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Direct calculation of electric and magnetic forces from potentials

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(Received 10 July 1989; accepted for publication 31 October 1989)

A new method for calculating electric and magnetic forces is presented. The method makes it possible to calculate forces directly from scalar and vector potentials rather than from field vectors. Several new force equations are derived. They are grouped in four categories: equations for calculating electric forces from electric scalar potentials; equations for calculating electric forces from electric vector potentials; equations for calculating magnetic forces from magnetic vector potentials; and equations for calculating magnetic forces from magnetic scalar potentials. Since the potentials are usually easier to compute than the corresponding fields, the new equations provide an effective alternative to the previously available techniques for force calculations. From the theoretical point of view, these equations reveal a physical significance of electric and magnetic potentials not heretofore apparent and provide a new insight into the nature of electric and magnetic forces. Illustrative examples on the use of the equations are given.

I. INTRODUCTION

In the conventional treatments of electromagnetic theory, electric and magnetic fields are defined from the very start in terms of forces, and potentials are either defined in terms of energy or are closely associated with it. Inadvertently, these definitions or associations result in an implicit conclusion that for calculating electric and magnetic forces one should use fields, and that for calculating electric and magnetic energies one should use potentials. Even when equations later appear in which electric and magnetic energy is expressed in terms of the fields, the initial conclusion prevails. And yet, it is precisely the association of the energy with fields rather than with potentials that has resulted in the discovery of such important phenomena as the propagation of energy by Poynting's vector, electromagnetic momentum, etc. These discoveries indicate that the conventional association of energy with potentials rather than with fields is nothing more than a historical circumstance and is not demanded by the nature of electric or magnetic phenomena.

It appears plausible, therefore, that the association of electric and magnetic forces with fields is also incidental and that relations should exist by means of which forces could be calculated directly from potentials rather than from fields.

These considerations have led me to explore in some detail the relations between forces and potentials. As I proceeded with this exploration, I became convinced that a number of useful and intriguing electric and magnetic force relations have been previously overlooked. The purpose of this article is to report and to discuss these relations.

Several illustrative examples are included in the article. Most are very simple; their intent is to illustrate the application of the theory rather than to obtain new results. However, some of the examples (such as Examples 3 and 9) are relatively difficult to solve by standard means; these examples serve not only to demonstrate the use of the new equations, but also to demonstrate the power of the equations.

II. CALCULATION OF ELECTRIC FORCES FROM SCALAR POTENTIALS

The electric force on a charge distribution ρ located in an external electric field[1] $\mathbf{E}'$ is given by

$$\mathbf{F} = \int_{\text{all charge}} \rho \mathbf{E}' \, dv. \tag{1}$$

Let us write this equation as

$$\mathbf{F} = \int_{\text{surface layer}} \rho \mathbf{E}' \, dv + \int_{\text{interior}} \rho \mathbf{E}' \, dv, \tag{2}$$

where the first integral is extended over the surface layer of the charge distribution, and the second integral is extended over the interior of the charge distribution. The volume of the surface layer may be assumed as small as we please, so that unless the internal charge is enclosed within a layer of surface charge (which we assume not to be the case), the first integral may be disregarded. We then have

$$\mathbf{F} = \int_{\text{interior}} \rho \mathbf{E}' \, dv. \tag{3}$$

Let us now replace $\mathbf{E}'$ in Eq. (3) by $-\nabla\phi'$, where ϕ' is the external scalar potential at the location of ρ, and let us then transform the integrand by means of vector identity (A1) (all vector identities used in this article are listed in the Appendix). We then have

$$\mathbf{F} = \int_{\text{interior}} \rho \mathbf{E}' \, dv$$

$$= -\int_{\text{interior}} \rho \, \nabla\phi' \, dv$$

$$= \int_{\text{interior}} \phi' \, \nabla\rho \, dv - \int_{\text{interior}} \nabla(\rho\phi') dv. \tag{4}$$

If we now transform the last integral by means of vector identity (A2), we obtain

$$\mathbf{F} = \int_{\text{interior}} \phi' \, \nabla\rho \, dv - \oint_{\text{surface}} \rho\phi' \, d\mathbf{S}, \tag{5}$$

where the second integral is extended over the surface of the charge distribution.[2]

A remarkable feature of Eq. (5) is that it associates the force on a charge distribution directly with the potential rather than with the field. The equation is immediately suspect, because the potential is determined only to within an additive constant, while the force must be a single-valued quantity. However, a closer examination of the equation shows that any additive constant appearing in ϕ' integrates out and has no effect on the force.[3]

If the charge distribution is constant, the first integral in Eq. (5) vanishes, and we have

$$\mathbf{F} = -\rho \oint \phi' \, d\mathbf{S}. \tag{6}$$

If the charge is confined to a thin layer, the surface integral in Eqs. (5) and (6) can be split into the integrals over the broad surface of the layer and over the rim of the layer. The latter integral contributes to the total force an amount

$$\mathbf{F}_{\text{rim}} = -\oint \rho\phi' t \, d\mathbf{l}_{\text{out}} = -\oint \sigma\phi' \, d\mathbf{l}_{\text{out}}, \tag{7}$$

where t is the thickness of the layer, σ is the surface charge density of the layer, and $d\mathbf{l}_{\text{out}}$ is a vector representing a length element of the rim and directed out of the charge distribution at right angles to the rim.

