences a sudden jump from a finite value inside the capacitor to zero just outside. Since the field changes in a direction perpendicular to the direction of the field, $\nabla \times \mathbf{E}$ is not zero anymore. Likewise, when one neglects end effects in a solenoid, one thereby creates $\nabla \cdot \mathbf{B} \neq 0$ at the solenoid's ends. Such simplifications of the systems, although usually perfectly safe, make therefore Maxwell stress tensors and stress integrals inapplicable to the systems.

This can be dramatically demonstrated by means of the following "first capacitor paradox." Consider a parallel-plate capacitor of plate separation d and depth b (into the page) with a voltage V applied to the capacitor, as shown in Fig. 1. Let us construct a closed surface ("Maxwellian surface") whose intersection with the plane of the drawing is *efgh*, and let us find the force on the volume enclosed by this surface by evaluating the Maxwell stress integral (the integral of the Maxwell stress tensor)

$$\mathbf{F} = -\frac{\epsilon_0}{2} \oint E^2 d\mathbf{S} + \epsilon_0 \oint \mathbf{E}(\mathbf{E} \cdot d\mathbf{S}) \tag{1}$$

over this surface. By symmetry, the contributions of the front and the back surfaces cancel each other and the contributions of the top and the bottom surfaces, *ef* and *gh*, also cancel. If the edge effects are neglected, which we assume to be the case, the surface *eh* contributes nothing, since $\mathbf{E} = 0$ outside the capacitor. Thus the only contribution to the force comes from the surface *fg* inside the capacitor. Since $E = V/d$ inside the capacitor, and since $\mathbf{E} \perp d\mathbf{S}$ on this surface, the force is

$$\mathbf{F} = -(\epsilon_0/2)(V^2/d^2)bt\mathbf{i}, \tag{2}$$

where $\mathbf{i}$ is a unit vector along the x axis and t is the distance between the two horizontal parts of the surface *efgh*.

Our solution is obviously wrong, since there can be no force on an empty volume. It is clear therefore that, as far as the present problem is concerned, something is wrong with Eq. (1) and, of course, with the stress tensor whose integral Eq. (1) represents.

The correct solution of the capacitor problem is given in Sec. II, where the proper use of stress equations for electric systems is discussed. In Sec. III the proper use of stress equations for magnetic systems is discussed. In Sec. IV the compatibility of various stress equations with some special electric and magnetic systems is examined.

II. ELECTRIC SYSTEMS

Although Eq. (1) can be obtained by integrating the Maxwell stress tensor for electric fields, it can also be obtained in a much simpler manner by using the vector identity

$$\oint (\mathbf{V} \cdot \mathbf{W}) d\mathbf{S} - \oint \mathbf{V}(\mathbf{W} \cdot d\mathbf{S}) - \oint \mathbf{W}(\mathbf{V} \cdot d\mathbf{S}) = \int \mathbf{V} \times (\nabla \times \mathbf{W}) dv + \int \mathbf{W} \times (\nabla \times \mathbf{V}) dv - \int \mathbf{V} \nabla \cdot \mathbf{W}\, dv - \int \mathbf{W} \nabla \cdot \mathbf{V}\, dv. \tag{3}$$

Since the force on a charge distribution in a vacuum is given by

$$\mathbf{F} = \int \rho \mathbf{E}\, dv = \int (\nabla \cdot \mathbf{D}) \mathbf{E} dv = \epsilon_0 \int (\nabla \cdot \mathbf{E}) \mathbf{E}\, dv, \tag{4}$$

Eq. (1) can be obtained from Eq. (3) by simply setting $\mathbf{V} = \mathbf{W} = \mathbf{E}$, multiplying Eq. (3) by ϵ_0, and dropping the two integrals with $\nabla \times \mathbf{E}$ (because $\nabla \times \mathbf{E} = 0$ in all real time-independent electric fields). As we shall presently see, it is this last step, dropping the integrals with $\nabla \times \mathbf{E}$, that is responsible for the capacitor paradox presented above. However, let us first examine what role is played by $\nabla \times \mathbf{E} = 0$ in deriving the Maxwell stress tensor for time-independent electric fields.

Although there are numerous variations in the deriva-

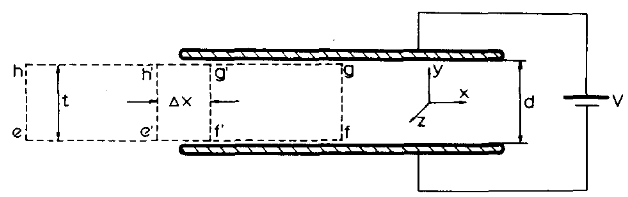

Fig. 1. Calculation of force acting on the empty space in a parallel-plate capacitor.

tion of the Maxwell stress tensor, they all belong to one of the following four basic types: (1) where the volume force $\rho\mathbf{E}$ is expressed as the divergence of a tensor[1,2]; (2) where the Maxwell stress tensor is postulated and where it is then shown that its divergence represents the volume force[3]; (3) where the electromagnetic momentum is found and then the time rate of momentum transfer is determined[4]; and (4) where the electromagnetic field tensor is derived and then the Maxwell stress tensor is obtained from it.[5]

The first two of the above methods make an explicit use of $\nabla\times\mathbf{E} = 0$. All the others also use it, although implicitly, and although it may not be easily apparent that they do. Observe, however, that in the derivations which start with time-dependent fields the Maxwell equation $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ is inevitably used. For stationary fields this equation reduces to $\nabla\times\mathbf{E} = 0$. Therefore the Maxwell stress tensor for stationary fields is, in fact, always based on $\nabla\times\mathbf{E} = 0$ whether or not this relation does explicitly appear in a particular derivation. Consequently $\nabla\times\mathbf{E} = 0$ is a fundamental assumption for Eq. (1) regardless whether it is obtained by integrating the stress tensor or directly from Eq. (3).

For practical applications the integrals of stress tensors, such as Eq. (1), are usually more important than the tensors themselves, and therefore we shall concentrate our attention here on such integrals.

Having established the role of $\nabla\times\mathbf{E} = 0$ in obtaining Eq. (1), we now suspect that by having neglected the edge effects in the system shown in Fig. 1 we have created $\nabla\times\mathbf{E}\neq 0$ and thus have made our system incompatible with Eq. (1).

As it follows from Eq. (3), the correct force equation that must be used if $\nabla\times\mathbf{E}\neq 0$ is

$$\mathbf{F} = -\frac{\epsilon_0}{2}\oint E^2 d\mathbf{S} + \epsilon_0\oint \mathbf{E}(\mathbf{E}\cdot d\mathbf{S}) + \epsilon_0\int \mathbf{E}\times(\nabla\times\mathbf{E})\,dv, \tag{5}$$

which differs from Eq. (1) by the presence of an extra integral with $\mathbf{E}\times(\nabla\times\mathbf{E})$. Let us evaluate this integral for the capacitor shown in Fig. 1. Since

$$\nabla\times\mathbf{E} = \mathbf{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right), \tag{6}$$

and since for our system $\mathbf{E} = -(V/d)\mathbf{j}$ inside the capacitor and $\mathbf{E} = 0$ outside, we have in the region $e'f'g'h'$ of width Δx

$$\nabla\times\mathbf{E} = \mathbf{k}\frac{\partial E_y}{\partial x} = \mathbf{k}\frac{E_{\text{inside}} - E_{\text{outside}}}{\Delta x} \tag{7}$$

or

$$\nabla\times\mathbf{E} = -\mathbf{k}(V/d\Delta x). \tag{8}$$

Taking into account that the average value of $\mathbf{E}$ inside $e'f'g'h'$ is

$$\mathbf{E}_{\text{av}} = -\tfrac{1}{2}(V/d)\mathbf{j} \tag{9}$$

and observing that the volume of the region is $tb\Delta x$, we obtain

$$\begin{aligned}\epsilon_0\int \mathbf{E}\times(\nabla\times\mathbf{E})dv &= \epsilon_0[\mathbf{E}_{\text{av}}\times(\nabla\times\mathbf{E})]tb\Delta x\\ &= -\frac{\epsilon_0}{2}\frac{V}{d}\left(-\frac{V}{d\Delta x}\right)tb\Delta x\mathbf{j}\times\mathbf{k}\\ &= \frac{\epsilon_0}{2}\left(\frac{V}{d}\right)^2 tb\mathbf{i}.\end{aligned} \tag{10}$$

When this correction term is added to Eq. (2) it makes the force equal to zero, thus resolving the capacitor paradox.

From Eq. (10) we can obtain a useful edge effect correction for parallel-plate capacitors that can be used in the future without repeated calculations of $\nabla\times\mathbf{E}$. This can be done as follows. If the edge effects in Fig. 1 are not neglected, the correction term given by Eq. (10) is due to the field outside the capacitor and represents the contribution of the *ef* and *gh* surfaces to the second integral of Eq. (1). Therefore the contribution of just one of these surfaces, e.g., *gh*, is one half that given by Eq. (10), or

$$\mathbf{F} = (\epsilon_0/2)(V/d)^2 t'b\mathbf{i}, \tag{11}$$

where t' is the distance between the part of the Maxwellian surface "experiencing" the edge effect force and the horizontal midplane of the capacitor.

