different from considering single events on the screen and then taking the average over repeated events. The present work, we hope, clarifies some of the points related to quantum interference, path integrals and single versus repeated events relationships.

[a]Professor Emeritus, Colorado State University, Fort Collins, Colorado, 500 Manhattan Drive, B4, Boulder, CO 80303


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# Solutions of Maxwell's equations for electric and magnetic fields in arbitrary media


Oleg D. Jefimenko
*Physics Department, West Virginia University, Morgantown, West Virginia 26506*



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Solutions of Maxwell's equations for fields in arbitrary media are derived in terms of charge density $\rho$, current density **J**, polarization **P**, and magnetization **M**. The solutions express **E**, **D**, **H**, and **B** as integrals of retarded $\rho$, **J**, **P**, **M**, and their spatial and temporal derivatives.


## I. INTRODUCTION

Most of the previously reported solutions of Maxwell's equations as integrals of retarded charge and current densities were limited, according to their authors, to a vacuum or to media of constant permittivity and permeability occupying all space.[1–3] Griffiths and Heald, who discussed such solutions and provided their own derivations,[4] stated in their footnote 22 that the solutions could be extended to fields in dielectric and magnetic media, if "one interprets **J** to include $\partial \mathbf{P}/\partial t$ and $\nabla\times\mathbf{M}$ in addition to the free current density $\mathbf{J}_{\text{free}}$ and $\rho$ to include $-\nabla\cdot\mathbf{P}$ in addition to the free charge density $\rho_{\text{free}}$." However, they did not actually provide or discuss the extended solutions.

The purpose of this paper is to present detailed derivations of the solutions of Maxwell's equations for fields in arbitrary media and to demonstrate their possible applications. Two sets of solutions are obtained. The first set expresses the fields **E**, **D**, **H**, and **B** in terms of retarded charge density $\rho$, retarded current density **J**, retarded polarization **P**, retarded magnetization **M** and retarded spatial and temporal derivatives of $\rho$, **J**, **P**, and **M**. The second set contains no spatial derivatives.

The solutions are general and impose no restrictions on any of the quantities involved except that **E**, **D**, **H**, and **B** are assumed to be regular at infinity and that $\rho$, **J**, **P**, and **M** are assumed to be confined to a finite region of space. Since these requirements are implicit in Maxwell's equations, the solutions are equivalent to Maxwell's equations.

## II. BASIC EQUATIONS AND DEFINITIONS

The equations to be solved are the four Maxwell's equations:

$$\nabla\cdot\mathbf{D}=\rho, \tag{1}$$

$$\nabla\cdot\mathbf{B}=0, \tag{2}$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \tag{3}$$

$$\nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}. \tag{4}$$

In order to solve these equations we also need equations correlating **D** with **E** and **B** with **H**. The most general equations of this type are those making use of the polarization **P** and magnetization **M**. They are

$$\mathbf{P}=\mathbf{D}-\epsilon_0\mathbf{E}, \tag{5}$$

$$\mathbf{M}=\mathbf{B}-\mu_0\mathbf{H}. \tag{6}$$

[If **P** and **M** are independently defined (as dipole moment densities, for example), then these equations constitute definitions of **D** and **B** (or **H**). If "cavity definitions" are used to define **E**, **D**, **H**, and **B**, then these equations define **P** and **M**].[5]

To represent the solutions as simply as possible, we shall make use of the "equivalent," or "fictitious," polarization and magnetization charge and current densities defined as[6–8]