It should be noted that the external potential ϕ' appearing in the above equations can be replaced by the total potential ϕ, because a self-potential cannot produce a net force on a charge distribution.

Example 1. A point charge q is located on the axis (z axis) of a thin-walled cylinder of uniform surface charge σ, length $2l$, and radius a. The distance between q and the center of the cylinder (assumed to be to the right of q) is z. Find the force exerted on the cylinder by the point charge.

Since the charge distribution is constant, only the surfaces of the cylinder contribute to the force, and, by the symmetry of the system, the only contribution comes from the two end surfaces (rims) of the cylinder. The external potential ϕ' at the end of the cylinder closest to the charge is[4] $\phi' = q/4\pi\epsilon_0[(z-l)^2+a^2]^{1/2}$, and that at the other end is $\phi' = q/4\pi\epsilon_0[(z+l)^2+a^2]^{1/2}$. By Eq. (7), taking into account that the integrand is a constant, the force is then

$$\mathbf{F} = \mathbf{k}\,\sigma q 2\pi a/4\pi\epsilon_0[(z-l)^2+a^2]^{1/2}$$

$$- \mathbf{k}\sigma q 2\pi a/4\pi\epsilon_0[(z+l)^2+a^2]^{1/2}$$

or

$$\mathbf{F} = \mathbf{k}\frac{q\sigma a}{2\epsilon_0}\left(\frac{1}{[(z-l)^2+a^2]^{1/2}} - \frac{1}{[(z+l)^2+a^2]^{1/2}}\right).$$

Example 2. A thin sheet of dielectric material of surface area S carrying a uniform surface charge σ is inserted between two large parallel horizontal grounded conducting plates, so that it is located at a distance a from the bottom plate and at a distance b from the top plate. Find the force on the sheet.

The sheet induces surface charges σ_1 and σ_2 on the lower and upper plate, respectively, so that

$$\sigma_1 = -\sigma b/(a+b), \quad \sigma_2 = -\sigma a/(a+b).$$

The potential produced by these charges in the space between the plates is

$$\phi' = [\sigma(b-a)/2\epsilon_0(a+b)]x,$$

where x is the distance up from the bottom plate. Let the thickness of the surface charge be t. The two surfaces of the charge are then at distances $x = a - t/2$ and $x = a + t/2$ from the bottom plate. By Eq. (6), the force is then

$$\mathbf{F} = \mathbf{i}\rho\frac{\sigma(b-a)}{2\epsilon_0(b+a)}\left(a - \frac{t}{2}\right)S$$

$$- \mathbf{i}\rho\frac{\sigma(b-a)}{2\epsilon_0(b+a)}\left(a + \frac{t}{2}\right)S,$$

or

$$\mathbf{F} = \mathbf{i}\rho t\,[\sigma(a-b)/2\epsilon_0(a+b)]S$$

$$= \mathbf{i}[\sigma^2(a-b)/2\epsilon_0(a+b)]\,S.$$

Example 3. A uniformly charged sphere of charge q and radius a consists of two separate hemispheres. Find the force between the hemispheres.

Since ρ is constant, we can use Eq. (6). However, we shall use this equation with the total potential ϕ, because ϕ' is difficult to compute. The total potential for $r \leqslant a$ is

$$\phi = (q/8\pi\epsilon_0 a^3)(3a^2 - r^2),$$

where r is the distance from the center of the sphere. Let us assume that the hemispheres are separated by a horizontal plane, and let us calculate the force on the upper hemisphere. The surface integral in Eq. (6) can be split into a part over the hemispherical surface and a part over the flat base of the upper hemisphere. Since the magnitude of $\int d\mathbf{S}$ over a hemispherical surface is just the area of the projection of the hemisphere on its base, the contribution of the hemispherical surface to the force is

$$\mathbf{F}_1 = \mathbf{i}\rho q\pi a^2/4\pi\epsilon_0 a = \mathbf{i}\rho q a/4\epsilon_0,$$

where $\mathbf{i}$ is a unit vector normal to the base and directed downward. The contribution of the base of the hemisphere is

$$\mathbf{F}_2 = -\mathbf{i}\rho\int_0^a \frac{q}{8\pi\epsilon_0 a^3}(3a^2 - r^2)2\pi r\,dr$$

$$= -\mathbf{i}\rho\left(\frac{3qa}{8\epsilon_0} - \frac{qa}{16\epsilon_0}\right) = -\mathbf{i}\frac{5\rho q a}{16\epsilon_0}.$$

The total force $\mathbf{F}_1 + \mathbf{F}_2$ is then

$$\mathbf{F} = -\mathbf{i}\rho q a/16\epsilon_0 = -\mathbf{i}3q^2/64\pi\epsilon_0 a^2.$$

III. CALCULATION OF ELECTRIC FORCES FROM VECTOR POTENTIALS

Electric fields in charge-free regions can be represented not only as gradients of scalar potentials but also as curls of vector potentials.[5]

Let us replace $\mathbf{E}'$ in Eq. (1) by $(1/\epsilon_0)\nabla\times\mathbf{A}'$, where $\mathbf{A}'$ is the vector potential due to the sources producing $\mathbf{E}'$ (the presence of ρ does not preclude the existence of the external vector potential $\mathbf{A}'$ at the location of ρ, since all sources of $\mathbf{A}'$ are outside of ρ). We have

$$\mathbf{F} = \frac{1}{\epsilon_0}\int_{\text{all charge}} \rho\nabla\times\mathbf{A}'\, dv. \quad (8)$$

