THE BALANCE EQUATIONS OF ENERGY AND MOMENTUM IN CLASSICAL ELECTRODYNAMICS

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1. Introduction

A physical theory is more than a consistent mathematical structure: it requires an interpretation that enables us to test it and to apply it to a partial understanding of the physical world. In general terms, controversy arises more frequently about interpretation than about the mathematical structure and its formal consequences. Moreover, the interpretation of a physical theory develops slowly as the theory is applied to exemplary cases, usually through models that specialize the theory in some respects, and feedback from experience is obtained. In fact, the mathematical structure of a theory, its interpretation, and its corroboration advance in a very complicated way, as the history of any particular theory can show. Likewise, a theory does not grow up in isolation, and therefore is influenced by advances in other theories and in turn influences other theories; of course a theory is also tested with the aid of other theories, which makes its acceptation or rejection a non-trivial issue.

The basic components of a physical theory are: i) its primitive concepts, that are not explicitly defined within the theory, and therefore must be learnt in context; ii) its axioms, that relate the primitive concepts, usually through differential equations; iii) explicit definitions in terms of primitive concepts and theorems deduced

from the axioms, and iv) an interpretation that links the theory to experience. Of course a theory may use some of these elements from other theories, which makes an explicit listing of all the assumptions perhaps an impossible task.

In the following we deal with some interpretative problems of Classical Electrodynamics (CED) that refer to the balance of energy and momentum in electrodynamical systems, as well as to the associated problem of the localization of energy in the electromagnetic field. Since the conception of localized energy was proposed by Poynting in 1884, controversy has divided physicists into those convinced of the soundness, or at least the utility, of the idea, and those with different degrees of skepticism.

Formally Poynting's theorem (in SI units) states that

$$\frac{\partial u}{\partial t} + \nabla \cdot (\vec{E} \times \vec{H}) = -\vec{j} \cdot \vec{E}. \tag{1}$$

$u = \frac{1}{2}(\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B})$ is interpreted as energy density in the electromagnetic field; $\vec{S} = \vec{E} \times \vec{H}$, the Poynting vector, is interpreted as a local flux of field energy, and $\vec{j} \cdot \vec{E}$ is the work done in unit time by the field on the current. This power may be positive or negative, depending on the relative directions of $\vec{j}$ and $\vec{E}$. $\vec{E}, \vec{D}, \vec{B}$ and $\vec{H}$ are the electromagnetics fields.

Some reasons for having doubts about the interpretation of Poynting's theorem as a true continuity equation for electromagnetic energy are the following. One problem is that an arbitrary divergenceless vector field can be added to it and equation (1) will still be satisfied. Thus alternative definitions of the energy flux have been proposed for a long time [1], and the Poynting vector appears only as the simplest possibility.

Another problem with the concept of localized energy is that in some static situations, for example a point charge and a magnet at relative rest, the theorem implies a flux of energy in closed paths [2]. If this flux is taken as a transport of energy in space and time, the transport must be through the fields, but these are static fields. This unobservable circulation of energy in static conditions gives energy a quality of quasi-substance that goes counter to intuition. Another objection usually raised against the interpretation of $\vec{S}$ as a local flux of energy is that, to some, it sometimes seems to point in the wrong direction [3]. For example, in the case of a conduction current in a wire the flux is perpendicular to the wire and not in the direction of the current. However, for a convection current (a moving charge) the energy flux is in the direction of the current; for some authors this is "more reasonable" [4].

As a last difficulty we mention a seeming violation of the energy balance [25]. In the classical theory of the electron, developed by Lorentz and others, the electron is modelled as a tiny spherical shell of charge. When the electron is moving with constant velocity $\vec{v}$, the momentum of the field calculated with Poynting's vector results $\frac{4}{3}\frac{U_o}{c^2}\vec{v}$, where U_o is the field energy of the charge at rest. Thus the moving charge seems to have gained an energy $\frac{1}{3}U_o$ "out of nothing". For these reasons we consider necessary a careful revision of the formal derivations of Poynting's Theorem and its possible interpretations consistent with the general interpretation of CED. It will be necessary to deal with controversial points, but controversy is the flavor of research.

2. An outline of Classical Electrodynamics

Before we engage in the particular discussion of energy and momentum balance in electrodynamic systems, we sum up the main features of CED as a field theory.

The formal basis of CED was established by Maxwell in the last century. His interpretation, however, did not survive [5]. The theory introduces as primitive concepts, on the one hand, four vector fields, $\vec{E}, \vec{D}, \vec{B}$, and $\vec{H}$, and on the other hand charge and current densities, ρ and $\vec{j}$, respectively. The fields are known as electric field intensity, electric displacement, magnetic induction, and magnetic field intensity, respectively, and are related to the charge and current densities through Maxwell's equations, that expressed in modern notation and SI units are

$$\begin{aligned} \nabla\cdot\vec{D}=\rho; \quad & \nabla\times\vec{H}-\frac{\partial\vec{D}}{\partial t}=\vec{j} \\ \nabla\cdot\vec{B}=0; \quad & \nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0 \end{aligned} \tag{2}$$

These equations refer to the total fields and the total charge and current densities, total in the sense that these fields include the fields associated with the charged bodies and any external fields, while the charges and currents includes the "free" as well as the "bound" charges and currents. The first pair of inhomogeneous equations link into a natural pair the fields $(\vec{D}, \vec{H})$, and relate it to the "sources", ρ

and $\vec{j}$. The second pair of homogeneous equations link into another natural pair the fields $(\vec{E}, \vec{B})$. This is a consequence of the non-existence of magnetic monopoles, otherwise in the right-hand sides of the second pair we would find the monopole density and monopole current density. Also, the first pair of equations implies the continuity equation, $\nabla\cdot\vec{j}+\partial_t\rho=0$, which expresses the conservation of charge.