As an illustration of the use of Eq. (11) let us find the force with which the edge field of a parallel-plate capacitor tends to pull the edges of the capacitor plates away from the capacitor. Let the capacitor be as shown in Fig. 2 (depth *b*, as before), and let us construct a Maxwellian surface *efgh* so that the part *ef* is at infinity and the part *eh* almost touches the upper plate. The horizontal force on the portion of the lower plate enclosed by this surface, as calculated from Eqs. (1) and (10), is then the sum of the contributions due to the surface *gh* (internal field)

$$F_{x1} = -(\epsilon_0/2)(V^2/d)b \tag{12}$$

and due to the surface *he* (edge effect)

$$F_{x2} = \frac{\epsilon_0}{2}\left(\frac{V}{d}\right)^2(t')b = \frac{\epsilon_0}{2}\left(\frac{V}{d}\right)^2\left(\frac{d}{2}\right)b = \frac{\epsilon_0}{4}\frac{V^2}{d}b, \tag{13}$$

the contributions of the surfaces *ef* and *fg* being zero because they are in the regions where the field is essentially absent. Thus the force is

$$F_x = F_{x1} + F_{x2} = -(\epsilon_0 V^2/4d)b. \tag{14}$$

(There is of course also a vertical component of the force on the plate, but it is of no interest to us in this example.)

We shall now turn to another problem where Maxwell stress tensor or integral may give mysterious results unless the problem is approached with some care.

Consider the capacitor shown in Fig. 3. As before, let the separation between the plates and the depth of the capaci-

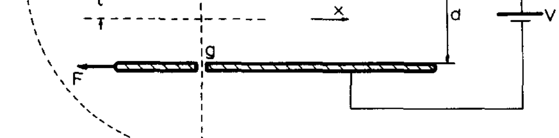

Fig. 2. Calculation of force acting on the edge of a plate in a parallel-plate capacitor.

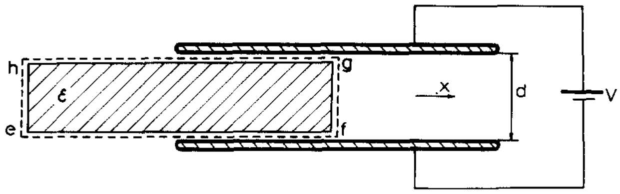

Fig. 3. Calculation of force acting on a dielectric inserted into a parallel-plate capacitor.

tor be d and b, respectively. Let there be a dielectric of dielectric constant ϵ and thickness t partially inserted into the capacitor. If we now try to find the force on this dielectric by neglecting the edge effects and using the corrected force equation, Eq. (5), we are confronted with the "second capacitor paradox."

Indeed, let us calculate the force by applying Eq. (5) to the Maxwellian surface *efgh* shown in Fig. 3. The field at the surface *fg* is V/d, exactly as in the empty capacitor of Fig. 1. The field in the dielectric at the capacitor edge is also V/d, exactly as in the empty capacitor; therefore $\nabla\times\mathbf{E}$ and $\mathbf{E}_{av}$ at the edge are the same, too. Hence the force as calculated from Eq. (5) will be also exactly the same as for the empty capacitor, that is, $\mathbf{F}=0$! This result is, of course, absurd, since in reality the dielectric is pulled into the capacitor.

To resolve this paradox we shall need to examine two things: first, the range of applicability of Eqs. (1) or (5), second, how closely the capacitor with the dielectric present, but end effects neglected, approximates the same capacitor with the edge effects not discarded.

Let us examine in some detail how Eqs. (1) and (5) are derived. Crucial to the derivation is the relation

$$\mathbf{F}=\epsilon_0\int(\nabla\cdot\mathbf{E})\mathbf{E}\,dv. \tag{15}$$

For a vacuum $\epsilon_0\nabla\cdot\mathbf{E}=\nabla\cdot\mathbf{D}=\rho$. For a dielectric, however, $\epsilon_0\nabla\cdot\mathbf{E}=\nabla\cdot(\mathbf{D}-\mathbf{P})=\rho+\rho_P$, where $\mathbf{P}$ is the polarization of the dielectric, ρ is the real electric charge, and $\rho_P=-\nabla\cdot\mathbf{P}$ is the "equivalent polarization charge" (the charge in a vacuum by which the dielectric can be replaced for calculating electric forces acting on it or for calculating fields produced by the dielectric[6]). Therefore Eqs. (1) and (5) are equally well applicable to systems containing dielectrics and to systems without dielectrics. However, when Eqs. (1) and (5) are used, the dielectrics must be treated only in terms of the equivalent polarization charges ρ_P because that is the only way in which dielectrics are represented by Eqs. (1) or (5) (another representation and equation for dielectrics will be discussed in Sec. IV).

Let us now consider what happens to the system shown in Fig. 3 when the edge effects are neglected. To make the problem more general, we shall assume that the dielectric is of some thickness $t<d$, as shown in Fig. 4(a). If the edge effects are not neglected and the dielectric is represented by the equivalent polarization charges, the system is as shown in Fig. 4(b), with the polarization charges present even outside the capacitor. But if the edge effects are neglected, the system degenerates to that shown in Fig. 4(c), with the polarization charges confined to the interior of the capacitor.

It is now clear why we have obtained a zero force for the system of Fig. 3. Having neglected the edge effects, we have altered $\nabla\cdot\mathbf{P}$ and have placed all polarization charges inside

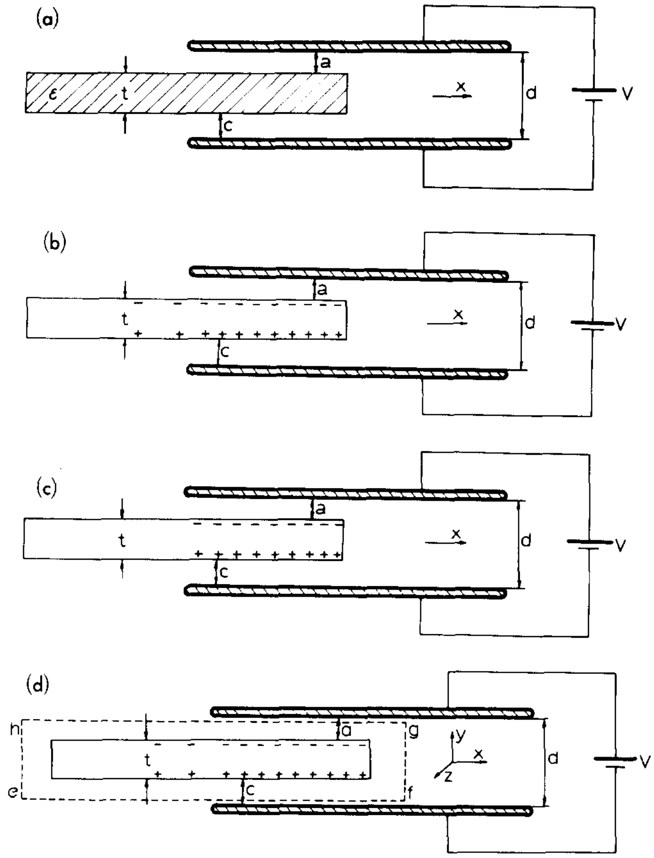

Fig. 4. (a) A dielectric in a parallel-plate capacitor. (b) The dielectric is replaced by equivalent polarization charges. (c) If edge effects are neglected, there are no polarization charges outside the capacitor. (d) Correct system for calculating the force on the dielectric.

the capacitor, that is, inside a homogeneous field. But a charge distribution in a homogeneous field cannot experience a force perpendicular to that field, and hence the force that we have found is zero. Thus, to obtain the correct force, we must represent our system as shown in Fig. 4(d). (This does not mean that one may not ignore edge effects when calculating forces from energy. In the present example the energy is then calculated as a function of penetration of the dielectric into the capacitor, and the edge effects are merely left out of the calculation, since for a sufficiently long dielectric the field configuration outside the capacitor is not affected by the degree of penetration.)

To find the force on the dielectric shown in Fig. 4(d), we need to know the electric fields inside and outside the dielectric. They are calculated by using the boundary condition $\epsilon_0E_{\text{vacuum}}=\epsilon_0\epsilon E_{\text{dielectric}}$ together with the voltage relation $E_{\text{dielectric}}t+E_{\text{vacuum}}(a+c)=V$. The field in the capacitor above and below the dielectric (not near its right edge) is

$$\mathbf{E}_a=\mathbf{E}_c=\{-\epsilon V/[t+\epsilon(d-t)]\}\mathbf{j}; \tag{16}$$

the field in the dielectric (also not near its right edge) is

$$\mathbf{E}_t=\{-V/[t+\epsilon(d-t)]\}\mathbf{j}; \tag{17}$$

and the field in the right side of the capacitor is

$$\mathbf{E}=(-V/d)\mathbf{j}. \tag{18}$$

The Maxwellian surface is *efgh*. Note that the horizontal parts of this surface are close to the plates and the vertical part *fg* is reasonably far from the dielectric; this is needed in order to avoid the region of the unknown inhomogeneous field in the vicinity of the dielectric's edge.

Unfortunately it is not at all clear how to take into account the polarization charges outside the capacitor, if the edge effects are neglected. Therefore we shall leave the edge field intact and shall solve the problem by using a modified version of Eq. (11) (another, more direct, solution of the problem will be given in Sec. IV).

Observe that the contribution made by a given part of a Maxwellian surface to the force depends only on the field at that surface. Therefore the contributions of the surfaces *ef* and *hg* in our system are exactly the same as would be made by these surfaces if they were in an empty parallel-plate capacitor having the same field and the same edge effects (field configuration) as actually present at these surfaces. But this field is

$$\mathbf{E}' = \mathbf{E}_a = \mathbf{E}_c = \{ -\epsilon V/[t + \epsilon(d - t)]\}\mathbf{j}. \tag{19}$$

The configuration of the edge field in an empty parallel-plate capacitor is controlled by the separation of the plates. Hence an empty capacitor with the field given by Eq. (19) and with the same edge effects as are actually present in the system under consideration must have a plate separation

$$d' = [t + \epsilon(d - t)]/\epsilon. \tag{20}$$

Therefore we can find the force on our dielectric by using Eq. (1) and two edge-effect forces (one for surface *ef*, the other for surface *hg*) given by Eq. (11) with V/d in it replaced by E' and t' replaced by $d'/2$. The result is

$$\mathbf{F} = -\frac{\epsilon_0}{2}\left(\frac{V}{d}\right)^2 bd\mathbf{i} + 2\frac{\epsilon_0}{2}\left[\frac{\epsilon V}{t + \epsilon(d - t)}\right]^2 \frac{[t + \epsilon(d - t)]}{2\epsilon} b\mathbf{i} \tag{21}$$

or, after obvious simplifications,

$$\mathbf{F} = \frac{\epsilon_0(\epsilon - 1)V^2 bt}{2d\,[t + \epsilon(d - t)]}\mathbf{i}, \tag{22}$$

which is the correct expression for the force.