Splitting the integral, as before, into an integral over the surface layer of the charge and an integral over the interior of the charge, and ignoring the first integral, we have

$$\mathbf{F} = \frac{1}{\epsilon_0}\int_{\text{interior}} \rho\,\nabla\times\mathbf{A}'\, dv. \quad (9)$$

Using now vector identity (A3), we have

$$\mathbf{F} = \frac{1}{\epsilon_0}\int_{\text{interior}} \rho\,\nabla\times\mathbf{A}'\, dv$$

$$= \frac{1}{\epsilon_0}\int_{\text{interior}} \nabla\times(\rho\mathbf{A}')dv$$

$$-\frac{1}{\epsilon_0}\int_{\text{interior}} \nabla\rho\times\mathbf{A}'\, dv,$$

and, using vector identity (A4), we obtain

$$\mathbf{F} = \frac{1}{\epsilon_0}\int_{\text{interior}} \mathbf{A}'\times\nabla\rho\, dv - \frac{1}{\epsilon_0}\oint_{\text{Surface}} \rho\mathbf{A}'\times d\mathbf{S}. \quad (10)$$

For constant ρ, Eq. (10) simplifies to

$$\mathbf{F} = -\frac{1}{\epsilon_0}\rho\oint \mathbf{A}'\times d\mathbf{S}. \quad (11)$$

For a layer of charge, the contribution of the rim of the layer to the total force is

$$\mathbf{F}_{\text{rim}} = -\frac{1}{\epsilon_0}\int \rho\mathbf{A}'\times t\, d\mathbf{l}_{\text{out}} = -\frac{1}{\epsilon_0}\int \sigma\mathbf{A}'\times d\mathbf{l}_{\text{out}}, \quad (12)$$

where t, σ, and $d\mathbf{l}_{\text{out}}$ are the same as in Eq. (7).

Note that, in contrast to the similar equations for scalar potentials, only the external vector potentials can be used in Eqs. (8)–(12), because an electric vector potential is defined only for regions of space external to the charges that produce the vector potential.

Example 4: Find the force between the plates of a parallel-plate capacitor whose plates are disks of radius a separated by a small distance d and carrying equal and opposite surface charges $\pm\sigma$.

Let us assume that the positive plate is the "force-producing" plate and that the negative plate is the "force-experiencing" plate. The vector potential due to the positive plate is, in cylindrical coordinates whose z axis coincides with the symmetry axis of the capacitor and is directed from the positive to the negative plate, $\mathbf{A}' = (\sigma/4)r\hat{\boldsymbol{\theta}}$.

Since σ is constant, Eq. (11) can be used to find the force. Only the rim of the charged layer makes a net contribution to the force (on the flat surfaces, $\mathbf{A}'\times d\mathbf{S}$ is radial and produces no net effect).

By Eq. (12), in lieu of Eq. (11), the force is then

$$\mathbf{F} = \frac{1}{\epsilon_0}\int \frac{\sigma^2}{4}a\hat{\boldsymbol{\theta}}\times d\mathbf{l}_{\text{out}} = -\mathbf{k}\frac{\sigma^2 a}{4\epsilon_0}2\pi a = -\mathbf{k}\frac{\sigma^2 S}{2\epsilon_0}.$$

[Although this expression has been derived for a capacitor with circular plates, the result is independent of the form of the plates. This can be shown by using the vector potential in rectangular coordinates $\mathbf{A}' = -\mathbf{i}\sigma y/4 + \mathbf{j}\sigma x/4$ and noting that $\frac{1}{2}\int(x\,dy - y\,dx)$ represents the area of the surface enclosed by the path of integration.]

Example 5: Find the force between a point charge q and a uniformly charged disk of charge Q and radius a located at a distance z from the point charge, if the surface of the disk is perpendicular to the line (z axis) joining the point charge with the center of the disk.

The vector potential of the point charge is, in spherical coordinates centered at the point charge,

$$\mathbf{A}' = (q/4\pi r)[(1-\cos\theta)/\sin\theta]\hat{\boldsymbol{\phi}},$$

where θ is the angle between r and the z axis. As in the preceding example, only the rim of the disk contributes to the force. Substituting $\cos\theta = z/(z^2 + a^2)^{1/2}$ and $r\sin\theta = a$, we obtain, by Eq. (12),

$$\mathbf{F} = \mathbf{k}\frac{1}{\epsilon_0}\oint \frac{\sigma q[1 - z/(z^2+a^2)^{1/2}]}{4\pi a}\, dl$$

$$= \mathbf{k}(1/\epsilon_0)\{\sigma q[1 - z/(z^2+a^2)^{1/2}]/4\pi a\}2\pi a$$

$$= \mathbf{k}(qQ/2\epsilon_0\pi a^2)[1 - z/(z^2+a^2)^{1/2}].$$

Example 6. An infinitely long line charge of density λ is placed along the z axis of rectangular coordinates. An infinite plane sheet of surface charge is placed parallel to the yz plane at the distance $x = a$ from the line charge; the center of the sheet being on the x axis. Find the force per unit length exerted by the line charge on the sheet, if the charge density of the sheet is

$$\sigma = -\sigma_0[a^2/(a^2+y^2)].$$

Since the charge is not constant, we must use Eq. (10). Assuming that the thickness of the surface charge is t, we have for $\nabla\rho$

$$\nabla\rho = \frac{1}{t}\nabla\sigma = \frac{1}{t}\sigma_0\frac{2ya^2}{(a^2+y^2)^2}\mathbf{j}.$$

The vector potential produced by the line charge is, in cylindrical coordinates,

$$\mathbf{A}' = (\lambda/2\pi)\theta\,\mathbf{k},$$

where θ is the angle around the z axis in the xy plane. By the symmetry of the system, the surface integral in Eq. (10) makes no contribution to the force (the contributions of the front and back surfaces cancel, and there is no contribution from the edges at infinity).