Since Maxwell's theory was originally proposed to account for electromagnetic phenomena in macroscopic bodies and media, it is really what we now call Macroscopic Electrodynamics, to distinguish it from Microscopic Electrodynamics, that takes into account the atomic structure of matter. Both theories have their own difficulties, as we will see. Since the number of equations is not enough to determine all the components of the four fields, constitutive relations are introduced by the equations $\vec{D}=\varepsilon(\vec{E})\,\vec{E}$ and $\vec{H}=\frac{1}{\mu(\vec{B})}\vec{B}$, where $\varepsilon(\vec{E})$ is the permitivity and $\mu(\vec{B})$ the permeability of the medium, which in general may be complicated tensor functions, or even functionals, of the fields $\vec{E}$ and $\vec{B}$. For linear media, however, ε and μ are independent of the fields $\vec{E}$ and $\vec{B}$, but may depend on thermodynamic variables, as density, temperature, pressure and time. These constitutive relations express the response of a medium to the fields $\vec{E}$ and $\vec{B}$. We have also the relation $\vec{j}=\sigma\vec{E}$ that states that a current density $\vec{j}$ arises in ohmic conductors as response to an external field $\vec{E}$. Since in nature there are not instantaneous responses, the most general relation between response f_{out} and exciting force f_{in}, consistent with linearity (weak fields) and time invariance is,

$$f_{out}=\int_{-\infty}^{\infty} I(t-t')f_{in}(t')dt'.$$

$I(t-t')$ is the impulse response function [6]. In order to comply with causality the response function must be zero for $t<t'$. f_{out} may be $\vec{D}, \vec{H}$ or $\vec{j}$, f_{in} may be $\vec{E}, \vec{B}$ or $\vec{E}$ and I (t - t') may be ε, μ or σ respectively [7, 8]. However, for weak fields that vary slowly with time we can use simpler relations. In vacuum we have $\vec{D}=\varepsilon_o\vec{E}$ and $\vec{H}=\frac{\vec{B}}{\mu_o}$, sometimes called the "ether relations", where ε_o and μ_o are universal constants, related to another universal constant, the speed of light in free space c, through $c^{-2}=\mu_o\varepsilon_o$. Here "ether" does not refer to the "luminiferous ether", but to a class of inertial frames. We will use "ether relations" as a shorter term for the constitutive relations of free space [9, 10].

Of course in other systems of units the permitivity and permeability of free space may have other values. For example in Gaussian units they are both defined as 1, so $\vec{D} = \vec{E}$ and $\vec{H} = \vec{B}$. However, what really matters is that the invariance of the ether relations define a class of reference frames, the Lorentz frames, since they are related through Lorentz transformations [9, 10]. That is, Maxwell's equations are generally covariant, but the invariance of the ether relations restrict the space-time transformations to Lorentz transformations [9]. Thus a Lorentz frame, or ether frame, is one with respect to which Maxwell's equations and the ether relations maintain their form, or equivalently, the wave equation maintains its form, or the velocity of light remains constant.

Since our basic equations relate the electromagnetic fields to the charge and current densities, and the constitutive relations link the fields $(\vec{D}, \vec{H})$ to the fields $(\vec{E}, \vec{B})$, the problems that can be posed are: given the sources, ρ and $\vec{j}$, and specific constitutive relations, to find out the electromagnetic fields, and given the fields to find out the charge and current densities. Usually the constitutive relations are postulated on the basis of experimental work or microscopic models are proposed for simple materials.

A particular case of the first kind of problem is when in a region there are no "sources"; that is, ρ and $\vec{j}$ are zero in the whole region. It happens that there are non trivial solutions of Maxwell's equations, and these solutions can be cast, in general, into the form of a superposition of plane waves. Hence in regions where there are no "sources" there may be electromagnetic fields, and therefore electromagnetic energy. The general solution of this kind of problem is the sum of a homogeneous solution, i.e., a solution of Maxwell's equations when ρ and $\vec{j}$ are zero, and a particular solution of the inhomogeneous equation, when ρ and $\vec{j}$ are given and known functions of $\vec{r}$ and t. It is this kind of problem that lends a basis for calling ρ and $\vec{j}$ sources of the electromagnetic field.

However, the second type of problem is more common in macroscopic CED, where ρ and $\vec{j}$ are often the unknowns. Indeed, Maxwell considered every kind of charge and current, even in vacuum, as resulting from field processes. That is, for Maxwell and his followers, like Poynting, charge was rather an epiphenomenon of a polarizable all-pervading medium: the electromagnetic ether [5]. This view lent plausibility to the conception that energy is localized in the fields, as potential energy is localized in a stressed medium. Thus, contrary to the modern concept of charge, for Maxwell charge was not a substance:

"In most theories on the subject, Electricity is treated as a substance, but

inasmuch as there are two kinds of electrification which, being combined annul each other, and since we cannot conceive of two substances annulling each other, a distinction has been drawn between Free Electricity and Combined Electricity" [11].

The advent of Electron Theory, however, changed this view. Helmholtz, Lorentz and others regarded charge as carried by "atoms of electricity", thus introducing a fundamental dualism of basic entities: charged bodies constituted by "electrons", and fields. In this way the Lorentz-Maxwell Theory gives a different reading to the equations: "electrons" are the real sources of electromagnetic fields, and the natural problem to be solved is that given the charged particles and their trajectories to find out the corresponding fields. Macroscopic bodies are regarded as assemblies of "electrons", and any macroscopic property, such as conductivity, permitivity or permeability, is to be derived by statistical averaging over these assemblies of point charges and their trajectories. Hence the Lorentz-Maxwell Theory, or microscopic CED, was conceived as the *fundamental* theory, from which macroscopic CED had to be derived. Yet there are some problems with this view, since microscopic CED cannot account for *stable* systems of charged particles, and the averaging must be done with unknown statistical distributions. Most work has been directed at obtaining general averaging methods, but at present we do not have a sufficiently general method [12]. Only Quantum Mechanics (QM) and Quantum Electrodynamics (QED) can explain to a certain extent these systems. Also Stochastic Electrodynamics, which is based on the assumptions of microscopic CED and the existence of a real, omnipresent, stochastic electromagnetic field of spectrum $\sim w^3$ has obtained some features of QM and QED, so it may account for dynamically stable configurations of charged points and fields [13].