$$\rho_P = -\nabla\cdot\mathbf{P}, \qquad (7)$$

$$\rho_M = -\nabla\cdot\mathbf{M}, \qquad (8)$$

$$\mathbf{J}_P = (1/\epsilon_0)\nabla\times\mathbf{P}, \qquad (9)$$

$$\mathbf{J}_M = (1/\mu_0)\nabla\times\mathbf{M}. \qquad (10)$$

Some explanation is needed in connection with the definitions of the fictitious magnetization charge density $\rho_M$ and fictitious polarization current density $\mathbf{J}_P$. The term "fictitious magnetization charge" refers to a fictitious charge by which a magnetized medium can be replaced for the purpose of calculating magnetic field inside and outside the medium.[6,7] The fictitious magnetization charge should not be confused with the true electric charge. Its dimensions are volt-sec (weber) rather than ampere-sec (coulomb), and $\rho_M$ and $\mathbf{J}_M$ do not satisfy the continuity equation. Likewise, the term "fictitious polarization current" refers to a fictitious current by which a polarized dielectric medium can be replaced for the purpose of calculating electric field inside and outside the medium.[8] The fictitious polarization current should not be confused with the true electric current. Its dimensions are volt rather than ampere, and $\rho_P$ and $\mathbf{J}_P$ do not satisfy the continuity equation. Nor should it be confused with the Maxwell–Heaviside "polarization current" (the usual polarization current) representing a motion of polarization charges within a polarized dielectric, whose density is defined as $\mathbf{J}_P = \partial\mathbf{P}/\partial t$. Note that the symbol $\mathbf{J}_P$ will be used in this paper to designate the fictitious polarization current density only. The usual polarization current density will be designated as $\partial\mathbf{P}/\partial t$. The fictitious (or "equivalent") charges and currents are very useful concepts, since they make it possible to treat electric and magnetic fields in the presence of magnetized or polarized media in a completely symmetric manner.

In some cases it may be desirable to use "equivalent" surface charge and current densities defined as

$$\sigma_P = -\mathbf{P}\cdot\mathbf{n}_{\text{in}}, \qquad (11)$$

$$\sigma_M = -\mathbf{M}\cdot\mathbf{n}_{\text{in}}, \qquad (12)$$

$$\mathbf{J}_P^{(s)} = (1/\epsilon_0)\mathbf{n}_{\text{in}}\times\mathbf{P}, \qquad (13)$$

$$\mathbf{J}_M^{(s)} = (1/\mu_0)\mathbf{n}_{\text{in}}\times\mathbf{M}, \qquad (14)$$

where $\mathbf{n}_{\text{in}}$ is a unit vector along an inward normal to the surface under consideration. Sometimes it may also be desirable to use the following relations between spatial and temporal derivatives involving polarization and magnetization vectors [these relations are derived from Eqs. (9) and (10)],

$$\nabla\times\frac{\partial\mathbf{P}}{\partial t} = \epsilon_0\frac{\partial\mathbf{J}_P}{\partial t}, \qquad (15)$$

$$\nabla\times\frac{\partial\mathbf{M}}{\partial t} = \mu_0\frac{\partial\mathbf{J}_M}{\partial t}. \qquad (16)$$

Using Eqs. (5)–(10), we can rewrite Maxwell's Eqs. (1)–(4) as

$$\nabla\cdot\mathbf{E} = (1/\epsilon_0)(\rho+\rho_P), \qquad (17)$$

$$\nabla\cdot\mathbf{H} = (1/\mu_0)\rho_M, \qquad (18)$$

$$\nabla\times\mathbf{D} = \epsilon_0\mathbf{J}_P - \epsilon_0\frac{\partial\mathbf{M}}{\partial t} - \frac{1}{c^2}\frac{\partial\mathbf{H}}{\partial t}, \qquad (19)$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\mathbf{J}_M + \mu_0\frac{\partial\mathbf{P}}{\partial t} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}, \qquad (20)$$

where we have also used $c^2 = 1/\epsilon_0\mu_0$.

## III. THE FIRST SET OF SOLUTIONS

Let us take the curl of Maxwell's Eq. (3) and let us eliminate **B** from the resulting equation by using Eq. (20). We have

$$\nabla\times(\nabla\times\mathbf{E}) = -\mu_0\frac{\partial\mathbf{J}}{\partial t} - \mu_0\frac{\partial\mathbf{J}_M}{\partial t} - \mu_0\frac{\partial^2\mathbf{P}}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}, \qquad (21)$$

or

$$\nabla\times(\nabla\times\mathbf{E}) + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = -\mu_0\frac{\partial\mathbf{J}}{\partial t} - \mu_0\frac{\partial\mathbf{J}_M}{\partial t} - \mu_0\frac{\partial^2\mathbf{P}}{\partial t^2}. \qquad (22)$$

The solution of this equation is[1,9]

$$\mathbf{E} = -\frac{1}{4\pi}\int\frac{\left[\nabla'(\nabla'\cdot\mathbf{E}) + \mu_0\frac{\partial}{\partial t}(\mathbf{J}+\mathbf{J}_M) + \mu_0\frac{\partial^2\mathbf{P}}{\partial t^2}\right]}{r}\,dv'. \qquad (23)$$