Expressing the gradient of the charge in terms of the angle θ, we have

$$\nabla\rho = (2\sigma_0/ta)\sin\theta\cos^3\theta\,\mathbf{j}.$$

The force per unit length is then

$$\mathbf{F}_l = -\mathbf{i}\frac{1}{\epsilon_0}\int_{-\infty}^{\infty}\frac{\lambda}{2\pi}\theta\frac{2\sigma_0}{ta}\sin\theta\cos^3\theta t\,dy$$
$$= -\mathbf{i}\frac{\lambda\sigma_0}{2\pi\epsilon_0}\int_{-\pi/2}^{\pi/2}2\theta\sin\theta\cos^3\theta\frac{d\theta}{\cos^2\theta}$$
$$= -\mathbf{i}\frac{\lambda\sigma_0}{2\pi\epsilon_0}\int_{-\pi/2}^{\pi/2}\theta\sin 2\theta\,d\theta = -\mathbf{i}\frac{\lambda\sigma_0}{4\epsilon_0}.$$

IV. CALCULATION OF MAGNETIC FORCES FROM VECTOR POTENTIAL

The magnetic force on a current distribution $\mathbf{J}$ due to an external field $\mathbf{B}'$ is given by

$$\mathbf{F} = \int_{\text{all current}} \mathbf{J}\times\mathbf{B}'\,dv. \tag{13}$$

Replacing $\mathbf{B}'$ in Eq. (13) by $\nabla\times\mathbf{A}'$, where $\mathbf{A}'$ is the external vector potential, and splitting the integral into an integral over the surface layer and an integral over the interior of the current, we have

$$\mathbf{F} = \int_{\text{surface layer}} \mathbf{J}\times(\nabla\times\mathbf{A}')\,dv + \int_{\text{interior}} \mathbf{J}\times(\nabla\times\mathbf{A}')\,dv. \tag{14}$$

As before, the integral over the surface layer can be ignored. The integral over the interior can be transformed by using vector identity (A5) to

$$\mathbf{F} = \int \mathbf{A}'(\nabla\cdot\mathbf{J})dv + \int \mathbf{J}(\nabla\cdot\mathbf{A}')dv + \oint(\mathbf{A}'\cdot\mathbf{J})d\mathbf{S} - \oint\mathbf{J}(\mathbf{A}'\cdot d\mathbf{S}) - \oint\mathbf{A}'(\mathbf{J}\cdot d\mathbf{S}) - \int\mathbf{A}'\times(\nabla\times\mathbf{J})dv. \tag{15}$$

But $\nabla\cdot\mathbf{J} = 0$, $\nabla\cdot\mathbf{A}' = 0$, and $\mathbf{J}\cdot d\mathbf{S} = 0$ (because a current is always parallel to its surface). Therefore, we have

$$\mathbf{F} = \oint(\mathbf{A}'\cdot\mathbf{J})d\mathbf{S} - \oint\mathbf{J}(\mathbf{A}'\cdot d\mathbf{S}) - \int\mathbf{A}'\times(\nabla\times\mathbf{J})dv. \tag{16}$$

We can also transform the second volume integral together with the second surface integral in Eq. (15) by using vector identity (A6). We then obtain the alternative force equation

$$\mathbf{F} = \oint(\mathbf{A}'\cdot\mathbf{J})d\mathbf{S} - \int(\mathbf{A}'\cdot\nabla)\,\mathbf{J}\,dv - \int\mathbf{A}'\times(\nabla\times\mathbf{J})dv. \tag{17}$$

From this equation we immediately see that for constant $\mathbf{J}$ the force is simply

$$\mathbf{F} = \oint(\mathbf{A}'\cdot\mathbf{J})d\mathbf{S}. \tag{18}$$

For a surface current of density $\mathbf{J}^{(s)} = t\,\mathbf{J}$ per unit width, where t is the thickness of the current, the edges (rim) of the current contribute

$$\mathbf{F}_{\text{rim}} = \oint(\mathbf{A}'\cdot\mathbf{J}^{(s)})d\mathbf{l}_{\text{out}} - \oint\mathbf{J}^{(s)}(\mathbf{A}'\cdot d\mathbf{l}_{\text{out}}) \tag{19}$$

to Eq. (16) and

$$\mathbf{F}_{\text{rim}} = \oint(\mathbf{A}'\cdot\mathbf{J}^{(s)})d\mathbf{l}_{\text{out}} \tag{20}$$

to Eqs. (17) or (18).

Although the above equations have been derived for the internal potential $\mathbf{A}'$, they remain valid if the total potential $\mathbf{A}$ is used in them, because the self-potential cannot create a net force on a current.