According to another point of view, microscopic CED is subsumed by macroscopic CED [14]. That is, from a logical point of view the equations of microscopic CED result from those of macroscopic CED when ρ and $\vec{j}$ take the particular forms $\rho(\vec{r},t) = q\delta(\vec{r} - \vec{r}_0(t))$ and $\vec{j} = $ q $\delta(\vec{r} - \vec{r}_0(t))\vec{v}$. Here $\delta(\vec{r} - \vec{r}_0(t))$ is Dirac's distribution, or delta function, and ρ and $\vec{j}$ represent a point charge q at position $\vec{r}_0(t)$ moving with velocity $\vec{v}$. On the other hand, in order to go from microscopic CED to macroscopic CED, it is necessary to average over assemblies of point charges. This averaging requires special techniques to blur the delta functions. One of the best techniques is that proposed by Robinson [12], that consists in taking the Fourier transform of the delta functions, introducing an appropriate cut–off frecuency. In this sense microscopic CED does not entail macroscopic CED, which does not contradict the program of founding macrophysics on microphysics. Indeed, a phenomenological approach to macroscopic CED (i.e. independent of the atomic constitution of matter) complements the microscopic approach. Hence macroscopic CED must be consistently correlated to other macroscopic theories, as continuum mechanics and thermodynamics. In this way a charged point particle is considered a kind of degenerate charged body without internal degrees of freedom [10]. An important consequence of this view is that, given the great diversity of media and constitutive relations modelling them, there is not a general expression for the electromagnetic force on a macroscopic body, and the force on a point charge emerges as a motivated postulate, the Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$, in which the fields are only the incident or external fields, that are independent of the presence of the point charge q. In other words, the field contributed by the charge q itself, its self-field, is to be excluded explicitly from the Lorentz force. We have therefore two very different theories to explore: microscopic CED and macroscopic CED, and following the present usage we begin our discussion with the first theory.

3. Balance of energy and momentum in microscopic CED

We have in this theory two systems in interaction: point charges and the fields $\vec{E}, \vec{D}, \vec{B}$ and $\vec{H}$, with the ether relations $\vec{D} = \varepsilon_0 \vec{E}$ and $\vec{H} = \frac{\vec{B}}{\mu_o}$; we have also the Lorentz force density $\vec{f} = \rho(\vec{E} + v \times \vec{B})$, where ρ is a charge density of the form $\rho = q\delta(\vec{r} - \vec{r}_o(t))$. In the interaction of a point charge with electromagnetic fields appears a problem pointed out by Einstein [15]: the point charge has a finite number of degrees of freedom, while the fields have an infinite number of degrees of freedom. This difference is exhibited by different transformation properties of energy and momentum of particle and self-field under Lorentz transformations, as will be seen below. Other difficulties arise from the divergencies in energy and stress in the self-field of a point charge. Let us discuss this point and fix our concept of field energy.

The notion of energy associated with an electromagnetic field emerges as the work necessary to create a given configuration of charges and currents. Thus the energy of the electrostatic field of point charges q_i fixed at positions $\vec{r_i}$ is the work expended to bring these charges from infinite distance to their final positions:

$$U = \frac{1}{4\pi\varepsilon_0} \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} \tag{3}$$

This may be called the mutual energy of the configuration, since the work required to create any charge q_i at position $\vec{r}_i$ has been excluded. At this stage the question of the localization of this energy is rather artificial. However, we can generalize the above expression to continuous distributions of charge:

$$U = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{2} \int \frac{dq dq'}{R}, \tag{4}$$

where R is the distance between the elements of charge dq and dq'. Applied to a spherical shell of charge of radius r_o the expression results in

$$U = \frac{1}{4\pi\varepsilon_0} \frac{1}{2} \frac{q^2}{r_o} \tag{5}$$

With the above interpretation, this energy is the work done to shrink a shell of infinite radius, with a charge q uniformely distributed over it, to a shell of radius r_o. Since after the contraction of the infinite shell an electrostatic field, the self-field, appears where there was none, this energy may be associated with the creation of the self-field. This same energy can also be obtained by integrating the expression $u = \frac{1}{2}\varepsilon_0 E^2$ over all space, and therefore u is interpreted as an energy density. Obviously if we take the limit $r_o \to 0$, the self-energy becomes infinite. If we have several point charges, the integral of u gives the mutual energy plus the sum of the self-energies of the point charges. Thus the energy associated with the electrostatic field through the energy density u is the work spent in creating the field, which includes the work necessary to create the charges themselves. In a similar way, an energy density $u = \frac{1}{2}\frac{B^2}{\mu_o}$ is associated with the magnetic field of a distribution of stationary currents. Thus we have an expression for the energy required to create electrostatic and magnetostatic fields, in the form of an energy density $u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$, at the expense of other forms of energy.

In order to deal with time-dependent fields, let us return to our basic equations. The ether relations permit us the elimination of the fields $\vec{D}$ and $\vec{H}$, and Maxwell's equations become

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0} \qquad \frac{1}{\mu_0}\nabla \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{j}$$
$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{6}$$

In microscopic CED we can set $\vec{j} = \rho\vec{v}$, with $\rho = \sum_i q_i \delta(\vec{r} - \vec{r}_i(t))$.

There are at least two ways of deriving Poynting's theorem, which is usually interpreted as expressing conservation of energy in electrodynamic systems. One derivation [7] begins with the power density developed by the Lorentz force. Our system is a region of space where there are electromagnetic fields and a point charge. Then

$$\vec{f} \cdot \vec{v} = \rho\left(\vec{E}^{ex} + \vec{v} \times \vec{B}^{ex}\right) \cdot \vec{v} = \rho\vec{v} \cdot \vec{E}^{ex} = \vec{j} \cdot \vec{E}^{ex}, \tag{7}$$

where we emphasize that the fields are external fields. Since these fields are independent of the presence of the charged particle, they must be solutions of the homogeneous Maxwell equations in the given region. The derivation proceeds then to substitute $\vec{j}$ as given by the Ampère-Maxwell law:

$$\vec{j} \cdot \vec{E}^{ex} = [\frac{1}{\mu_0}\nabla \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}] \cdot \vec{E}^{ex} = \frac{1}{\mu_0}(\nabla \times \vec{B}) \cdot \vec{E}^{ex} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \cdot \vec{E}^{ex}. \tag{8}$$

Here we find a *faux pas* [16], since $\vec{E}$ and $\vec{B}$ are the ***total*** fields, which include the homogeneous solution, and the inhomogeneous solution related to the presence of the charge in the region. Yet let us continue the derivation, using the vector identity $\nabla \cdot (\vec{E}^{ex} \times \vec{B}) = \vec{B} \cdot \nabla \times \vec{E}^{ex} - E^{ex} \cdot \nabla \times \vec{B}$. Then,

$$\vec{j} \cdot \vec{E}^{ex} = \frac{1}{\mu_0}[\vec{B} \cdot \nabla \times \vec{E}^{ex} - \nabla \cdot (\vec{E}^{ex} \times \vec{B})] - \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \cdot \vec{E}^{ex}, \tag{9}$$

and using Faraday's law, $\nabla \times \vec{E}^{ex} = -\frac{\partial \vec{B}^{ex}}{\partial t}$, we obtain

$$\vec{j} \cdot \vec{E}^{ex} = -\frac{1}{\mu_0}\frac{\partial \vec{B}^{ex}}{\partial t} \cdot \vec{B}^{ex} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \cdot \vec{E}^{ex} - \frac{1}{\mu_0}\nabla \cdot (\vec{E}^{ex} \times \vec{B}). \tag{10}$$