In this integral, the square brackets are the "retardation symbol" indicating that the quantities between the brackets are to be evaluated for the time $t-r/c$, where $t$ is the time for which the field **E** is evaluated, the primed operator del operates on the source point coordinates $x',y',z'$ only, and $r$ is the distance between the field point $x,y,z$ and the source point $x',y',z'$ (volume element $dv'$); the integration in this and in all other integrals appearing in this paper is over all space, except when stated otherwise. Using Eq. (17), we then have

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\int\frac{\left[\nabla'(\rho+\rho_P) + \frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{J}+\mathbf{J}_M) + \frac{1}{c^2}\frac{\partial^2\mathbf{P}}{\partial t^2}\right]}{r}\,dv'. \qquad (24)$$

Similarly, from Eqs. (19), (4), and (10), we obtain

$$\mathbf{D} = -\frac{1}{4\pi}\int\frac{\left[\nabla'\rho - \epsilon_0\nabla'\times\mathbf{J}_P + \frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{J}+\mathbf{J}_M)\right]}{r}\,dv'. \qquad (25)$$

From Eqs. (4), (19), and (18), we obtain

$$\mathbf{H} = -\frac{1}{4\pi\mu_0}\int\frac{\left[\nabla'\rho_M - \mu_0\nabla'\times\mathbf{J} - \frac{1}{c^2}\frac{\partial\mathbf{J}_P}{\partial t} + \frac{1}{c^2}\frac{\partial^2\mathbf{M}}{\partial t^2}\right]}{r}\,dv'. \qquad (26)$$

And from Eqs. (20), (3), and (9), we obtain

$$\mathbf{B}=\frac{1}{4\pi}\int\frac{\left[\mu_0\nabla'\times(\mathbf{J}+\mathbf{J}_M)+\frac{1}{c^2}\frac{\partial\mathbf{J}_P}{\partial t}\right]}{r}dv'. \tag{27}$$

Equations (24), (25), (26), and (27) constitute the first set of our solutions of Maxwell's equations.[10]

## IV. THE SECOND SET OF SOLUTIONS

The integrals representing the solutions of Maxwell's equations obtained in the preceding section contain both spatial and temporal derivatives. Another set of solutions can be obtained by eliminating the spatial derivatives from these integrals. This can be done by using the following relations valid for any scalar point function $\rho$ and any vector point function $\mathbf{J}$ confined to a finite region of space[11]

$$\int\frac{[\nabla'\rho]}{r}dv'=-\int\left\{\frac{[\rho]}{r^2}+\frac{1}{rc}\frac{\partial[\rho]}{\partial t}\right\}\mathbf{r}_u dv' \tag{28}$$

$$\int\frac{[\nabla'\times\mathbf{J}]}{r}dv'=\int\left\{\frac{[\mathbf{J}]}{r^2}+\frac{1}{rc}\frac{\partial[\mathbf{J}]}{\partial t}\right\}\times\mathbf{r}_u dv', \tag{29}$$

where $\mathbf{r}_u$ is a unit vector directed from the field point to the source point (volume element $dv'$), and the integrals are extended over all space, as stated above.

Applying Eqs. (28) and (29) to Eqs. (24), (25), (26), and (27), we have

$$\mathbf{E}=\frac{1}{4\pi\epsilon_0}\int\left\{\frac{[\rho+\rho_P]}{r^2}+\frac{1}{rc}\frac{\partial[\rho+\rho_P]}{\partial t}\right\}\mathbf{r}_u dv' -\frac{1}{4\pi\epsilon_0 c^2}\int\frac{1}{r}\left[\frac{\partial}{\partial t}(\mathbf{J}+\mathbf{J}_M)+\frac{\partial^2\mathbf{P}}{\partial t^2}\right]dv', \tag{30}$$

$$\mathbf{D}=\frac{1}{4\pi}\int\left\{\frac{[\rho]}{r^2}+\frac{1}{rc}\frac{\partial[\rho]}{\partial t}\right\}\mathbf{r}_u dv'+\frac{\epsilon_0}{4\pi}\int\left\{\frac{[\mathbf{J}_P]}{r^2}+\frac{1}{rc}\frac{\partial[\mathbf{J}_P]}{\partial t}\right\}\times\mathbf{r}_u dv'-\frac{1}{4\pi c^2}\int\frac{1}{r}\left[\frac{\partial}{\partial t}(\mathbf{J}+\mathbf{J}_M)\right]dv', \tag{31}$$