Example 7. A long straight wire is placed along the z axis. It carries a current I' in the direction of z. A straight current-carrying strip of length L and width a is placed parallel to the wire in the xz plane. The current in the strip is I, also in the z direction, and the distance of its front edge from the wire is d. Find the force on the strip.

We can use Eqs. (18) and (20) to solve the problem. The surface current density in the strip is $\mathbf{J}^{(s)} = \mathbf{k}\,I/a$. The vector potential produced by the wire is, in cylindrical coordinates,

$$\mathbf{A}' = -(\mu_0 I'/2\pi)\ln r\,\mathbf{k}.$$

By the symmetry of the system, the contributions of the two flat surfaces of the strip cancel each other so that only the contributions of the edges remain. By Eq. (20), noting that the integrand is a constant, we then have

$$\mathbf{F} = \mathbf{i}\frac{\mu_0 I'}{2\pi}\ln d\left(\frac{I}{a}\right)L - \mathbf{i}\frac{\mu_0 I'}{2\pi}\ln(d+a)\left(\frac{I}{a}\right)L$$
$$= \mathbf{i}\frac{\mu_0 II'L}{2\pi a}\ln\frac{d}{d+a}.$$

Example 8. A solenoid of n_1 turns, radius a, length l_1, and current I_1, is partially inserted into a larger solenoid of n_2 turns, radius b, length l_2, and current I_2. Neglecting end effects (that is, assuming that the magnetic field of each solenoid is confined to the interior of the solenoid), find the force on the smaller solenoid.

We can use Eqs. (18) and (20) to solve the problem. Let us assume that the axes of the two solenoids coincide with the z axis (directed left to right), that the smaller solenoid is to the right of the larger solenoid, and that both currents I_1 and I_2 are right-handed relative to z. The vector potential produced by the larger solenoid is then, in cylindrical coordinates,

$$\mathbf{A}' = (\mu_0 n_2 I_2/2l_2)\,r\,\hat{\boldsymbol{\theta}}.$$

By the symmetry of the system, only the end of the smaller solenoid inside the larger solenoid contributes to the force. The surface current of the smaller solenoid is $\mathbf{J}^{(s)} = (n_1 I_1/l_1)\hat{\boldsymbol{\theta}}$. Hence, the force is, by Eq. (20),

$$\mathbf{F} = \oint\frac{\mu_0 n_2 I_2}{2l_2}a\frac{n_1 I_1}{l_1}d\mathbf{l}_{\text{out}} = -\frac{\mu_0 n_1 n_2 I_1 I_2}{l_1 l_2}\pi a^2\mathbf{k}.$$

Example 9. A uniformly charged spherical shell of charge density ρ, inner radius a, and outer radius b consists of two separate hemispheres. The shell rotates with angular velocity ω about its vertical symmetry axis (z axis) passing at right angles to the equatorial plane separating the two hemispheres. Find the force between the two hemispheres.

The current density in the shell is, in spherical coordinates centered at the center of the shell, $\mathbf{J} = \rho\boldsymbol{\omega}\times\mathbf{r}$. The vector potential inside the shell is

$$\mathbf{A} = (\mu_0\rho\omega/30)(5b^2 - 3r^2 - 2a^5/r^3)r\sin\theta\,\hat{\phi}.$$

Let us find the force on the upper hemisphere. To do so we can use Eq. (17) with the total potential **A**. Since $\nabla\times\mathbf{J} = 2\rho\boldsymbol{\omega}$, $\mathbf{A}\times(\nabla\times\mathbf{J})$ does not contribute to the net force, so that only the surface integrals need to be considered. Since **J** is not a function of ϕ, the second integral vanishes. Thus the force is all due to the first integral of Eq. (17). The contribution of the flat base of the hemisphere is

$$\mathbf{F}_1 = -\mathbf{k}\int_a^b \frac{\mu_0\rho\omega}{30}\left(5b^2 - 3r^2 - \frac{2a^5}{r^3}\right) r\rho\omega r 2\pi r\,dr$$

$$= -\mathbf{k}\frac{\mu_0\rho^2\omega^2\pi}{60}(3b^6 - 5b^2a^4 - 8ba^5 + 10a^6).$$

Only the vertical component of $d\mathbf{S}$ makes a net contribution to the integral over the outer hemispherical surface. Hence this surface contributes

$$\mathbf{F}_2 = \mathbf{k}\int_0^{\pi/2} \frac{\mu_0\rho\omega}{30}\left(5b^2 - 3b^2 - \frac{2a^5}{b^3}\right) b\sin\theta\,\rho\omega b$$

$$\times\sin\theta\, 2\pi b^2\sin\theta\cos\theta\,d\theta$$

$$= \mathbf{k}(\mu_0\rho^2\omega^2\pi/30)(b^5 - a^5)b.$$

The inner surface of the upper hemisphere contributes, similarly,

$$\mathbf{F}_3 = -\mathbf{k}\int_0^{\pi/2}(\mu_0\rho\omega/6)(b^2 - a^2)a\sin\theta\,\rho\omega a$$