In order to get the usual result that appears in textbooks we must take the risk of being inconsistent, since further reduction of the preceding equation requires neglecting the distinction between the external fields and the total fields. Then either we discard the inhomogeneous part of the total fields, or we postulate that the fields in the Lorentz force are the total fields. Both alternatives are possible only in the limit that the magnitude of the charge tends to zero, since in that case the inhomogeneous fields also tend to zero. With these reservations we finally obtain

$$-\vec{j} \cdot \vec{E} = \frac{1}{2\mu_0}\frac{\partial B^2}{\partial t} + \frac{1}{2}\varepsilon_0 \frac{\partial E^2}{\partial t} + \frac{1}{\mu_0}\nabla \cdot (\vec{E} \times \vec{B}), \tag{11}$$

which under the assumption that the energy density of the time-dependent fields is the same as that of the static and stationary fields becomes

$$-\vec{j} \cdot \vec{E} = \frac{\partial u}{\partial t} + \frac{1}{\mu_0}\nabla \cdot (\vec{E} \times \vec{B}) = \frac{\partial u}{\partial t} + \nabla \cdot (\vec{E} \times \vec{H}) \tag{12}$$

with the interpretation given in the introduction referring only to external fields. In other words, the theorem with its interpretation, as derived above, holds only for *test charges*.

The other derivation [17] is more general and amounts to a formal transformation of Maxwell's equations, that is a mathematical theorem. It begins with the scalar multiplication of the Ampère-Maxwell law by the field $\vec{E}$,

$$\frac{1}{\mu_0}\vec{E} \cdot \nabla \times \vec{B} - \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \vec{j} \cdot \vec{E}. \tag{13}$$

With the vector identity used before we get

$$\frac{1}{\mu_0}[\vec{B} \cdot \nabla \times \vec{E} - \nabla \cdot (\vec{E} \times \vec{B})] - \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \vec{j} \cdot \vec{E}, \tag{14}$$

and substituting $\nabla \times \vec{E}$ from Faraday's law results in

$$\frac{1}{\mu_0}[-\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \nabla \cdot (\vec{E} \times \vec{B})] - \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \vec{j} \cdot \vec{E}, \tag{15}$$

which with the identifications made in the other derivation yields

$$\frac{\partial}{\partial t}\frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) + \frac{1}{\mu_0}\nabla \cdot (\vec{E} \times \vec{B}) = -\vec{j} \cdot \vec{E} = \frac{\partial u}{\partial t} + \frac{1}{\mu_0}\nabla \cdot (\vec{E} \times \vec{B}). \tag{16}$$

Now the total fields appear in a consistent way in the final result of the theorem, since it is a consequence of two of Maxwell's equations. Accordingly, the energy density and the energy flux are those corresponding to the total fields, but now the power $\vec{j}\cdot\vec{E}$ is also the power developed by the total field $\vec{E}$, which includes the inhomogeneous field associated with $\vec{j}$. Hence the term $\vec{j}\cdot\vec{E}$ represents a self-interaction besides the external work, unless we assume that the self-field cannot do work on the charge. This may be at best an approximation, since the radiation reaction is usually regarded as a self-interaction [18].

We have thus two versions of Poynting's theorem, that can be made to coincide only if the inhomogeneous part of the total field, which contains the self-field, is discarded. To see if the theorem can be consistently interpreted as an energy balance, let us apply it to some simple cases. This requires integrating the balance equation over the region we are interested in. It is usually argued that since it is in its integral form that the balance equation receives its physical interpretation, it is not really a drawback that the flux of energy, $\vec{S}=\vec{E}\times\vec{H}$, is not unique. However, this non-uniqueness of the energy flux is still a drawback for the concept of localized energy, or transport of energy in space and time.

Let us consider now a simple case, a point test charge in a given external field. Then the Lorentz force applies and reflects adequately the charge-field interaction, since the self-field is left aside. The integrated form of the balance equation is

$$\frac{d}{dt}\int u dv+\int\nabla\cdot(\vec{E}\times\vec{H})dv=-q\vec{v}\cdot\vec{E} \tag{17}$$

The volume integral of the divergence of $\vec{S}$ can be changed to a surface integral of $\vec{S}$. Since the radiation of a test charge is negligible, the contribution of this surface integral can be made zero by taking the enclosing surface far away from the charge. Then we have

$$\frac{d}{dt}\int u dv=-q\vec{v}\cdot\vec{E}=-\vec{v}\cdot\vec{F}=-\frac{d}{dt}\frac{1}{2}mv^2 \tag{18}$$

Therefore we have that the sum of the field energy and the kinetic energy of the test charge remains constant. Thus the charge gets kinetic energy at the cost of field energy. However, this simple case becomes unwieldy if the magnitude of the charge is great enough, so that the radiation field resulting from the motion of the charge cannot be neglected. Then the Lorentz force is just part of the charge-field interaction, the other part being the radiation reaction force, of order q^2. In this case the fields in the balance equation must be the total fields.

We have noted before that only two types of problems can be solved in CED: given a medium, if the charge-current distribution is given then the fields can be calculated from the field equations, or if the fields are given then the motion of the charge can be calculated from the Lorentz force and Newton's second law. The simple example sketched above corresponds to the second type of problem. Let us see what happens with the energy balance in the first type of problem. Now the fields and the charge do not constitute a closed or isolated system, since an external force must intervene to produce the given current, $\vec{j}^{(e)}$, also known as the external current, subject to conservation of charge, of course. In this case the term $-\vec{j}^{(e)}\cdot\vec{E}$ is work per unit time done against the field, so it is positive. Thus in a region where there is a charge driven by an external force, energy is injected into the region by this force at the rate $-\vec{j}^{(e)}\cdot\vec{E}$, where $\vec{E}$ is the total field, and this energy may change the energy density of the electromagnetic field inside the region, may escape from the region as radiation at a rate given by the surface integral of $\vec{S}=\vec{E}\times\vec{H}$, and may change the kinetic energy of the charge. Indeed, one of the fundamental problems of CED lies in understanding the term $-\vec{j}\cdot\vec{E}$, when $\vec{E}$ is the total field [14]. The interaction between charge and field is intricate, since the motion of the charge under the action of the field alters the field with which the charge interacts. At present we have only an approximate expression for this interaction, one part being the Lorentz force and other the radiation reaction force.