Finally, let us consider one more system for which Eq. (5) rather than Eq. (1) must be used although no edge effects are involved. The system is the so-called "slot-effect" electret transducer shown in Fig. 5. It consists of an electret of thickness t placed between a grounded bottom plate and a parallel slotted electrode located at a distance d above the electret. If a voltage V is applied across the slot, the electret experiences a horizontal force. Our problem is to find this force.

Let the remanent polarization of the electret be $\mathbf{P}_r$ and let the real surface charge on the horizontal surfaces of the electret be $\pm\sigma$. It is customary to describe electrets in terms of their "effective surface charge" defined as

$$\sigma_e = \sigma + P_r, \tag{23}$$

and we shall follow this custom here. The induced polarization of the electret is

$$\mathbf{P}_i = \epsilon_0(\epsilon - 1)\mathbf{E}_1, \tag{24}$$

where ϵ is the permittivity of the electret material and $\mathbf{E}_1$ is the electric field in the electret (region 1). The boundary conditions for $\mathbf{D}$ at the top surface of the electret require that (unless otherwise stated we shall speak here only of the y components of the fields)

$$D_2 - D_1 = \sigma, \tag{25}$$

where D_2 is the displacement outside the electret (region 2) and D_1 is the displacement inside the electret (region 1). Since $\mathbf{D} = \mathbf{P} + \epsilon_0\mathbf{E}$, and since $\mathbf{P}$ inside the electret is $\mathbf{P}_1 = \mathbf{P}_r + \mathbf{P}_i$, while $\mathbf{P}_2 = 0$ outside, Eqs. (25), (24), and (23) yield

$$\epsilon_0 E_2 - \epsilon_0\epsilon E_1 = \sigma_e. \tag{26}$$

Designating the fields in the left half of the system by subscript l, and those in the right half by subscript r, we also have for the voltages between the bottom and top electrodes

$$E_{1l}t + E_{2l}d = -\tfrac{1}{2}V \tag{27}$$

and

$$E_{1r}t + E_{2r}d = \tfrac{1}{2}V. \tag{28}$$

Solving Eqs. (28), (27), and (26), we obtain for the fields

$$E_{1l} = -\frac{2\sigma_e d + \epsilon_0 V}{2\epsilon_0(\epsilon d + t)}, \tag{29}$$

$$E_{1r} = -\frac{2\sigma_e d - \epsilon_0 V}{2\epsilon_0(\epsilon d + t)}, \tag{30}$$

$$E_{2l} = \frac{\sigma_e}{\epsilon_0} - \frac{\epsilon(2\sigma_e d + \epsilon_0 V)}{2\epsilon_0(\epsilon d + t)}, \tag{31}$$

$$E_{2r} = \frac{\sigma_e}{\epsilon_0} - \frac{\epsilon(2\sigma_e d - \epsilon_0 V)}{2\epsilon_0(\epsilon d + t)}. \tag{32}$$

By inspection we also have for regions 3 and 4

$$E_3 = -V/2(d + t) \tag{33}$$

and

$$E_4 = V/2(d + t). \tag{34}$$

Let us now find the horizontal force on the electret by applying Eq. (1) to the Maxwellian surface *efgh* shown in Fig. 5. Rewritten for the x component, Eq. (1) is

$$F_x = -\frac{\epsilon_0}{2}\oint E^2 dS_x + \epsilon_0\oint E_x(\mathbf{E}\cdot d\mathbf{S}). \tag{35}$$

By symmetry, the contributions of the surfaces *eh* and *fg* to the two integrals of Eq. (35) cancel each other. The contribution from *ef* is zero, because there is neither dS_x nor E_x at this surface. Thus the only nonvanishing contribution is made by the surface *hg* and is made to the second integral, because on the surface *hg* there is E_x due to the voltage V applied to the slot. However, this E_x is different from zero only near the slot, e.g., between the points h' and g'. Hence we have for the force

$$F_x = \epsilon_0\int_{h'g'} E_x(\mathbf{E}\cdot d\mathbf{S}). \tag{36}$$

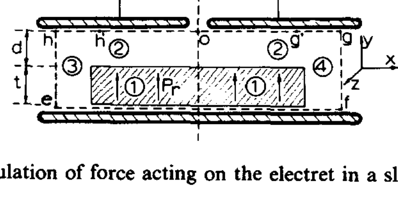

Fig. 5. Calculation of force acting on the electret in a slot-effect transducer.

But on this surface $\mathbf{E}\cdot d\mathbf{S} = E_{2l}dS$ in the left half of the system (between h' and 0), and $\mathbf{E}\cdot d\mathbf{S} = E_{2r}\, dS$ in the right half (between 0 and g'). Since E_{2l} and E_{2r} are constant, and since $dS = b\, dx$ (as before b is the length, or depth, of the electrodes into the page), we have

$$F_x = \epsilon_0 E_{2l} b \int_{h'}^{0} E_x\, dx + \epsilon_0 E_{2r} b \int_{0}^{g'} E_x\, dx. \quad (37)$$

The two line integrals represent the potential differences $\varphi_{h'} - \varphi_0$ and $\varphi_0 - \varphi_{g'}$, respectively. By symmetry, these potential differences are just $V/2$. Using these relations we obtain from Eq. (37) upon substituting E_{2l} and E_{2r} and simplifying

$$F_x = V\sigma_e tb/(\epsilon d + t). \quad (38)$$

This result is however absolutely wrong. In fact it does not agree with the expression obtained for the force (derived by a different method) when the slot effect was first described,[7] and does not agree with the actual force measurements.[8]

The error in the above calculations is a subtle one. It is due to the fact that in obtaining our solutions for the fields in the left and right halves of the system we did not allow these fields to change gradually across the middle plane of the system. Instead, we made them jump suddenly from one value to another. Thereby we have inadvertently created $\nabla\times\mathbf{E}$ at the middle plane, and have made our system incompatible with Eq. (1). To obtain the correct solution we should have used Eq. (5). However, since Eq. (5) differs from Eq. (1) only by the presence of the volume integral with $\nabla\times\mathbf{E}$, we can obtain the correct solution by merely evaluating this integral and adding it to Eq. (38).

Assuming that the fields at the middle plane change over a short distance Δx, we find, essentially as in the previous examples,

$$\nabla\times\mathbf{E}_1 = \mathbf{k}\frac{E_{1r} - E_{1l}}{\Delta x} = \mathbf{k}\frac{V}{(\epsilon d + t)\Delta x}, \quad (39)$$

$$\mathbf{E}_{1\text{av}} = \mathbf{j}\frac{E_{1r} + E_{1l}}{2} = -\mathbf{j}\frac{\sigma_e d}{\epsilon_0(\epsilon d + t)}, \quad (40)$$

$$\nabla\times\mathbf{E}_2 = \mathbf{k}\frac{E_{2r} - E_{2l}}{\Delta x} = \mathbf{k}\frac{\epsilon V}{(\epsilon d + t)\Delta x}, \quad (41)$$

$$\mathbf{E}_{2\text{av}} = \mathbf{j}\frac{E_{2r} + E_{2l}}{2} = \mathbf{j}\frac{\sigma_e t}{\epsilon_0(\epsilon d + t)}. \quad (42)$$

The volume integral of Eq. (5) contributes a force

$$\mathbf{F}' = \epsilon_0 \int \mathbf{E}\times(\nabla\times\mathbf{E})dv = \epsilon_0[\mathbf{E}_{1\text{av}}\times(\nabla\times\mathbf{E}_1)]tb\Delta x + \epsilon_0[\mathbf{E}_{2\text{av}}\times(\nabla\times\mathbf{E}_2)]db\Delta x, \quad (43)$$

which, upon substituting Eqs. (39), (40), (41), and (42) reduces to

$$F'_x = (\epsilon - 1)V\sigma_e tbd/(\epsilon d + t)^2. \quad (44)$$

Adding now Eq. (44) to Eq. (38), we finally obtain the correct result

$$F_x = V\sigma_e bt(t + d)/(\epsilon d + t)^2 \quad (45)$$

for the slot-effect force.