$$\mathbf{H}=\frac{1}{4\pi\mu_0}\int\left\{\frac{[\rho_M]}{r^2}+\frac{1}{rc}\frac{\partial[\rho_M]}{\partial t}\right\}\mathbf{r}_u dv'+\frac{1}{4\pi}\int\left\{\frac{[\mathbf{J}]}{r^2}+\frac{1}{rc}\frac{\partial[\mathbf{J}]}{\partial t}\right\}\times\mathbf{r}_u dv'+\frac{1}{4\pi\mu_0 c^2}\int\frac{1}{r}\left[\frac{\partial\mathbf{J}_P}{\partial t}-\frac{\partial^2\mathbf{M}}{\partial t^2}\right]dv', \tag{32}$$

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int\left\{\frac{[\mathbf{J}+\mathbf{J}_M]}{r^2}+\frac{1}{rc}\frac{\partial[\mathbf{J}+\mathbf{J}_M]}{\partial t}\right\}\times\mathbf{r}_u dv' +\frac{1}{4\pi c^2}\int\frac{1}{r}\left[\frac{\partial\mathbf{J}_P}{\partial t}\right]dv'. \tag{33}$$

Equations (30)–(33) constitute the second set of our solutions of Maxwell's equations. Both sets can be written in a somewhat different form by using Eqs. (15) and (16).

## V. AN ILLUSTRATIVE EXAMPLE

We shall now demonstrate how the above solutions of Maxwell's equations can be used for computing time-dependent electric and magnetic fields (for time-independent systems, these solutions reduce to the well-known standard electrostatic and magnetostatic expressions).

The illustrative example that we shall use may at first seem somewhat "exotic." However, it involves a variety of typical computations that need to be performed when Eqs. (24)–(27) and (30)–(33) are used, and, as we shall see from a forthcoming footnote (Ref. 12), it is much more meaningful and universal than it appears to be.

Consider a thin, small disk-shaped electret of thickness $d$ and radius $a$ having a uniform polarization $\mathbf{P}$ normal to the flat surfaces of the disk. The polarization of the electret suddenly changes its magnitude (for our purpose it is not important how this may happen; in principle this can happen due to a sudden application of heat or due to a piezoelectric effect). Let us find the time-variable electric and magnetic fields produced by the electret at a large distance $r \gg a \gg d$ from it.

We shall solve this problem by using Eqs. (26) and (31). For the system under consideration, Eq. (26) reduces to

$$\mathbf{H}=\frac{1}{4\pi\mu_0 c^2}\frac{\partial}{\partial t}\int\frac{[\mathbf{J}_P]}{r}dv'. \tag{34}$$

Since the polarization of the electret is uniform, the only contribution to $\mathbf{J}_P$ comes from the rim of the electret, where the "equivalent" polarization current is a surface current given by Eq. (13). But since the electret is thin, this current may be considered to be a filamentary ring-current $\mathbf{I}_P$ given by[8,12]

$$\mathbf{I}_P=\mathbf{J}_P^{(s)}d=(1/\epsilon_0)Pd\boldsymbol{\phi}_u, \tag{35}$$

along the rim of the electret; the direction of the current (represented by the unit vector $\boldsymbol{\phi}_u$) is, by Eq. (13), right-handed relative to $\mathbf{P}$. The volume integral of Eq. (34) can therefore be replaced by a line integral along the rim of the electret, so that

$$\mathbf{H}=\frac{1}{4\pi\mu_0 c^2}\frac{\partial}{\partial t}\oint\frac{[I_P]}{r}d\mathbf{l}'. \tag{36}$$

Transforming this integral into a surface integral, we have

$$\mathbf{H}=-\frac{1}{4\pi\mu_0 c^2}\frac{\partial}{\partial t}\int\nabla'\frac{[I_P]}{r}\times d\mathbf{S}'. \tag{37}$$

where the integration is over one flat surface of the electret ($d\mathbf{S}'$ is parallel to $\mathbf{P}$). Differentiating the integrand and taking into account that $I_P$ is not a function of space coordinates, we have[13]

$$\nabla'\frac{[I_P]}{r}=-\frac{\mathbf{r}'_u[I_P]}{r^2}+\frac{\nabla'[I_P]}{r}=\frac{\mathbf{r}_u[I_P]}{r^2}+\frac{\mathbf{r}_u}{rc}\frac{\partial[I_P]}{\partial t}. \tag{38}$$