The field is indeed a formidable system, and in order to be the site of localized energy, Maxwell endowed it with a state of stress. In analogy with continuum mechanics, from this stress a force density can be derived through $\vec{f}=\nabla\cdot\mathbf{T}$, where $\mathbf{T}$ is the stress tensor. The balance of momentum for the charge-field system can be obtained by substituting ρ and $\vec{j}$ in the Lorentz force density from the inhomogeneous Maxwell equations, though in this way we run into inconsistencies similar to those mentioned in the derivation of Poynting's theorem from the Lorentz force. These can be avoided by a consistent manipulation of Maxwell's equations. After vector manipulations one gets [17]

$$\frac{\partial}{\partial t}\frac{(\vec{E}\times\vec{H})}{c^2}+\rho(\vec{E}+\vec{v}\times\vec{B})=\nabla\cdot\mathbf{T}^M, \tag{19}$$

where $\mathbf{T}^M$ is Maxwell's stress tensor given by $T^M_{ij}=\varepsilon_0E_iE_j+\frac{1}{\mu_0}B_iB_j-\delta_{ij}u$, u being the energy density, $u=\frac{1}{2}(\varepsilon_0\vec{E}^2+\frac{1}{\mu_0}\vec{B}^2)$. We see that a momentum density $\vec{g}=\frac{\vec{S}}{c^2}=\varepsilon_0\vec{E}\times\vec{B}$ can be associated with the field, so that Eq. (19) can be written,

$$\frac{\partial}{\partial t}(\vec{p}+\vec{g}) = \nabla \cdot \mathbf{T}^M \tag{20}$$

where $\vec{p}$ is the mechanical momentum density, whose time derivative is equal to the Lorentz force density, if the fields are external fields. However, in the expression similar to the Lorentz force we have now the total fields and then we cannot link directly the theory with Newtonian mechanics, since in this theory change of momentum is associated only with external forces. It is important to note that we have two Lorentz forces, one that contains only external fields and expresses the ponderomotive effect of the fields. The other expression, derived from Maxwell's equations, involves the total fields, and therefore implies self-interaction. The role of this last expression is at present uncertain. This completes the view of the electromagnetic field as a generalized continuous mechanical system, endowed with energy and momentum densities, and stress.

3.1 Covariant Formulation

Relativity theory emerges from imposing the invariance group of Maxwell's equation and the ether relations, that is, the Lorentz group, to the structure of space and time. Thus Newtonian mechanics, being invariant under Galilean transformations of space and time, requires a deeper modification than CED. However, an explicitly covariant formulation of CED leads to the unification of several concepts, for instance space and time in the first place. Also the vector fields $\vec{E}$ and $\vec{B}$ merge into a second rank four-tensor, $F^{\mu\nu}$, the field tensor, and some laws merge into more general laws.

Thus the balance equations of energy and momentum can be linked in a covariant formulation of CED. In this case we have a four-tensor of energy, momentum and stress, from which a four-vector force density is obtained by applying the four-divergence operator:

$$f^\nu = -\frac{\partial T^{\mu\nu}}{\partial x^\nu} \tag{21}$$

where

$$f^\mu = \rho\left(\frac{\vec{E}\cdot\vec{v}}{c}, \vec{E}+\vec{v}\times\vec{B}\right), \quad T^{\mu\nu} = \begin{pmatrix} u & \vec{s}/c \\ c\vec{g} & -\mathbf{T}^M \end{pmatrix}, \tag{22}$$

and $x^\nu = (ct, \vec{x})$. μ and ν are 0, 1, 2 or 3. We use the metric tensor $g_{\mu\nu}$, with $g_{00} = 1$ $g_{11} = g_{22} = g_{33} = -1$, and $g_{\mu\nu} = 0$ if $\mu \neq \nu$.

The stress-energy tensor is symmetric [19], so that $\frac{\vec{s}}{c} = c\vec{g}$, or $\vec{S} = c^2\vec{g}$. Thus a flux of energy is equivalent to a momentum density, or in other words "the inertia of a system depends on its content of energy" [20].

The electromagnetic field can also be set in this language, as well as the charge and current densities, so that

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{c}E_x & -\frac{1}{c}E_y & -\frac{1}{c}E_z \\ \frac{1}{c}E_x & 0 & -B_z & B_y \\ \frac{1}{c}E_y & B_z & 0 & -B_x \\ \frac{1}{c}E_z & -B_y & B_x & 0 \end{pmatrix} \tag{23}$$

and $j^\mu = (c\rho, \vec{j})$; Maxwell's inhomogeneous equation become

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_o j^\mu \tag{24}$$

With this notation the energy-momentum balance can be written as

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = -F^{\mu\nu}j_\nu = -F^\mu, \tag{25}$$

The energy balance corresponds to

$$\frac{\partial T^{0\mu}}{\partial x^\mu} = \frac{\partial u}{c\partial t} + \frac{1}{c}\nabla\cdot\vec{s} = -f^0 = -\rho\frac{\vec{E}\cdot\vec{v}}{c}, \tag{26}$$

while the momentum balance is

$$-\frac{\partial T^{i\mu}}{\partial x^\mu} = f^i\ ; \quad i = x, y, z, \tag{27}$$

or

$$-\frac{\partial c\vec{g}}{c\partial t} + \nabla\cdot\mathbf{T}^M = \rho(\vec{E}+\vec{v}\times\vec{B}). \tag{28}$$

In this way we obtain a generatization of Cauchy's first law of motion of continuum mechanics: $\rho\ddot{\vec{X}} = \nabla\cdot\mathbf{T} + \rho\vec{b}$, where ρ is the mass density, $\mathbf{T}$ the stress tensor, and $\vec{b}$ the body force. Thus a very general expression of the balance of energy-momentum is

$$\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = 0 \tag{29}$$

This equation defines a closed or isolated system [21], that is, one in which all possible interactions have been taken into account. The subsystems of this closed system can be characterized by partial stress-energy tensors, so that

$$\frac{\partial(T_1^{\mu\nu} + T_2^{\mu\nu} + ...)}{\partial x^{\nu}} = 0. \tag{30}$$

Hence our equation $\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = -f^{\mu}$ expresses the fact that the electromagnetic fields do not constitute a closed system in the presence of a charge-current distribution, as do the free fields.