$$\times\sin\theta\,2\pi a^2\sin\theta\cos\theta\,d\theta$$

$$= -\mathbf{k}(\mu_0\rho^2\omega^2\pi/12)(b^2 - a^2)a^4.$$

The total force is then

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

$$= -\mathbf{k}(\mu_0\rho^2\omega^2\pi/60)(b^6 - 6ba^5 + 5a^6).$$

V. CALCULATION OF MAGNETIC FORCES FROM SCALAR POTENTIALS

Let us replace the flux density vector $\mathbf{B}'$ in Eq. (13) by $-\mu_0\nabla\phi'$, where ϕ' is the external magnetic scalar potential. Transforming the equation as before, we have

$$\mathbf{F} = \int \mathbf{J}\times\mathbf{B}'\,dv = -\mu_0\int \mathbf{J}\times\nabla\phi'\,dv$$

$$= -\mu_0\int_{\text{surface layer}}\mathbf{J}\times\nabla\phi'\,dv - \mu_0\int_{\text{interior}}\mathbf{J}\times\nabla\phi'\,dv. \quad (21)$$

Disregarding the integral over the surface layer and using vector identity (A3), we obtain

$$\mathbf{F} = -\mu_0\int_{\text{interior}}\mathbf{J}\times\nabla\phi'\,dv$$

$$= \mu_0\int_{\text{interior}}\nabla\times(\phi'\mathbf{J})dv$$

$$-\mu_0\int_{\text{interior}}\phi'\nabla\times\mathbf{J}\,dv. \quad (22)$$

The first integral on the right can be transformed by using vector identity (A4) into a surface integral, so that the force becomes

$$\mathbf{F} = -\mu_0\oint\phi'\mathbf{J}\times d\mathbf{S} - \mu_0\int\phi'\nabla\times\mathbf{J}\,dv. \quad (23)$$

For a surface current $\mathbf{J}^{(s)}$ the contribution of the rim surface is

$$\mathbf{F}_{\text{rim}} = -\mu_0\oint\phi'\mathbf{J}^{(s)}\times d\mathbf{l}_{\text{out}}. \quad (24)$$

Note that only the external potential can be used in the above equations because the magnetic scalar potential is defined only for regions of space external to the source of the potential.

Example 10. An infinitely long wire carries a current I' along the z axis of rectangular coordinates. A conducting bar of square cross section and thickness $2a$ is placed parallel to the wire so that the center line of the bar is in the xz plane at a distance d from the wire. The surfaces of the bar are parallel to the yz and xz planes. The bar carries a current I in the z direction. Find the force exerted by the wire on the bar, if the length of the bar is l.

The potential of the wire is, in rectangular coordinates,

$$\phi' = -(I'/2\pi)\tan^{-1}(y/x), \quad \text{for } y>0$$

and

$$\phi' = (I'/2\pi)\tan^{-1}(-y/x), \quad \text{for } y<0.$$

Since the current density is constant, only the surface integral in Eq. (23) needs to be used. Also, by the symmetry of the system, only the horizontal surfaces of the bar contribute to the force, both surfaces making equal contributions. Hence we have, integrating over the upper surface,

$$\mathbf{F} = -\mathbf{i}2\int_{d-a}^{d+a}\frac{I'}{2\pi}\tan^{-1}\left(\frac{a}{x}\right)\left(\frac{I}{4a^2}\right)l\,dx$$

$$= -\mathbf{i}(II'l/4\pi a^2)\{(d+a)\theta_2 - (d-a)\theta_1$$

$$+ (a/2)\ln[a^2 + (d+a)^2]$$

$$- (a/2)\ln[a^2 + (d-a)^2]\},$$

where θ_1 is the angle between the x axis and the line joining the wire with the near edge of the bar, and θ_2 is the angle between the x axis and the line joining the wire with the far edge of the bar, both lines being in the xy plane.

Example 11. A thin disk of uniform charge density ρ, radius a, and thickness t rotates with angular velocity ω about its symmetry axis, which is also the z axis of cylindrical coordinates. Also on the z axis and perpendicular to it is a distant ring of surface area S carrying a current I. The distance between the ring and the disk is z. Assuming that the current in the ring and the rotation of the disk are right-handed relative to the z axis, find the force exerted by the ring on the disk.

The potential of the ring is, in cylindrical coordinates,

$$\phi' = (IS/4\pi)[z/(z^2 + r^2)^{3/2}],$$

where z is the distance from the ring and r is the distance from the z axis.

The rotating disk constitutes a current distribution $\mathbf{J} = \rho\omega r\hat{\theta}$ for which $\nabla\times\mathbf{J} = 2\rho\omega\mathbf{k}$. Using Eqs. (23) and (24), and taking into account that, by the symmetry of the system, only the rim of the ring contributes to the surface integral, and that the potential and the current density are constant at the rim, we have

$$\mathbf{F} = \mathbf{k}\mu_0 [ISz/4\pi(z^2 + a^2)^{3/2}]\rho\omega at\, 2\pi a$$
$$- \mathbf{k}\,\mu_0 \int_0^a [2ISz\rho\omega/4\pi(z^2 + r^2)^{3/2}]\, t\, 2\pi r\, dr$$
$$= \mathbf{k}\mu_0 IS\rho\omega t\, [a^2 z/2(z^2 + a^2)^{3/2}$$
$$+ z/(z^2 + a^2)^{1/2} - 1].$$

Example 12. Find the force between the wire and the conducting strip described in Example 7.