III. MAGNETIC SYSTEMS

Just like forces in electric systems, forces in magnetic systems can be calculated by using a Maxwell stress tensor or stress integral. The stress integral can be obtained by integrating the stress tensor or, more directly, from the vector identity given by Eq. (3). Since the force on a current distribution in a vacuum can be expressed as

$$\mathbf{F} = \int \mathbf{J}\times\mathbf{B}\, dv = \int (\nabla\times\mathbf{H})\times\mathbf{B}\, dv = \mu_0 \int (\nabla\times\mathbf{H})\times\mathbf{H}\, dv, \quad (46)$$

we obtain from Eq. (3) by setting in it $\mathbf{V} = \mathbf{W} = \mathbf{H}$, multiplying it by μ_0, and dropping the two integrals with $\nabla\cdot\mathbf{H}$ [because in a vacuum $\nabla\cdot\mathbf{H} = (\nabla\cdot\mathbf{B})/\mu_0 = 0$]

$$\mathbf{F} = -\frac{\mu_0}{2}\oint H^2\, d\mathbf{S} + \mu_0 \oint \mathbf{H}(\mathbf{H}\cdot d\mathbf{S}). \quad (47)$$

Let us use Eq. (47) to calculate the force on the volume enclosed by a cylindrical Maxwellian surface *efgh* in the solenoid shown in Fig. 6. The current in the solenoid is I, the length is l, and the number of turns is n. Let the radius of the Maxwellian surface be r. If the end effects of the solenoid are neglected, the field inside the solenoid is

$$\mathbf{H} = (nI/l)\mathbf{i} \quad (48)$$

and the field outside the solenoid is zero. By the geometry of the system, only the vertical parts of *efgh* can then make a contribution to the force. But on *eh* the field is zero, thus the only contribution comes from *fg* and is (noting that $\mathbf{H}$ is parallel to $d\mathbf{S}$ on this surface)

$$\mathbf{F} = \mathbf{i}(\mu_0/2)(nI/l)^2\pi r^2. \quad (49)$$

Once again we get a force on an empty space! The cause of this "solenoid paradox" is the fact that in deriving Eq. (47) we have assumed $\nabla\cdot\mathbf{H} = 0$, whereas by neglecting the end effects of the solenoid we have inadvertently created $\nabla\cdot\mathbf{H}\neq 0$ at the solenoid end. Indeed, since

$$\nabla\cdot\mathbf{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}, \quad (50)$$

we have at the left end of the solenoid

$$\nabla\cdot\mathbf{H} = \frac{(H_{\text{inside}} - H_{\text{outside}})}{\Delta x} = \frac{nI}{l\Delta x}, \quad (51)$$

where Δx is the distance over which the change in $\mathbf{H}$ takes place. Had we retained the integrals with $\nabla\cdot\mathbf{H}$, the force equation would have been

$$\mathbf{F} = -\frac{\mu_0}{2}\oint H^2\, d\mathbf{S} + \mu_0 \oint \mathbf{H}(\mathbf{H}\cdot d\mathbf{S}) - \mu_0 \int \mathbf{H}(\nabla\cdot\mathbf{H})dv. \quad (52)$$

For the solenoid under consideration the last integral in

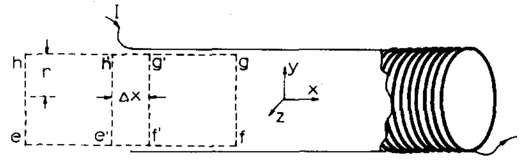

Fig. 6. Calculation of force acting on the empty space in a solenoid.

Eq. (52) is (only the region $e'f'g'h'$ contributes)

$$\mu_0 \int \mathbf{H}(\nabla\cdot\mathbf{H})dv = \mathbf{i}\mu_0 H_{av}(\nabla\cdot\mathbf{H})\pi r^2 \Delta x$$

$$= \mathbf{i}\mu_0 \frac{nI}{2l}\cdot\frac{nI}{l\Delta x}\pi r^2 \Delta x$$

$$= \mathbf{i}\frac{\mu_0}{2}\left(\frac{nI}{l}\right)^2 \pi r^2, \qquad (53)$$

where $H_{av} = nI/2l$ is the average magnetic field in $e'f'g'h'$. Subtracting Eq. (53) from Eq. (49) we obtain $\mathbf{F} = 0$, the correct answer.

Before going any farther, we must determine whether or not Eq. (47) could have possibly been derived without assuming $\nabla\cdot\mathbf{H} = 0$. In particular, we must clarify whether this assumption is necessary if Eq. (47) is obtained by integrating the Maxwell stress tensor for magnetic fields.

As in the case of electric fields, there are several variations for deriving the stress tensor.[2-5] But all these variations make use of the Maxwell equation $\nabla\cdot\mathbf{B} = 0$, which for a vacuum reduces to $\nabla\cdot\mathbf{H} = 0$. Thus $\nabla\cdot\mathbf{H} = 0$ (or, more properly, $\nabla\cdot\mathbf{B} = 0$) is already built into the Maxwell stress tensor and therefore Eq. (47) is based on this relation no matter how it is derived. Hence, it is Eq. (52), rather than Eq. (47), that must be used for calculating forces in magnetic systems where, for whatever reason, $\nabla\cdot\mathbf{H} = 0$ does not hold.

It should be noted however, that magnetic systems are usually much more difficult to treat in terms of the Maxwell stress tensor or stress integral than electric systems. In magnetic systems there is no counterpart of perfect conductors within which the fields are zero and at the surface of which they are normal. Therefore it is usually more difficult to find Maxwellian surfaces for which the direction and magnitude of magnetic fields are everywhere known. For this reason one should use for magnetic force calculations only those surfaces on which the fields are very well defined, because otherwise it is difficult to determine what correction terms are needed and where they should be used.

Consider, for example, the system shown in Fig. 7. It consists of a large solenoid into which a small solenoid is partially inserted. The current, the number of turns, the radius, and the length of the large and the small solenoids are I_1, n_1, r_1, l_1, and I_2, n_2, r_2, and l_2, respectively. Let us find the force with which the small solenoid is pulled in (or expelled from) the large solenoid. If we try to find the force by enclosing the entire small solenoid into the Maxwellian surface and using Eq. (52), we end up with incomprehensible results: on such a surface there are too many regions where the fields are not clearly defined and where $\nabla\cdot\mathbf{H}$ is in doubt. The best way to solve this problem is to use a cylindrical Maxwellian surface enclosing only the winding of one of the solenoids but not the empty space within the solenoids. Such a surface is shown in Fig. 7. Let us neglect the end effects of the large solenoid and let us apply Eq. (47) to this surface. The vertical parts of the surface can be made as small as we please and therefore make no contribution to the force. By symmetry, the first integral makes no contribution, either. The x component of the magnetic field due to the large solenoid is present at the interior surface only and is

$$H_{1x} = n_1 I_1 / l_1. \qquad (54)$$

The x components of the field due to the small solenoid are the same on both surfaces, so that their effect cancels (the normal component of the field also being the same on both surfaces). Thus the x component of the force is due to the second integral of Eq. (47) evaluated over the interior surface and is

$$F_x = \mu_0 \int H_{1x}(\mathbf{H}\cdot d\mathbf{S}) = H_{1x}\int \mu_0 \mathbf{H}\cdot d\mathbf{S} = H_{1x}\int \mathbf{B}\cdot d\mathbf{S}, \qquad (55)$$

where we have factored out the constant H_{1x} and replaced $\mu_0\mathbf{H}$ with $\mathbf{B}$. Since $d\mathbf{S}$ has the direction of an inward normal, $\int\mathbf{B}\cdot d\mathbf{S}$ is the negative of the magnetic flux escaping through the wall of the large solenoid. But this flux is entirely due to the enclosed end of the small solenoid (we have neglected the end effects of the large solenoid and thereby have made its field strictly horizontal). If the large solenoid is sufficiently long and if the small solenoid is sufficiently far inside it, the escaping flux is equal to the total flux of the small solenoid so that

$$\int \mathbf{B}\cdot d\mathbf{S} = \mp \mu_0 H_2 \pi r_2^2, \qquad (56)$$

where H_2 is the field produced by the small solenoid ("+" sign is used if H_2 is opposite to H_1). Thus the force on the small solenoid, being opposite to that on the large one, is

$$F_x = \pm \mu_0 (n_1 I_1 / l_1)(n_2 I_2 / l_2)\pi r_2^2. \qquad (57)$$

[Observe that although we had neglected the end effects of the large solenoid, we did not need Eq. (52) because the regions with $\nabla\cdot\mathbf{H} \neq 0$ were not enclosed by the Maxwellian surface.]

Let us next consider still another type of error that may be encountered when calculating magnetic forces from the Maxwell stress tensor or integral.

Let us suppose now that instead of the small solenoid, a plunger of permeability μ and radius $r_2 < r_1$ is inserted into the large solenoid, as shown in Fig. 8. To find the force on it we would probably try Eq. (52) with some reasonable expression for $\nabla\cdot\mathbf{H}$ at the left end of the solenoid. However, strange as it may be, Eq. (52) cannot possibly give us the correct solution for the problem, because it is totally incompatible with the system under consideration! Indeed, a

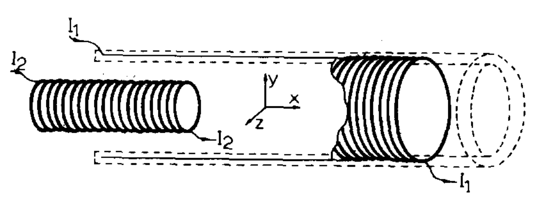

Fig. 7. Calculation of force acting on a small solenoid inserted into a large one.

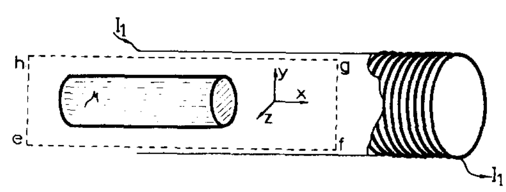

Fig. 8. Calculation of force acting on a plunger inserted into a solenoid.

crucial step in deriving Eq. (47) was the use of the relation

$$\mathbf{F}=\mu_0\int(\nabla\times\mathbf{H})\times\mathbf{H}\,dv. \tag{58}$$

But this relation is valid only for forces exerted by a magnetic field on currents in a vacuum and does not represent forces on magnetic materials. How then can one find the force on the plunger by using a Maxwell stress equation? We shall give the answer in Sec. IV.

IV. DIFFERENT TYPES OF STRESS INTEGRALS

We shall now derive a number of stress integrals designed for specific electric and magnetic systems. Our starting equation will always be the vector identity given by Eq. (3). We shall use, however, different force equations, depending on the system to be considered. In these equations ρ, ρ_P, and $\mathbf{P}$ are real charges, polarization charges, and polarization vector, respectively; J, J_M, and $\mathbf{M}$ are real current, magnetization current, and magnetization vector, respectively.