Of course we can write formally a stress-energy tensor such that

$$\frac{\partial t^{\mu\nu}}{\partial x^{\nu}} = f^{\mu} = F^{\mu\nu}j_{\nu}, \tag{31}$$

and in this way we recover the general conservation law

$$\frac{\partial(T^{\mu\nu} + t^{\mu\nu})}{\partial x^{\nu}} = 0 \tag{32}$$

However, only certain charge-current distributions can be cast into this form. One particular case is a swarm of charged particles, with a stress-energy tensor $T^{\mu\nu} = \sum m_i u_i^{\mu} u_i^{\nu}$, which after the application of the four–divergence operator and the assumption that each particle obeys

$$m\frac{du^{\mu}}{d\tau} = qF^{\mu\nu}u_{\nu} \tag{33}$$

that is, the generalization of Newton's second law with the Lorentz force as external force, gives the negative of the Lorentz force. Thus one obtains that the four-divergence of the stress-energy tensor of the electromagnetic field plus the stress-energy tensor of non-interacting charged particles is zero. Again, it must be noted that this result is an approximation, valid only for test charged particles, since the interaction with the self-field has been neglected.

3.2 Self-interaction and balance of energy and momentum

We have insisted on the fact that the interaction of a charge with the total fields includes the interaction with the self-field. This self-interaction has two main aspects: the electromagnetic inertia and the radiation reaction force. In both aspects we find problems with the balance of energy and momentum. Let us take first the problem of the electromagnetic mass.

Larmor conceived the electromagnetic mass in analogy with the apparent increase of mass that undergoes an object moving inside a fluid: the object is analogous to the charged particle and the fluid is analogous to the self-field. Here we find the germ of a velocity-dependent mass, that developed with the theory of relativity. Now, relativity theory unifies several concepts and laws in a four-dimensional space-time structure, as we have seen. In particular, energy and momentum merge into a four-vector, $P^{\mu} = \gamma(mc, m\,\vec{v})$, for the case of a neutral particle. Note that $p^0 c = \gamma\, mc^2$ is the relativistic kinetic energy of the particle, that includes the rest-mass energy mc^2. Since the field (and the continuous medium) is characterized by a stress-energy-momentum tensor, the question arises if the four-momentum

$$P^{\mu} = \int T^{\mu\nu} ds_{\nu}, \tag{34}$$

where ds_{ν} represents an element of a three-dimensional hypersurface, behaves as a four-vector. The answer is that in general it does not. A necessary condition for this to happen is that $\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = 0$ everywhere. This is a consequence of Gauss's theorem in four dimensions, that in this case implies that the integral that defines p^{μ} is independent of the hypersurface over which $T^{\mu\nu}$ is integrated. Thus one (space-like) hypersurface may be all space at t = constant, and other hypersurface may be all space at t' = constant in other reference frame related to the first by a Lorentz transformation.

Since in the presence of charges the four-divergence of the stress-energy tensor of the electromagnetic field is not zero, but equals the Lorentz force, P^{μ} as defined above is not independent of the hypersurface, and therefore P^{u} does not behave as a four-vector. Two approaches have been proposed to overcome this problem. One, initially proposed by Fermi and rediscovered by Rohrlich, [22, 23] consists in restricting the definition of P^{μ} to hypersurfaces orthogonal to the world-line of the particle. The other consists in considering a closed system such that

$\frac{\partial}{\partial x^\nu}(T^{\mu\nu}_{elm} + T^{\mu\nu}_{coh}) = 0$, where $T^{\mu\nu}_{coh}$ is a cohesion tensor that annuls the Coulomb repulsion that otherwise would make the charged particle to explode. This approach was proposed by Poincaré, and the cohesion forces are known as "Poincaré stresses".

If we take as a model of a classical electron a spherical shell of radius r uniformly charged, the energy associated with the electric field is, as we have seen before, $U = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{2r}$. This happens in the rest frame of the charge, where the associated momentum is zero, and then the four-momentum is $P^0 = \int T^{00} d^3x = U, P^1 = P^2 = P^3 = 0$. A Lorentz transformation of the stress-energy tensor to a frame moving in the x direction gives

$$T'^{00} = L^0_\mu L^0_\nu T^{\mu\nu} = \gamma^2 T^{00} - \beta\gamma^2(T^{01} - T^{10}) + \beta^2\gamma^2 T^{11}, \quad (35)$$

but since in the rest frame $T^{01} = T^{10} = 0$, and $T^{11} = \frac{1}{2}\,\varepsilon_0(E^2 - 2E_x^2)$,

$$P'^0 = \int T'^{00} d^3x' = \int (\gamma^2 T^{00} + \beta^2\gamma^2 T^{11})\frac{d^3x}{\gamma} = \gamma\int (T^{00} + \beta^2 T^{11}) d^3x. \quad (36)$$

Now, because of the spherical simmetry in the rest frame,

$$\int \varepsilon_0 E_x^2 d_x^3 = \int \varepsilon_0 E_y^2 d_x^3 = \int \varepsilon_0 E_z^2 d^3x = \int \frac{\varepsilon_0}{3} E^2 d^3x, \quad (37)$$

and therefore

$$\int T^{11} d^3x = \int \left(\frac{1}{2}\,\varepsilon_0 E^2 - 2E_x^2\right) d^3x = \frac{1}{3}\int \frac{1}{2}\,\varepsilon_0 E^2 d^3x = \frac{1}{3}\,U, \quad (38)$$

Thus

$$P'^0 = \gamma\left(U + \frac{1}{3}\beta^2 U\right) = \gamma\left(mc^2 + \frac{1}{3}\,mv^2\right) \quad (39)$$

In a similar way,

$$T'^{10} = L^0_\mu L^1_\nu T^{\mu\nu} = -\beta\gamma^2 T^{00} + \gamma^2 T^{01} - \beta^2\gamma^2 T^{10} - \beta\gamma^2 T^{11}. \quad (40)$$

Again T^{01} and T^{10} are zero in the rest frame, so

$$P'^1 = -\frac{1}{c}\int T'^{01}\, d^3x' = \gamma^2\frac{\beta}{c}\int (T^{00} + T^{11})\frac{d^3x}{\gamma} = \frac{\gamma\beta}{c}(U + \frac{1}{3}U) = \gamma\,\frac{4}{3}\,m\vec{v}. \quad (41)$$

Hence we find that although the momentum of a convection current is in the direction of motion, there seems to be a violation of the conservation of the purely electromagnetic energy, since an extra energy appears equal to one third of the rest electromagnetic energy.