The potential of the wire is, in cylindrical coordinates,

$$\phi' = -(I'/2\pi)\theta$$

above the xz plane and

$$\phi' = (I'/2\pi)\theta$$

below the xz plane. By the symmetry of the system, only the flat surfaces of the strip contribute to the force, each contributing the same amount. Let the thickness of the strip be $2t$. Since the strip is thin, θ at these surfaces can be expressed as t/x. The current density is $I/2ta$. By Eq. (23) we then have, integrating over the upper surface,

$$\mathbf{F} = -2\mathbf{i}\mu_0 \int_x^{x+d} \frac{I'}{2\pi} \frac{t}{x} \frac{I}{2at} L\, dx$$
$$= \mathbf{i}\, \frac{\mu_0 II'L}{2\pi a} \ln \frac{d}{d+a}.$$

VI. FORCE EQUATIONS FOR TIME-DEPENDENT FIELDS

The force equations derived in the preceding sections can be easily extended to time-dependent fields by using retarded potentials and by taking into account the following three considerations.

(1) In order to treat the forces as purely electric or purely magnetic, all force-experiencing charges should be stationary, and all force-experiencing currents should be neutral and should have zero divergence (otherwise net charges are required by the continuity condition).

(2) Instead of the scalar potentials, mixed potentials should be used for representing $\mathbf{E}'$ and $\mathbf{B}'$, that is,

$$\mathbf{E}' = -\nabla\phi_e'^* - \frac{\partial \mathbf{A}_m'^*}{\partial t},$$

and

$$\mathbf{B}' = -\mu_0\, \nabla\phi_m'^* + \mu_0 \frac{\partial \mathbf{A}_e'^*}{\partial t},$$

where the asterisks indicate that the potentials are retarded, and the subscripts e and m stand for electric and magnetic, respectively.

(3) In deriving magnetic force equations, one should take into account that $\nabla\cdot\mathbf{A}_m'^* = -\epsilon_0\mu_0\, \partial\phi_e'^*/\partial t$ (Lorentz's condition).

Using these considerations and repeating the derivations employed for obtaining Eqs. (5), (10), (16), (17), and (23), we then obtain the corresponding time-dependent equations:

$$\mathbf{F} = \int \phi_e'^*\, \nabla\rho\, dv - \int \rho\left(\frac{\partial \mathbf{A}_m'^*}{\partial t}\right) dv - \oint \rho\phi'^*\, d\mathbf{S}, \quad (5a)$$

$$\mathbf{F} = \left(\frac{1}{\epsilon_0}\right)\int \mathbf{A}_e'^* \times \nabla\rho\, dv - \left(\frac{1}{\epsilon_0}\right)\oint \rho\mathbf{A}_e'^* \times d\mathbf{S}, \quad (10a)$$

$$\mathbf{F} = \oint (\mathbf{A}_m'^*\cdot\mathbf{J})\, d\mathbf{S} - \oint \mathbf{J}(\mathbf{A}_m'^*\cdot d\mathbf{S})$$
$$- \epsilon_0\mu_0 \int \mathbf{J}\left(\frac{\partial\phi_e'^*}{\partial t}\right) dv - \int \mathbf{A}_m'^* \times (\nabla\times\mathbf{J})\, dv, \quad (16a)$$

$$\mathbf{F} = \oint (\mathbf{A}_m'^*\cdot\mathbf{J})\, d\mathbf{S} - \int \mathbf{A}_m'^* \times (\nabla\times\mathbf{J})\, dv$$
$$- \int (\mathbf{A}_m'^*\cdot\nabla)\mathbf{J}\, dv, \quad (17a)$$

$$\mathbf{F} = -\mu_0 \oint \phi_m'^* \mathbf{J}\times d\mathbf{S}$$
$$- \mu_0 \int \phi_m'^* \nabla\times\mathbf{J}\, dv + \mu_0 \int \mathbf{J}\times\left(\frac{\partial \mathbf{A}_e'^*}{\partial t}\right) dv. \quad (23a)$$

VII. CONCLUSION

The possibility of calculating electric and magnetic forces directly from potentials has been explored. Several new force equations have been derived. The equations considerably expand the methods available for electric and magnetic force calculations. By directly associating forces with electric and magnetic potentials, these equations reveal an entirely new aspect of the potentials and give them a physical significance not previously apparent.

It is clear that potentials can be used for force calculations just as well as the fields can be. There is no objective reason to assign great physical significance to equations expressing forces in terms of fields than to equations expressing forces in terms of potentials. Therefore our traditional view of electric and magnetic fields as being primarily responsible for force effects and of potentials as being primarily associated with electric and magnetic energy is merely a historical circumstance rather than a consequence of an unbiased interpretation of the nature of electric and magnetic phenomena.

The illustrative examples presented in the article provide a new insight into the nature of electric and magnetic forces. Consciously or subconsciously we associate these forces with some invisible "threads" (after Faraday's "physical lines of force") that "attach" themselves to electric charges and currents. Even if we profess to reject such a mechanical picture of the forces, we nevertheless do associate the forces with certain specific locations. For example, the force on the smaller solenoid of Example 8 is conventionally attributed to the end effects of the larger solenoid (because only then an axial force can be obtained from $\int\mathbf{J}\times\mathbf{B}'\, dv$). We say "the force *acts on the part* of the smaller solenoid *located in the end region* of the larger solenoid." But does the force really *act* there? Not according to Example 8! According to that example, the force *acts* on the *end* of the small solenoid *located well within* the large one. Or consider Examples 7 and 12. They both deal with exactly the same physical system. Yet, according to Example 7, the force is entirely due to the rim of the current-carrying strip, while according to Example 12 it is entirely due to the broad surfaces of the strip. And, of course, according to the usual equation, $\mathbf{F} = \int\mathbf{J}\times\mathbf{B}'\, dv$, the force is due to the interior of the strip.