A. Stress integrals for electric systems

(1) Force on real charges in a vacuum:

$$\mathbf{F}=\int\rho\mathbf{E}\,dv=\frac{1}{\epsilon_0}\int(\nabla\cdot\mathbf{D})\mathbf{D}\,dv. \tag{59}$$

Setting in Eq. (3) $\mathbf{V}=\mathbf{W}=\mathbf{D}$, we obtain

$$\mathbf{F}=-\frac{1}{2\epsilon_0}\oint D^2 d\mathbf{S}+\frac{1}{\epsilon_0}\oint\mathbf{D}(\mathbf{D}\cdot d\mathbf{S})+\frac{1}{\epsilon_0}\int\mathbf{D}\times(\nabla\times\mathbf{D})dv. \tag{60}$$

(2) Force on real charges only, even if they are embedded in dielectrics:

$$\mathbf{F}=\int\rho\mathbf{E}\,dv=\int(\nabla\cdot\mathbf{D})\mathbf{E}\,dv. \tag{61}$$

Setting in Eq. (3) $\mathbf{V}=\mathbf{E}$, $\mathbf{W}=\mathbf{D}$, we obtain

$$\mathbf{F}=-\oint(\mathbf{E}\cdot\mathbf{D})d\mathbf{S}+\oint\mathbf{E}(\mathbf{D}\cdot d\mathbf{S})+\oint\mathbf{D}(\mathbf{E}\cdot d\mathbf{S})+\int[\mathbf{E}\times(\nabla\times\mathbf{D})+\mathbf{D}\times(\nabla\times\mathbf{E})-\mathbf{D}\nabla\cdot\mathbf{E}]dv. \tag{62}$$

(3) Force on real charges and/or dielectrics (considered as equivalent charges) in a vacuum:

$$\mathbf{F}=\int(\rho+\rho_P)\mathbf{E}\,dv=\epsilon_0\int(\nabla\cdot\mathbf{E})\mathbf{E}\,dv. \tag{63}$$

Setting in Eq. (3) $\mathbf{V}=\mathbf{W}=\mathbf{E}$, we obtain

$$\mathbf{F}=-\frac{\epsilon_0}{2}\oint E^2 d\mathbf{S}+\epsilon_0\oint\mathbf{E}(\mathbf{E}\cdot d\mathbf{S})+\epsilon_0\int\mathbf{E}\times(\nabla\times\mathbf{E})dv. \tag{64}$$

(4) Force on dielectrics in a vacuum due to an external field $\mathbf{E}'$:

$$\mathbf{F}=\int\rho_P\,\mathbf{E}'\,dv=-\int(\nabla\cdot\mathbf{P})\mathbf{E}'dv. \tag{65}$$

Setting in Eq. (3) $\mathbf{V}=\mathbf{E}'$, $\mathbf{W}=\mathbf{P}$, we obtain

$$\mathbf{F}=\oint(\mathbf{P}\cdot\mathbf{E}')d\mathbf{S}-\oint\mathbf{P}(\mathbf{E}'\cdot d\mathbf{S})-\oint\mathbf{E}'(\mathbf{P}\cdot d\mathbf{S})-\int[\mathbf{P}\times(\nabla\times\mathbf{E}')+\mathbf{E}'\times(\nabla\times\mathbf{P})-\mathbf{P}(\nabla\cdot\mathbf{E}')]dv. \tag{66}$$

B. Stress integrals for magnetic systems

(1) Force on real currents in a vacuum:

$$\mathbf{F}=\int\mathbf{J}\times\mathbf{B}\,dv=\mu_0\int(\nabla\times\mathbf{H})\times\mathbf{H}\,dv. \tag{67}$$

Setting in Eq. (3) $\mathbf{V}=\mathbf{W}=\mathbf{H}$, we obtain

$$\mathbf{F}=-\frac{\mu_0}{2}\oint H^2 d\mathbf{S}+\mu_0\oint\mathbf{H}(\mathbf{H}\cdot d\mathbf{S})-\mu_0\int\mathbf{H}(\nabla\cdot\mathbf{H})dv. \tag{68}$$

(2) Force on real currents only, even if they are within magnetic materials:

$$\mathbf{F}=\int\mathbf{J}\times\mathbf{B}\,dv=\int(\nabla\times\mathbf{H})\times\mathbf{B}\,dv. \tag{69}$$

Setting in Eq. (3) $\mathbf{V}=\mathbf{H}$, $\mathbf{W}=\mathbf{B}$, we obtain

$$\mathbf{F}=-\oint(\mathbf{H}\cdot\mathbf{B})d\mathbf{S}+\oint\mathbf{H}(\mathbf{B}\cdot d\mathbf{S})+\oint\mathbf{B}(\mathbf{H}\cdot d\mathbf{S})+\int[\mathbf{H}\times(\nabla\times\mathbf{B})-\mathbf{H}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{H})]dv. \tag{70}$$

(3) Force on real currents and/or magnetic materials (considered as equivalent currents)[9] in a vacuum

$$\mathbf{F}=\int(\mathbf{J}+\mathbf{J}_M)\times\mathbf{B}\,dv=\frac{1}{\mu_0}\int(\nabla\times\mathbf{B})\times\mathbf{B}\,dv. \tag{71}$$

Setting in Eq. (3) $\mathbf{V}=\mathbf{W}=\mathbf{B}$, we obtain

$$\mathbf{F}=-\frac{1}{2\mu_0}\oint B^2 d\mathbf{S}+\frac{1}{\mu_0}\oint\mathbf{B}(\mathbf{B}\cdot d\mathbf{S})-\frac{1}{\mu_0}\int\mathbf{B}(\nabla\cdot\mathbf{B})dv. \tag{72}$$

(4) Force on magnetic materials in a vacuum due to an external field $\mathbf{B}'$

$$\mathbf{F}=\int\mathbf{J}_M\times\mathbf{B}'\,dv=\frac{1}{\mu_0}\int(\nabla\times\mathbf{M})\times\mathbf{B}'\,dv. \tag{73}$$

Setting in Eq. (3) $\mathbf{V}=\mathbf{M}$, $\mathbf{W}=\mathbf{B}'$ we obtain

$$\mathbf{F}=-\frac{1}{\mu_0}\oint(\mathbf{M}\cdot\mathbf{B}')d\mathbf{S}+\frac{1}{\mu_0}\oint\mathbf{M}(\mathbf{B}\cdot d\mathbf{S})+\frac{1}{\mu_0}\oint\mathbf{B}'(\mathbf{M}\cdot d\mathbf{S})+\frac{1}{\mu_0}\int[\mathbf{M}\times(\nabla\times\mathbf{B}')-\mathbf{M}(\nabla\cdot\mathbf{B}')-\mathbf{B}'(\nabla\cdot\mathbf{M})]dv. \tag{74}$$

(Other similar equations can naturally be derived in an analogous manner.)

We shall illustrate the use of the above equations with three examples.

As the first example, consider a parallel-plate capacitor of plate area S and plate separation d with a voltage V applied between the plates. Constructing a tightly fitting Maxwellian surface around one of the plates (top plate, for example) and using Eq. (64) we find that the attractive force between the plates is

$$F=(\epsilon_0/2)(V/d)^2S. \tag{75}$$

Suppose now that the capacitor is submerged in a liquid dielectric. Since the electric field between the plates remains the same, Eq. (64) gives exactly the same force as before, which clearly is a wrong result. The reason for the error is in the fact that Eq. (64) is incompatible with the second system—its purpose is to find forces on bodies located in a vacuum. What we need is the force on a body located in a dielectric, and that force is given by Eq. (62). The contribution of the surface integrals of Eq. (62) to the force on the top plate is (assuming that the y axis is directed upward)

$$\mathbf{F}' = -\epsilon_0\epsilon(V/d)^2 S\mathbf{j}. \tag{76}$$

There is, however, also a contribution from the volume integral:

$$\mathbf{F}'' = -\int \mathbf{D}\nabla\cdot\mathbf{E}\,dv. \tag{77}$$

Since inside the metal plates $\mathbf{E} = 0$ while inside the capacitor $\mathbf{E} = -(V/d)\mathbf{j}$ (assuming that the top plate is positive), we have at the inner surface of the top plate $\nabla\cdot\mathbf{E} = (V/d)\Delta y$, where Δy is the distance over which $\mathbf{E}$ changes. Since the average $\mathbf{D}$ within this distance is

$$\frac{\mathbf{D}_{\text{plate}} + \mathbf{D}_{\text{capacitor}}}{2} = \frac{-\epsilon_0\epsilon V}{2d}\mathbf{j}, \tag{78}$$

we have

$$\mathbf{F}'' = (\epsilon_0\epsilon/2)(V/d)^2\mathbf{j}, \tag{79}$$

so that the total force, $\mathbf{F}' + \mathbf{F}''$, is

$$\mathbf{F} = -(\epsilon_0\epsilon/2)(V/d)^2 S\mathbf{j}, \tag{80}$$

which is the correct result.