As noted above, one solution to this problem is to change the definition of electromagnetic energy-momentum $P^\nu = \int T^{0\nu} d^3x$ (independent of the hypersurface only for free fields). The proposed definition is [23].

$$P^\mu = \frac{1}{c^2}\int_{(\sigma)} T^{\mu\nu} v_\nu d\sigma, \quad (42)$$

where $d\sigma^\mu = \frac{v^\mu}{c} d\sigma$ is an element of a hypersurface that is always orthogonal to the world-line of the particle. The integral leaves out the volume of the charge, a sphere in the rest frame and the Lorentz transformation of a sphere in any other frame; this is what the notation $\int_{(\sigma)}$ means. This definition yields an energy-momentum four-vector given by

$$U = \gamma\int_{(\sigma)} u d\sigma - \frac{\gamma}{c^2}\int_{(\sigma)} \vec{S}\cdot\vec{v} d\sigma, \quad (43)$$
$$\vec{P} = \frac{\gamma}{c^2}\int_{(\sigma)} \vec{S} d\sigma + \frac{\gamma}{c^2}\int_{(\sigma)} T\cdot\vec{v} d\sigma,$$

$$P^\mu = \left(\frac{U}{c}, \vec{P}\right) \quad (44)$$

With this definition the factor $\frac{4}{3}$ in equation [41] is reduced to 1, and so the conservation of the purely electromagnetic energy is re-established, but at the cost of making the definition of energy and momentum surface-dependent.

The other point of view, Poincaré's, takes into account the fact that the spherical shell of charge is not a closed system: we have not considered the constraint forces that keep the charge distribution stable. Thus besides the electromagnetic energy we have energy associated with the non electromagnetic forces that make the existence of the charge distribution possible. Poincaré showed that this energy can be accounted for as an energy $U_{coh} = pV$, where p is a uniform pressure $p = \frac{1}{2}\varepsilon_0 E^2$ inside the spherical shell that balances the Coulomb repulsion. Then

$$U_{coh} = \frac{1}{2}\,\varepsilon_0 \frac{e^2}{(4\pi\varepsilon_0)^2\, r^4} \cdot \frac{4}{3}\,\pi r^3 = \frac{e^2}{4\pi\varepsilon_0} \cdot \frac{1}{3} \cdot \frac{1}{2r} = \frac{1}{3}\,U. \tag{45}$$

Thus the *total* energy, electromagnetic plus non electromagnetic energy, is $\frac{4}{3}\,U$, corresponding to a mass $\frac{4}{3}\,\frac{U}{c^2}$. Therefore there is no violation of conservation of energy. What happens is that this energy is seen as purely electromagnetic energy in the moving charge, and therefore the electromagnetic mass is $\frac{4}{3}\frac{U}{c^2}$. In other words, although the constraint forces do no work in the rest frame, they may develop power in other frames of reference, and so produce a flux of energy that, in this case, appears as the electromagnetic energy of the moving charge. This power is necessary to mantain the mechanical equilibrium of the charge distribution, and thus the total energy and momentum of the shell of charge constitute a four-vector if and only if it is in mechanical equilibrium in every frame of reference.

This is the content of von Laue's Theorem [24]. This view, however, is not well accepted because the nature of these cohesive forces is unknown in the case of the electron, and therefore little progress can be made in the development of a fundamental classical theory of charged elementary particles, as conceived by Lorentz and others.

It is worthwhile to note that, if we define the total energy of the charge as its electrostatic rest energy, the energy transfered from the cohesive stresses is substracted from the energy of the charge in motion and then we obtain a factor 1 in the electromagnetic mass instead of the factor 4/3. Thus for a closed system we can have either factor and the electromagnetic energy and momentum constitute a four-vector independently of our space-like hypersurface [25].

In the case of the radiation reaction force the problems are even worse. We find that just tapping a charge causes it to flee, with ever increasing speed in the non–relativistic theory, or approaching the speed of light in the relativistic theory. In any case the charge gets an infinite amount of energy of unknown origin. These unphysical predictions arise from the fact that the radiation reaction force is proportional to the time derivative of the acceleration, thus eluding Newtonian mechanics or its relativistic generalization. We cannot discuss here all the intricacies of this problem [21]. We can just give a practical rule: never use the radiation reaction force as the only external force.

4. The balance equations in macroscopic classical electrodynamics

We have noted that Maxwell's equations refer to macroscopic bodies in interaction with electromagnetic fields. As a consequence of this interaction, charge and current distribution may be generated in these bodies or materials, which in turn alter the electromagnetic fields. The charge and bound current that arise from the initially neutral material are called bound charge and bound current, to distinguish them from charges and currents that may be studded through the material; these are called free or true charges and currents. The bound charge and current densities are related to new fields $\vec{P}$ and $\vec{M}$, which characterize the response of the material through the equation

$$\rho_b = -\nabla\cdot\vec{P} \quad \text{and} \quad \vec{j}_b = \nabla\times\vec{M} + \frac{\partial\vec{P}}{\partial t}, \tag{46}$$

and of course ρ_b and $\vec{j}_b$ satisfy the continuity equation.

Since the total charge and current densities are the sum of the corresponding bound and free densities, $\rho = \rho_b + \rho_f$ and $\vec{j} = \vec{j}_b + \vec{j}_f$, we can rewrite Maxwell's equations in terms only of the free densities, defining new fields

$$\vec{H}_f = \vec{H} - \vec{M} = \frac{\vec{B}}{\mu_0} - \vec{M} \quad \text{and} \quad \vec{D}_f = \vec{D} + \vec{P} = \varepsilon_0\vec{E} + \vec{P}. \tag{47}$$

Then the inhomogeneous equations become

$$\nabla\cdot\vec{D}_f = \rho_f; \quad \text{and} \quad \nabla\times\vec{H}_f - \frac{\partial\vec{D}_f}{\partial t} = \vec{j}_f, \tag{48}$$

while the homogeneous equations remain the same. We can drop the subindex f from the fields $\vec{D}_f$ and $\vec{H}_f$ if we bear in mind that these fields are related to the free charge and current densities, while the fields $\vec{D}$ and $\vec{H}$ that appear in the ether relations refer to the total charge and current densities.