Hence,

$$\mathbf{H}=-\frac{1}{4\pi\mu_0 c^2}\frac{\partial}{\partial t}\int\left\{\frac{[I_P]}{r^2}+\frac{1}{rc}\frac{\partial[I_P]}{\partial t}\right\}\mathbf{r}_u\times d\mathbf{S}'. \tag{39}$$

If the polarization changes sufficiently fast (which we assume to be the case), the first term in the integral of Eq. (39) can be neglected. We then have

$$\mathbf{H}=-\frac{1}{4\pi\mu_0c^3}\frac{\partial^2}{\partial t^2}\int\frac{[I_P]}{r}\mathbf{r}_u\times d\mathbf{S}'. \quad (40)$$

Since the electret is small ($a \ll r$), the integrand is essentially constant over the surface of the electret, and we can replace the integral by the product of the integrand and the surface area of the electret. Substituting $I_P$ from Eq. (35), noting that $d\mathbf{S}'$ is parallel to $\mathbf{P}$ and replacing $1/\epsilon_0\mu_0$ by $c^2$, we then have

$$\mathbf{H}=-\frac{1}{4\pi\mu_0c^3}\frac{\pi a^2d}{\epsilon_0 r}\mathbf{r}_u\times\frac{\partial^2[\mathbf{P}]}{\partial t^2}=-\frac{a^2d}{4rc}\mathbf{r}_u\times\frac{\partial^2[\mathbf{P}]}{\partial t^2}. \quad (41)$$

Rewriting the last expression in terms of spherical coordinates whose polar axis coincides with the symmetry axis of the electret, we finally obtain

$$\mathbf{H}=\frac{a^2d}{4rc}\frac{\partial^2[P]}{\partial t^2}\sin\theta\,\boldsymbol{\phi}_u, \quad (42)$$

where $\boldsymbol{\phi}_u$ is a unit vector in the azimuthal direction [the same as in Eq. (35)].

To find the electric field of the electret, we shall use Eq. (31). Since the medium outside the electret is a vacuum, we have

$$\mathbf{E}=\frac{\mathbf{D}}{\epsilon_0}$$
$$=\frac{1}{4\pi}\int\frac{1}{rc}\frac{\partial[\mathbf{J}_P]}{\partial t}\times\mathbf{r}_u dv'$$
$$=-\frac{1}{4\pi c}\frac{\partial}{\partial t}\oint\frac{[I_P]}{r}\mathbf{r}_u\times d\mathbf{l}'. \quad (43)$$

Since the electret is small, we may factor out $\mathbf{r}_u$, so that

$$\mathbf{E}=-\frac{1}{4\pi c}\frac{\partial}{\partial t}\mathbf{r}_u\times\oint\frac{[I_P]}{r}d\mathbf{l}'. \quad (44)$$

But the integral in Eq. (44) is the same as in Eq. (36). Hence, Eq. (44) reduces to

$$\mathbf{E}=-\frac{a^2d}{4\epsilon_0rc^2}\frac{\partial^2[P]}{\partial t^2}\sin\theta\,\mathbf{r}_u\times\boldsymbol{\phi}_u. \quad (45)$$

The final result is therefore

$$\mathbf{E}=\frac{a^2d}{4\epsilon_0rc^2}\frac{\partial^2[P]}{\partial t^2}\sin\theta\,\boldsymbol{\theta}_u, \quad (46)$$

where $\boldsymbol{\theta}_u$ is a unit vector in the direction of the increasing polar angle $\theta$.

## VI. THE RANGE OF APPLICABILITY

The solutions of Maxwell's equations obtained above are general and their range of applicability is the same as that of Maxwell's equations themselves [provided that the supplementary Eqs. (5) and (6) are fully compatible with Maxwell's equations]. This means that Maxwell's equations can be derived from Eqs. (24)–(27) as well as from Eqs. (30)–(33). We shall not demonstrate such reversed derivations here. Similar derivations for fields in a vacuum have been presented in Ref. 4. They are easily extendible to the equations obtained in this paper.