As the second example consider once again the capacitor with dielectric shown in Fig. 4(d). An equation compatible with the system is Eq. (66). Since $\mathbf{P} = 0$ outside the dielectric, there is no contribution from the surface integrals. The only contribution comes from the volume integral. If we do not neglect the edge effects of the capacitor, $\nabla\times\mathbf{E}' = 0$. Also, $\nabla\cdot\mathbf{E}' = \nabla\cdot\mathbf{D}'/\epsilon_0 = 0$ since there are no real charges in the space within the capacitor. Thus the force is all due to the term $\mathbf{E}'\times(\nabla\times\mathbf{P})$. There are two regions where $\nabla\times\mathbf{P}\neq 0$: the right edge of the dielectric, and the edge of the capacitor. By Eq. (17), the polarization in the dielectric is

$$\mathbf{P} = \epsilon_0(\epsilon - 1)\mathbf{E} = -\frac{\epsilon_0(\epsilon-1)V}{t + \epsilon(d-t)}\mathbf{j}. \tag{81}$$

Assuming that it changes to zero over a small distance Δx both at the right edge of the dielectric and at the edge of the capacitor, we have for the two regions

$$\nabla\times\mathbf{P} = \pm\frac{\epsilon_0(\epsilon-1)V}{[t+\epsilon(d-t)]\Delta x}\mathbf{k}. \tag{82}$$

The external field at the right edge is $\mathbf{E}' = -(V/d)\mathbf{j}$, and at the edge of the capacitor it is $\mathbf{E}'_{\text{av}} = -(V/2d)\mathbf{j}$ (the average between the field inside and outside the capacitor). Hence the force is

$$\mathbf{F} = \left\{\left(\frac{V}{d}\right)\frac{\epsilon_0(\epsilon-1)V}{[t+\epsilon(d-t)]\Delta x} - \left(\frac{V}{2d}\right)\frac{\epsilon_0(\epsilon-1)V}{[t+\epsilon(d-t)]\Delta x}\right\}\Delta x b t\,(\mathbf{j}\times\mathbf{k}) \tag{83}$$

or

$$\mathbf{F} = \frac{\epsilon_0(\epsilon-1)V^2 bt}{2d\,[t+\epsilon(d-t)]}\mathbf{i}, \tag{84}$$

which is identical with Eq. (22).

As the last example, consider once again the force on the plunger in the solenoid shown in Fig. 8. An equation compatible with the system is Eq. (74). On the Maxwellian surface $efgh$, $\mathbf{M} = 0$. Thus all surface integrals are zero. If the end effects of the large solenoid are not neglected, $\nabla\cdot\mathbf{B}' = 0$. Since there are no currents inside the solenoid, $\nabla\times\mathbf{B}' = 0$. Thus the force is

$$\mathbf{F} = -\frac{1}{\mu_0}\int \mathbf{B}'(\nabla\cdot\mathbf{M})dv. \tag{85}$$

Just as $\nabla\times\mathbf{P}$ in the preceding example, $\nabla\cdot\mathbf{M}\neq 0$ at the right end of the plunger and at the edge of the solenoid. Since the field in the plunger is $\mathbf{H} = \mathbf{i}n_1 I_1/l_1$, the magnetization is $\mathbf{M} = \mathbf{i}\mu_0(\mu-1)n_1 I_1/l_1$ inside the plunger. It is zero outside. Hence, in the two regions,

$$\nabla\cdot\mathbf{M} = \mp\mu_0(\mu-1)n_1 I_1/l_1\Delta x. \tag{86}$$

The original field in the solenoid is $\mathbf{B}' = \mathbf{i}\mu_0 n_1 I_1/l_1$, and at the edge of the solenoid $\mathbf{B}'_{\text{av}} = \mathbf{i}\mu_0 n_1 I_1/2l_1$ (the average field). The force is therefore

$$\mathbf{F} = -\frac{1}{\mu_0}\{(\mu_0 n_1 I_1/l_1)[-\mu_0(\mu-1)n_1 I_1/l_1\Delta x] + (\mu_0 n_1 I_1/2l_1)[\mu_0(\mu-1)n_1 I_1/l_1\Delta x]\}\Delta x\pi r_2^2\mathbf{i} \tag{87}$$

or

$$\mathbf{F} = \frac{\mu_0(\mu-1)}{2}\left(\frac{n_1 I_1}{l_1}\right)^2\pi r_2^2\mathbf{i}. \tag{88}$$

V. CONCLUSIONS

Maxwell stress equations for electric and magnetic fields frequently become invalid when certain, usually perfectly safe, approximations are used. Typical examples of such approximations are simplified fields which, for the purpose of calculations, are assumed to experience sudden changes that do not occur in reality. The effects of such approximations on Maxwell stress equations are subtle and are frequently overlooked. The situation may be further complicated by the fact that different stress equations are compatible with some systems but are incompatible with the others. One must be careful, therefore, to select only those stress equations that apply to the system under consideration. Even then, correction terms may be needed to compensate for certain approximations. The correction terms are, however, easy to identify and easy to compute. Therefore the need for them should not be considered as an impediment to the use of the stress equations. On the contrary, the correction terms actually constitute a useful tool which allows one to use the stress equations on a scale much larger than that which is possible without them.

[1]W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Cambridge, MA, 1955), pp. 95–97.

[2]R. Becker, *Electromagnetic Fields and Interactions* (Blaisdell, New York, 1964), Vol. I, pp. 130, 131 (electric tensor), pp. 224, 225 (magnetic tensor).

[3]D. J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Englewood Cliffs, NJ, 1981), pp. 292, 293.

[4]E. J. Konopinski, *Electromagnetic Fields and Relativistic Particles* (McGraw-Hill, New York, 1981), pp. 160–164.

[5]J. B. Marion, *Classical Electromagnetic Radiation* (Academic, New York, 1965), pp. 418, 419, 437, 439.

[6]The charge ρ_P is better known in a different context as the "bound charge" of a dielectric. Its use as the "equivalent polarizaiton charge" for calculating fields and forces in the presence of dielectric media is discussed in detail in O. D. Jefimenko, *Electricity and Magnetism* (Appleton-Century-Crofts, New York, 1966), pp. 247–256.

[7]O. D. Jefimenko, Proc. W. Va. Acad. Sci. **40**, 345–348 (1968).

[8]O. D. Jefimenko and D. K. Walker, Proc. W. Va. Acad. Sci. **40**, 334–344 (1968).

[9]The use of "equivalent magnetization currents" for calculating magnetic field and forces is discussed in detail in Ref. 6, pp. 476–480.

Causality paradoxes and nonparadoxes: Classical superluminal signals and quantum measurements

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A potential causality paradox in quantum mechanics involving the measurement of a correlated system of two atoms that have moved far apart is constructed along the lines of a classical causality paradox involving the exchange of superluminal signals between distant observers. It is shown that the quantum measurement example does not lead to a paradox, but that the results have implications for the interpretation of quantum mechanics. The thought experiments presented have been used in undergraduate courses in special relativity and quantum mechanics.

I. INTRODUCTION

Paradoxes are part of the established lore of special relativity and of quantum mechanics. Discussions on how to resolve them have deepened our insight into the two theories. They also point to some of the fundamental features of these theories in such a provocative and graphic way that they make stimulating classroom presentations and discussions. Well-known examples include the twin paradox[1] and the pole-and-barn paradox[2] in special relativity, and the paradoxes of Schrödinger's cat[3] and Wigner's friend[4] in quantum mechanics.

Causality paradoxes are especially fascinating. These are paradoxes in which someone travels to the past or sends signals to the past, so that the conditions which made possible the departure or transmission may be changed: the paradigm is "killing your own grandfather as a boy." It has long been known that one may be able to signal the past if it is possible to send superluminal messages.[5] Whether the paradoxes that arise as a consequence are real or only apparent has been a matter of controversy, depending to some extent upon the properties with which one endows the hypothetical signal.[5–7]

A potential superluminal signal arises from the collapse of wave functions in quantum-mechanical measurements. If a measurement is made on one of a pair of widely separated but correlated particles, a resulting instantaneous collapse of the wave function of the pair might be expected to have some instant influence on the other particle. One would like to know whether such an effectively infinite-velocity influence can be used to construct a causality paradox.

In this article we analyze two parallelly constructed sets of thought experiments. One set involves the exchange of classical superluminal signals (tachyons), and the other involves the collapse of correlated wave functions in a quantum measurement. The tachyon experiments are described in Sec. II; the quantum measurement experiments are described in Sec. III. Several possible interpretations of the quantum measurement results are discussed in Sec. IV. A brief summary of the conclusions is given in Sec. V.

The thought experiments described here have been used in undergraduate courses in special relativity and quantum mechanics.[8] We have chosen to use specific numbers in the examples, which keep the arithmetic simple and which stress the symmetry between the two observers. One could, of course, choose different numbers or proceed algebraically instead.

II. A CLASSICAL TACHYON PARADOX

Hypothetical classical superluminal particles can be used to construct a causality paradox if they have certain properties.[5] We make the following assumptions[9]:

(1) Pulses of tachyons with any speed $v > c$ can be transmitted and received by observers in any of the inertial reference frames envisioned in special relativity.[10]

(2) Transmitted pulses can be modulated with the sender's call sign, so there is no confusion as to who sends a signal and who receives it.[11]

(3) All signals are strong and unequivocal.[12]

(4) The human observers who transmit and receive can equally well be preprogrammed automata, so that ques-

Erratum: "Correct use of Maxwell stress equations for electric and magnetic fields" [Am. J. Phys. 51, 988–996 (1983)]

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Several equations between Eqs. (44) and (45) have been left out of the article, "Correct use of Maxwell stress equations ... ." The equations are analogous to Eqs. (39)–(44) and produce correction terms due to $\nabla \times \mathbf{E}$ at the boundaries (3, 1), (3, 2), (2, 4), and (1, 4) of Fig. 5, where the vertical field is discontinuous. Two more groups of equations produce correction terms due to $\nabla \times \mathbf{E}$ created by the horizontal field E_x; this field generates $\nabla \times \mathbf{E}$ throughout regions 1 and 2. When all the correction terms are added to Eq. (38), one obtains Eq. (45).