So where exactly do electric and magnetic forces act? To what are they applied? For that matter, what are electric and magnetic forces? All we can actually say about electric and magnetic systems subjected to forces is that electric and magnetic fields affect the state of motion (or the shape) of charges and currents located in these fields. We can account for these changes by evaluating certain integrals over the interior or over the surface of charges and currents so affected. Or we can account for these changes by evaluating certain integrals (Maxwell's stress integrals) over surfaces (Maxwellian surfaces) passing through empty space around the affected charge and currents. Certainly, there is no objective reason to ascribe to any one of these integrals a greater physical significance than to any other. But then we must revise our concept of the localization of forces, and, very likely, of the electric and magnetic forces themselves. Our view of the forces as field–charge or field–current interactions appears to be not so well grounded after all. A plausible alternative is field–field interactions, the charges or currents merely manifesting these interactions but not experiencing them directly. Our force equations are then merely a means of predicting the outcome of these interactions, rather than a revelation of forces as a physical reality.[6,7]

APPENDIX: VECTOR IDENTITIES

In the following vector identities, light-face letters are scalars, boldface letters are vectors. The letters do not represent any particular electric or magnetic quantities.

$$\nabla(\varphi U) = \varphi \nabla U + U \nabla \varphi, \tag{A1}$$

$$\oint U\, d\mathbf{S} = \int \nabla U\, dv, \tag{A2}$$

$$\nabla \times (\varphi \mathbf{A}) = \varphi \nabla \times \mathbf{A} + \nabla \varphi \times \mathbf{A}, \tag{A3}$$

$$\oint \mathbf{A} \times d\mathbf{S} = -\int \nabla \times \mathbf{A}\, dv, \tag{A4}$$

$$\oint (\mathbf{A}\cdot\mathbf{B})d\mathbf{S} - \oint \mathbf{B}(\mathbf{A}\cdot d\mathbf{S}) - \oint \mathbf{A}(\mathbf{B}\cdot d\mathbf{S}) = \int [\mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) - \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})]dv, \tag{A5}$$

$$\oint \mathbf{A}(\mathbf{B}\cdot d\mathbf{S}) = \int [(\nabla\cdot\mathbf{B})\mathbf{A} + (\mathbf{B}\cdot\nabla)\mathbf{A}]dv. \tag{A6}$$

[1] All basic force equations in this article are derived for external, rather than for total, electric and magnetic fields. This is done for two reasons: First, only external fields produce net forces on charges and currents; second, only external fields can always be associated with scalar as well as with vector potentials.

[2] This formula has been previously derived (although in a different context) in Oleg D. Jefimenko *Electricity and Magnetism* (Electret Scientific, Star City, WV, 1989), 2nd ed., pp. 210 and 211. The derivation is repeated here for the sake of completeness of the presentation.

[3] This can be easily shown by replacing the potential in Eq. (5) by a constant.

[4] In order not to dilute the presentation by excessive details, the potentials are stated here without derivations.

[5] The possibility of expressing electrostatic fields by vector potentials is not well known. J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941), pp. 25–28, indicates such a possibility without, however, making any use of it.

[6] A discussion of force equations would be incomplete without a discussion of torque calculations. The force equations presented in this article cannot be used for calculating torques: The arbitrary additive constants in the potentials result in indeterminate values for the torques. However, general equations for calculating torques from potentials, similar to the corresponding force equations, can be derived by using techniques not much different from those by which the force equations have been derived here; see Oleg D. Jefimenko, "Direct calculation of electric and magnetic torques from potentials," to be submitted to Am. J. Phys.

[7] The electric force equations obtained in this article can easily be converted to gravitational force equations. This can be done by simply using mass densities and gravitational potentials (scalar or vector) in Eqs. (5)–(7) and (10)–(12) and removing ϵ_0 from Eqs. (8)–(12).

A rotating U-tube experiment

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(Received 14 July 1989; accepted for publication 24 September 1989)

A rotating U-tube experiment is described in which the axis of rotation lies between a vertical arm and the axis of symmetry of the tube. As the rotational frequency is slowly varied, the equilibrium position of the liquid in the tube can abruptly change, resulting in a hysteresis loop. The effect is not due to friction, which causes a much smaller amount of hysteresis. The data agree well with the theory.

I. INTRODUCTION

In a previous article,[1] we theoretically considered the equilibrium states of a liquid in a rotating U-tube (Fig. 1). If the frequency is slowly increased from zero, the height of the "near" end of the liquid (the end that is closer to the axis of rotation) continuously decreases. When this end is in the near corner, the liquid can forward jump to a configuration in which the near end is in the horizontal segment of the tube. If the frequency is now slowly decreased, the liquid temporarily remains in such a configuration, and then backward jumps to a configuration in which the near end is in the vertical segment of the near arm. We showed that this hysteresis occurs for a range of locations of the