[1]M. Javid and P. M. Brown, *Field Analysis and Electromagnetics* (McGraw-Hill, New York, 1963) p. 63.

[2]Oleg D. Jefimenko, *Electricity and Magnetism* (Appleton-Century-Crofts, New York, 1966) p. 516 (or the same page in the 2nd ed. published by Electret Scientific, Star City, 1989)

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[4]D. J. Griffiths and M. A. Heald, "Time-dependent generalizations of the Biot–Savart and Coulomb laws," Am. J. Phys. **59**, 111–117 (1991).

[5]Equation (6) defines the magnetization as $\mathbf{M}=\mathbf{B}-\mu_0\mathbf{H}$. Many textbooks define it as $\mathbf{M}=\mathbf{B}/\mu_0-\mathbf{H}$. The definition given by Eq. (6) results in greater symmetry between electric and magnetic field equations which will appear later in this paper.

[6]The use and applications of $\rho_P$, $\rho_M$ and $\mathbf{J}_M$ are well known. See, for example, Ref. 2, 2nd ed., pp. 249–256, 471–480; David J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Englewood Cliffs, NJ, 1989), 2nd ed., pp. 165, 177–178, 183, 253–263; John R. Reitz, Fredereick J. Milford and Robert W. Christy, *Foundations of Electromagnetic Theory* (Addison-Wesley, Reading, MA, 1979), 3rd ed., pp. 78–84, 96–97, 189–194, 216–217.

[7]For more advanced applications of magnetization currents and charges see Basilio Carrascal, Gentil A. Estévez and Vicente Lorenzo, "Equivalent magnetization approach for calculating magnetic fields of localized steady-state current distributions," Am. J. Phys. **59**, 233–235 (1991).

[8]The use and applications of the "fictitious polarization current" $\mathbf{J}_P$ are not well known. See Oleg D. Jefimenko, "New method for calculating electric and magnetic fields and forces," Am. J. Phys. **51**, 545–551 (1983) and Ref. 2, 2nd ed., pp. 586–588.

[9]Ref. 2, 2nd ed., pp. 46–47, 515–516.

[10]If the field point $x,y,z$ is in a vacuum, then the integrals in Eqs. (24) and (25) give identical results (as, of course, they should), although the two integrals seem quite different. Likewise, if the field point is in a vacuum, then the integrals in Eqs. (26) and (27) give identical results. The proof that the results are identical can be obtained by using Eqs. (5), (6), (7), (8), (9), and (10) together with the vector identity,

$$\mathbf{V}=-\frac{1}{4\pi}\int\frac{\left[\nabla'(\nabla'\cdot\mathbf{V})-\nabla'\times(\nabla'\times\mathbf{V})-\frac{1}{c^2}\frac{\partial^2\mathbf{V}}{\partial t^2}\right]}{r}dv'$$

with $\mathbf{V}$ replaced by $\mathbf{P}$ or $\mathbf{M}$ [this vector identity is derived in Oleg D. Jefimenko, *Causality, Electromagnetic Induction, and Gravitation* (Electret Scientific, Star City, 1992) pp. 164–165]. Observe that if $\mathbf{V}$ (or $\mathbf{P}$, or $\mathbf{M}$) is zero at the field point, then the integral in this identity is also zero. Essentially the same proof can be used in connection with Eqs. (30)–(31) and (32)–(33).

[11]Ref. 2, 2nd ed., pp. 51–52, 516.

[12]In spite of the fact that we are using an electret in this example, the example actually embraces several electromagnetic systems. As it is shown in Ref. 8, a real charge distribution can be replaced for the purpose of field calculations by an "equivalent" polarized dielectric, and a real current distribution can be replaced by an "equivalent" magnetized body (see also Ref. 7); furthermore, mathematical operations involving "equivalent" polarization and magnetization charges and currents are the same as those involving real charges and currents. Therefore our electret system is equivalent to a thin parallel-plate capacitor with time-variable surface charges given by $\pm P$ ("radiating electric dipole antenna"), and, except for the symbols, is equivalent to a magnetized disk with a time-variable magnetization or to a circular loop carrying a time-variable current ("radiating magnetic dipole antenna").

[13]For operations with retarded quantities, see Ref. 2, 2nd ed., pp. 46–52.