SOLUTION TO THE PROBLEM ON PAGE 437

In Fig. 1, *CD* is the virtual image of the trousers *AB*. *CD* is of the same length as *AB* and is situated at the same distance x behind the mirror as *AB* is in front of it. The angle subtended by the trousers' image is given by

$$\theta = \alpha - \beta = \tan^{-1}(h/2x) - \tan^{-1}[(h-l)/2x].$$

For the best view, we maximize θ from conditions $d\theta/dx = 0$ and $d^2\theta/dx^2 < 0$. After a little algebra, we get

$$x = \tfrac{1}{2}[h(h-l)]^{1/2}.$$

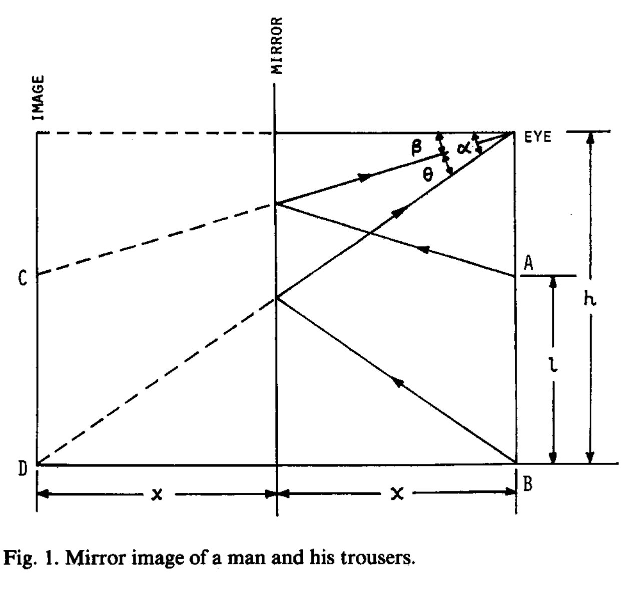

Fig. 1. Mirror image of a man and his trousers.

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Comment on "Correct use of Maxwell stress equations for electric and magnetic fields"

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"This result is however absolutely wrong." Thus, in his recent article,[1] Jefimenko brands his Eq. (38)

$$F_x = V\sigma_e tb/(\epsilon d + t) \tag{C1}$$

for the horizontal force F_x on the charged dielectric (or electret) in a slit-effect device according to his Fig. 5. Unfortunately, it is not mentioned in Ref. 1 that Eq. (C1) was reported by several authors[2–5] to be the correct force equation for the slit-effect transducer of Fig. 5 [Eq. (15) of Ref. 2, Eq. (11) of Ref. 4, and Eq. (16) of Ref. 5]. In some of these articles,[2,3] Jefimenko's early slit-effect papers[6–8] are directly criticized and shown to be partly inconsistent.

It will be demonstrated in the following that the derivation of the supposedly correct[1] Eq. (45) is muddled by a simple sign error as well as by an unjustified omission of the electret-edge contributions to the total force. From the result of these calculations, it will become clear that Eq. (C1) is indeed the correct force equation for the particular slit-effect arrangement discussed here, as long as only the electret contribution to the horizontal force is considered (cf. Refs. 2–5). The fundamental aspects of force calculations on dielectrics are not discussed in the present context.

In Ref. 1, the horizontal force F_x calculated from the horizontal field E_x [Eq. (38)] is "corrected" by addition of a force F'_x [Eq. (44)] resulting from the edge effects at the middle plane between the left and the right halves of the device. Correct addition of Eqs. (38) and (44), however, results in

$$F_x = V\sigma_e bt(2\epsilon d - d + t)/(\epsilon d + t)^2, \tag{C2}$$

which differs from Jefimenko's[1] Eq. (45) by $2F'_x$ [Eq. (44)] and does not agree with his original force expression.[6] Furthermore, Eq. (C2) does not contain the nonvanishing contributions from the edge effects of the electret itself [between regions 3 and 2 as well as 3 and 1 in the left half of the device and between regions 2 and 4 as well as 1 and 4 in its right half (cf. Fig. 5 of Ref. 1)].

Following the formalism proposed in Jefimenko's paper,[1] these additional contributions to the horizontal force can be easily calculated. In analogy to the derivation of Eq. (44), we obtain (1) for the lower left (ll) edge effect between regions 3 and 1, using Eqs. (29) and (33),

$$F''_{ll} = -\frac{\epsilon_0 V^2 dtb(\epsilon^2 d - d + 2\epsilon t - 2t)}{8(\epsilon d + t)^2(d + t)^2} + \frac{tb(\sigma_e d\epsilon_0 V + \sigma_e^2 d^2)}{2\epsilon_0(\epsilon d + t)^2}; \tag{C3}$$

(2) for the lower right (lr) edge effect between regions 1 and 4, using Eqs. (30) and (34),

$$F''_{lr} = +\frac{\epsilon_0 V^2 dtb(\epsilon^2 d - d + 2\epsilon t - 2t)}{8(\epsilon d + t)^2(d + t)^2} + \frac{tb(\sigma_e d\epsilon_0 V - \sigma_e^2 d^2)}{2\epsilon_0(\epsilon d + t)^2}; \tag{C4}$$

(3) for the upper left (ul) edge effect between regions 3 and 2, using Eqs. (31) and (33),

$$F''_{ul} = +\frac{\epsilon_0 V^2 dtb(2\epsilon^2 d + \epsilon^2 t - 2\epsilon d - t)}{8(\epsilon d + t)^2(d + t)^2} - \frac{db(\sigma_e \epsilon t\epsilon_0 V - \sigma_e^2 t^2)}{2\epsilon_0(\epsilon d + t)^2}; \text{ and} \tag{C5}$$

(4) for the upper right (ur) edge effect between regions 2 and 4, using Eqs. (32) and 34),

$$F''_{ur} = -\frac{\epsilon_0 V^2 dtb(2\epsilon^2 d + \epsilon^2 t - 2\epsilon d - t)}{8(\epsilon d + t)^2(d + t)^2} - \frac{db(\sigma_e \epsilon t\epsilon_0 V + \sigma_e^2 t^2)}{2\epsilon_0(\epsilon d + t)^2}. \tag{C6}$$

Addition of equations (C3)–(C6) yields the total horizontal-force contribution F''_x of the electret edges

$$F''_x = F''_{ll} + F''_{lr} + F''_{ul} + F''_{ur} = -(\epsilon - 1)V\sigma_e tbd/(\epsilon d + t)^2, \tag{C7}$$

which exactly cancels the contribution F'_x of the edge effects at the middle plane of the transducer [Eq. (44)].

Thus, by adding Eqs. (38), (44), and (C7) [or Eqs. (C2) and (C7)], the horizontal force F_x on the electret in Jefimenko's[1] slit-effect device is found to be correctly represented by Eq. (38). The result demonstrates that the correct use of the Maxwell stress equations leads to the same slit-effect equations as the use of other theoretical approaches.[2–5] Jefimenko's[1] observation, that sudden field changes have to be treated carefully in Maxwell stress formulations, is strongly supported by this agreement.

In an independent approach, Eq. (38) [or (C1)] can also be obtained by restricting the integration volume to regions 1 and 2 only (cf. Fig. 5 of Ref. 1). In this case, the contributions of the left (l) and right (r) vertical surfaces of the integration volume do not cancel each other. Their sum F''_x is given by

$$F''_x = -\frac{\epsilon_0}{2}\left(\oint (E_r^2 - E_l^2)\,dS_x\right) = -\frac{\epsilon_0}{2} b\,[d(E_{2r}^2 - E_{2l}^2) + t(E_{1r}^2 - E_{1l}^2)] = -(\epsilon - 1)V\sigma_e tdb/(\epsilon d + t)^2. \tag{C8}$$

This result, which was calculated by use of Eqs. (29)–(32) and (35), has to be added to Eq. (C2) in order to find the total horizontal force. Because Eqs. (C7) and (C8) are identical, the addition again yields the same force equation [(38) or (C1)].

In conclusion, the correct use of Maxwell stress equations according to Jefimenko's[1] suggestion leads to the same slit-effect equations as the application of other principles,[2–5] if all edge effects are properly included. Thus Jefimenko's original force expression[6] as well as all its later variations[1,7–10] have to be corrected as stated before.[2,3,5]

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[1]O. D.Jefimenko, Am. J. Phys. **51**, 988 (1983).
[2]T. B. Jones, IEEE Trans. Acoust. Speech Signal Process. **ASSP-22**, 141 (1974); **ASSP-23**, 498 (1975).
[3]J. F. Hoburg and J. R. Melcher, IEEE Trans. Acoust. Speech Signal Process. **ASSP-23**, 500 (1975).
[4]G. Morgenstern, Appl. Phys. **11**, 371 (1976).
[5]R. Gerhard-Multhaupt, J. Phys. D **17**, 649 (1984).
[6]O. Jefimenko, Proc. W. V. Acad. Sci. **40**, 345 (1968).
[7]O. Jefimenko and D. K. Walker, Proc. W. V. Acad. Sci. **40**, 338 (1968).
[8]O. D. Jefimenko, IEEE Trans. Acoust. Speech Signal Process. **ASSP-23**, 497 (1975).
[9]A. Abazi and O. D. Jefimenko, J. Appl. Phys. **54**, 4076 (1983).
[10]O. D. Jefimenko and A. Abazi, Rev. Sci. Instrum. **53**, 1746 (1982).

Response to "Comment on 'Correct use of Maxwell stress equations for electric and magnetic fields' "

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It is true that Eq. (45) in the paper under consideration[1] is improperly derived. Several intermediate equations similar to Eqs. (39)–(44) have been inadvertently left out and should be inserted between Eqs. (44) and (45). However, the error has no effect on any equations or on any conclusions reached in the paper. In particular, it does not change the fact that Eq. (38),

$$F_x = V\sigma_e tb/(\epsilon d + t), \tag{R1}$$

does not represent any force in a slot-effect transducer (except when $d = 0$) and that Eq. (45),

$$F_x = V\sigma_e tb(t + d)/(\epsilon d + t)^2, \tag{R2}$$

is the correct equation for the force on the electret in the transducer shown in Fig. 5 of Ref. 1.

To restore the missing equations without an unnecessary duplication of computations, we shall make use of some of the calculations presented in Gerhard-Multhaupt's comment. As it is shown in the Comment, after four more regions with $\nabla\times\mathbf{E}\neq 0$ [boundaries (3,1) (3,2), (2,4), and (1,4)] are taken into account, the correction terms cancel the contribution of $\nabla\times\mathbf{E}$ at the middle plane, so that Eq. (R1) is obtained once again. There is, however, one additional region where $\nabla\times\mathbf{E}\neq 0$. This region is the entire volume of the transducer! The additional $\nabla\times\mathbf{E}$ is created there by the horizontal component of the field E_x as a consequence of the simplifying assumptions about the vertical components E_y that have been made for deriving Eqs. (29)–(34) of Ref. 1.[2]

Let us consider in some detail the role and the behavior of E_x in the slot-effect transducer. First of all, we notice that according to the basic force equation

$$\mathbf{F} = \int (\rho + \rho_p)\,\mathbf{E}\,dv, \tag{R3}$$

the electret will not experience a horizontal force unless there is a horizontal component of the field at the location of the electret [represented in Eq. (R3) by the real and polarization charges ρ and ρ_p].[3] Second, we notice that $E_x = 0$ at the bottom electrode, because only a normal field can exist at the surface of a conductor. Third, we notice that since E_y is constant in all subregions of the transducer shown in Fig. 5 of Ref. 1, E_x may not be a function of x; otherwise

$$\nabla\cdot\mathbf{E} = \frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y \tag{R4}$$

will not vanish, so that electric charges will be present throughout the transducer.

Thus, as a result of approximations used for the vertical field, we have created a unidirectional field E_x which varies in a direction perpendicular to the field direction. In such a field $\nabla\times\mathbf{E}\neq 0$. Furthermore, since E_x is not a function of x, it satisfies the relation

$$E_x L = \phi_a - \phi_b, \tag{R5}$$

where ϕ_a and ϕ_b are the potentials of two points located at the same distance y above the bottom electrode ($\phi = 0$), and where L is the horizontal distance between these points. But since the vertical components of the field are constant in their respective regions, $\phi_a - \phi_b$ is a linear function of y. Hence E_x is also a linear function of y. Therefore $\nabla\times(E_x\mathbf{i})$, or, for simplicity, $\nabla\times\mathbf{E}_{(x)}$, must be constant throughout regions 1_l, 1_r, 2_l, and 2_r.

To find $\nabla\times\mathbf{E}_{(x)}$, we proceed as follows. The potential of a point on the top surface at the left edge of the electret is $\phi_l = -E_{1l}t$. The potential of a point on the top surface at the right edge is $\phi_r = -E_{1r}t$. Hence the potential difference is

$$\phi_l - \phi_r = (E_{1r} - E_{1l})t, \tag{R6}$$

or, substituting Eqs. (29) and (30) of Ref. 1,

$$\phi_l - \phi_r = Vt/(\epsilon d + t). \tag{R7}$$

Remembering that E_x is a linear function of y and using Eqs. (R5) and (R7), we therefore have for $\nabla\times\mathbf{E}_{(x)}$ in regions 1 and 2

$$\begin{aligned}\nabla\times\mathbf{E}_{1(x)} &= -\mathbf{k}(E_{x,y=t} - E_{x,y=0})/t \\ &= -\mathbf{k}V/L(\epsilon d + t),\end{aligned} \tag{R8}$$

$$\nabla\times\mathbf{E}_{2(x)} = -\mathbf{k}(E_{x,y=d+t} - E_{x,y=t})/d$$
$$= -\mathbf{k}\epsilon V/L(\epsilon d + t), \quad \text{(R9)}$$

where L is the length of the electret. The correction term due to $\nabla\times\mathbf{E}_{(x)}$ is then

$$\mathbf{F}' = \epsilon_0 \int \mathbf{E}\times[\nabla\times\mathbf{E}_{(x)}]\, dv, \quad \text{(R10)}$$

where the integration needs to be extended only over regions 1 and 2, since regions 3 and 4 can be made as small as one pleases. We have therefore

$$F'_x = -\epsilon_0\{[E_{1l}V/L(\epsilon d + t)]btL/2 + [E_{1r}V/L(\epsilon d + t)]btL/2 + [E_{2l}\epsilon V/L(\epsilon d + t)]bdL/2 + [E_{2r}\epsilon V/L(\epsilon d + t)]bdL/2\}, \quad \text{(R11)}$$

which, after substituting Eqs. (29)–(32), becomes

$$F'_x = -(\epsilon - 1)V\sigma_e tbd/(\epsilon d + t)^2. \quad \text{(R12)}$$

Adding Eq. (R12) to Eq. (R1), we finally obtain Eq. (R2), which is the same as Eq. (45) of Ref. 1 and the same as the force equation originally reported by the author.[4]

The slot-effect transducer was used in Ref. 1 merely as an illustration of the proper application of the Maxwell stress integral to this particular system. For practical purposes one can find the force on the electret much easier by using the basic force relation Eq. (R3) directly. Since this method also reveals the true nature of Eq. (R1), we shall demonstrate it here.[5]

Observe that, in contrast to Maxwell stress equations, Eq. (R3) does not depend on $\nabla\times\mathbf{E}$ or on $\mathbf{E}$ outside the charge distribution, so that fewer approximations are needed when this equation is used. Observe also that Maxwell stress equations for electric fields are derived directly or indirectly from Eq. (R3) or from its differential form. Therefore Eq. (R3) must give either a more accurate or an equally accurate result as that obtained from the Maxwell stress equations.

To apply Eq. (R3) to our system, we replace the electret by an equivalent surface charge distribution[3] $\sigma_t = \sigma + \sigma_r + \sigma_i$, where σ is the real surface charge, σ_r is the remanent polarization surface charge, and σ_i is the induced polarization surface charge ($\sigma_r + \sigma_i = -\mathbf{P}\cdot\mathbf{n}_{in}$, where $\mathbf{P} = \mathbf{P}_r + \mathbf{P}_i$ is the sum of the remanent and induced polarization of the electret, and $\mathbf{n}_{in}$ is a unit vector directed into the electret). For the left and the right half of the top surface of the electret (other surfaces do not contribute to the force) these charges are given by the "surface divergence" of $\mathbf{E}$

$$\sigma_{tl} = E_{2l} - E_{1l}, \quad \text{(R13)}$$

and

$$\sigma_{tr} = E_{2r} - E_{1r}, \quad \text{(R14)}$$

where the fields are given by Eqs. (29)–(32) of Ref. 1.

The force can now be found by evaluating

$$F_x = \int \rho_t E_x\, dv = \int \sigma_t E_x\, dS = \int_l^r \sigma_t E_x b\, dx$$
$$= \sigma_{tl}\, b\int_l^m E_x\, dx + \sigma_{tr}\, b\int_m^r E_x\, dx, \quad \text{(R15)}$$

where l indicates the left edge of the electret surface, r indicates the right edge of the electret surface, and m is the middle point of the surface directly below point o in Fig. 5 of Ref. 1. The last two integrals are just the potential differences $\phi_l - \phi_m$ and $\phi_m - \phi_r$, each equal to $(\phi_l - \phi_r)/2$ by the assumed symmetry of the system. Combining Eqs. (R13), (R14), (R15), and (R7), we promptly obtain Eq. (R2) once again.

It is interesting to note that by the same method of direct force calculation one can also obtain Eq. (R1) if instead of the total charge $\sigma_t = \sigma + \sigma_r + \sigma_i$ one uses just $\sigma + \sigma_r$ (thus ignoring the induced charge σ_i). Consequently, Eq. (R1), which Gerhard-Multhaupt erroneously assumes to be correct, represents merely an imperfect version of Eq. (R2).

The original "criticism" of Eq. (R2) to which Gerhard-Multhaupt refers was prompted by the failure of the critics[6–8] to derive Eq. (R2). They derived Eq. (R1) instead, which they obtained by an improper use of energy relations,[6] by an improper use of Maxwell stress tensor,[6–8] and by falsely supposing that Eq. (R3) was less reliable than Maxwell stress equations.[8] As it follows from Ref. 1 and from the present discussion, this criticism was completely groundless. The newer critics[9] merely duplicated old errors and quoted some of the expletives with which the original criticism was embellished, without attempting a new analysis of the validity of methods used to obtain Eq. (R1) and Eq. (R2).

Naturally, as it was pointed out by the author from the very start,[4] Eq. (R2) is only an approximate force expression. The final decision on its validity must come either from a more detailed theoretical analysis or from accurate experimental measurements. The author is not aware of any theoretical analysis more detailed than that presented here and in Ref. 4. And the only accurate measurements of the force known to him were done by Walker and himself.[10] These measurements involved meticulous determinations of ϵ, a study of the effect of the variation of t on the force, and precise measurements of the force for a wide range of V and σ_e values. The results of the measurements were in excellent agreement with Eq. (R2).

Gerhard-Multhaupt's concluding suggestion that Eq. (R2) "as well as all its later variations will have to be corrected" is interesting, but its implementation will have to wait until somebody produces a better equation.

[1] O. D. Jefimenko, Am. J. Phys. **51**, 988 (1983).

[2] The vertical field in the transducer can not be exactly homogeneous. Such a field has horizontal equipotential surfaces that make a horizontal field impossible.

[3] O. D. Jefimenko, *Electricity and Magnetism* (Appleton-Century-Crofts, New York, 1966), pp. 245–256.

[4] O. D. Jefimenko, Proc. W. V. Acad. Sci. **40**, 345 (1968).

[5] An alternative way toward a simpler solution is to use a Maxwellian surface enclosing region 1 only.

[6] T. B. Jones, IEEE Trans. Acoust. Speech Signal Process. **ASSP-22**, 141 (1974).

[7] T. B. Jones, IEEE Trans. Acoust. Speech Signal Process. **ASSP-23**, 498 (1975).

[8] J. F. Hoburg and J. R. Melcher, IEEE Trans. Acoust. Speech Signal Process. **ASSP-23**, 500 (1975).

[9] R. Gerhard-Multhaupt, J. Phys. D. **17**, 649 (1984).

[10] O. D. Jefimenko and D. K. Walker, Proc. W. V. Acad. Sci. **40**, 338 (1968).