In quasi–static conditions the response of materials is expressed by the fields $\vec{P}$ and $\vec{M}$ through the equations $\vec{P} = \chi_e\varepsilon_0\vec{E}$ and $\vec{M} = \chi_m\frac{\vec{B}}{\mu_0}$, where χ_e and χ_m are the electric and magnetic susceptibilities, respectively. The constitutive relations, again in the quasi-static approximation, can be expressed in the form $\vec{D} = \varepsilon\,\varepsilon_0\vec{E} = \varepsilon_0\vec{E} + \vec{D} = (1+\chi_e)\varepsilon_0\vec{E}$, and $\vec{H} = \frac{1}{\mu\,\mu_0}\vec{B} = \frac{\vec{B}}{\mu_o} - \vec{M} = (1-\chi_m)\frac{\vec{B}}{\mu_0}$, so $\varepsilon = 1 + \chi_e$ and $\frac{1}{\mu} = 1 - \chi_m$, where now ε is the relative electric permitivity and μ is the relative magnetic permeability. The absolute permitivity and permeability, the relative permitivity and permeability, or the susceptibilities are equivalent ways of characterizing the different materials and media found in nature.

These properties may depend on many factors, such as the thermodynamical state, the mechanical state of stress and strain, the fields in the case of non–linear media, the frequency of the fields in dispersive media, and even history in the case of materials that exhibit hysteresis. Such a complexity makes the development of a general theory a very difficult enterprise that requires a consistent composition of electrodynamics, thermodynamics, and continuum mechanics. In the following we limit our treatment to the simplest constitutive relations that suffice for the quasi-static approximation.

4.1 Energy Balance

In the case of macroscopic CED we cannot derive the energy balance from the Lorentz force density, even though there are now no difficulties associated with the self-field, since we are dealing only with non–singular charge and current densities. The difficulties arise rather from the fact that the Lorentz force is just one of the forces acting on an element of volume of material.

We can, however, transform Maxwell's equations into an identity analogous to Poynting's theorem in microscopic CED. Indeed, this is Poynting's original derivation, since he was a committed Maxwellian. We do not repeat all the steps; we just point out two important differences with respect to the microscopic case. One is the expression for the energy density that must be obtained from $\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}+\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}$. The usual expression $u=\frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H})$ can be obtained only if the medium is linear, that is, if ε and μ do not depend on $\vec{E}$ and $\vec{B}$, respectively. Besides, ε and μ must be independent of time or of any factor that depends on time. These restrictions leave out many interesting media and rapidly varying fields. The other point to note is that now we have a term $-\vec{j}_f\cdot\vec{E}$, which expresses the power developed by the total electric field and the free current density. The final form of Poynting's theorem in macroscopic CED is

$$\frac{\partial u}{\partial t}+\nabla\cdot(\vec{E}\times\vec{H})=-\vec{j}_f\cdot\vec{E}. \tag{49}$$

$\vec{j}_f$ includes the conduction current density and the convection current density $\rho\vec{v}$.

The theorem was applied by Poynting to the understanding of the conduction current in Maxwellian terms, that is, not as motion of matter but as the result of field processes [5]. This example is used in textbooks to illustrate the balance of energy in a conductor carrying a current, some commenting [2], [26] on the strange direction of the energy flow given by the Poynting vector, since now energy flows in a direction perpendicular to the conduction current, and not in the direction of the current as is the case with a convention current. Thus what now seems strange to the authors was quite natural for Poynting, who considered that

"A conduction current then may be said to consist of the inward flow of energy with its accompanying magnetic and electromotive forces, and the transformation of the energy into heat within the conductor" [5].

We see here again the view of Maxwell and his followers that charge and current are effects of the fields, contrary to the present view that the fields are effects of the charges and currents.

This same example can be seen in a different way. Since we have a stationary condition, $\frac{\partial u}{\partial t}=0$; also, we enclose our whole system within such a large closed surface that the flux of $\vec{S}$ is zero. Then we have $-\vec{j}_f\cdot\vec{E}=0$. This seems absurd, but let us remember that metallic conductors obey ohm's law, that in its generalized form is $\vec{j}_f=\sigma(\vec{E}+\vec{E}^{(e)})$, where $\vec{E}^{(e)}$ represents the external electromotive force, in this case provided by a battery, and σ is the conductivity. This is another example of a constitutive relation, in the quasi-static approximation, which characterizes the response of a metallic conductor to an external electric field. Then the balance reduces to

$$-\int\vec{j}_f\cdot\vec{E}\,d^3x=-\int(\sigma\vec{E}^2+\sigma\vec{E}\cdot\vec{E}^{(e)})d^3x=0 \tag{50}$$

Or, $\int\sigma\vec{E}^2\,d^3x=-\int\vec{E}^{(e)}\cdot\vec{j}\,d^3x$; the left-hand integral is Joule's heat, while the right-hand is the power developed by the external emf. Thus the energy dissipated as heat in the conductor is provided by the battery. We have therefore two interpretations of the same phenomenon. In one, energy is conceived as flowing from the battery, going into space and then flowing into the conductor, perpendicularly to its surface, where it is dissipated as heat, while at the same time produces the conduction current. In the other, the battery establishes a difference of potential in the conductor, or equivalently, an electric field of constant magnitude in the conductor.

That is, the battery puts the conductor in a state out of electrostatic equilibrium. This requires a constant input of energy. The electric field sets the free charges of the conductor into motion, thus producing the current. The current gives rise to the magnetic field, and to the Joule heat.

4.2 Momentum Balance

We have seen in microscopic CED that a momentum density must be associated with an electromagnetic field. The time derivative of this momentum density is equivalent to a force density, which is part of the momentum balance. In macroscopic CED it is more difficult to obtain a general expression for the force density. Let us see how far we can go guided by the general principles.

The aim is to obtain the equivalent of the Lorentz force density from the macroscopic Maxwell equations. Thus

$$\begin{aligned} \vec{E}\nabla\cdot\vec{D} &= \rho_f\vec{E} \\ \vec{H}\nabla\cdot\vec{B} &= 0 \\ (\nabla\times H)\times\vec{B} - \frac{\partial\vec{D}}{\partial t}\times\vec{B} &= \vec{j}_f\times\vec{B} \\ (\nabla\times\vec{E})\times\vec{D} + \frac{\partial\vec{B}}{\partial t}\times\vec{D} &= 0, \end{aligned} \tag{51}$$

and adding the four equations, with a rearrangement of some terms, we get

$$\vec{E}\nabla\cdot\vec{D} - \vec{D}\times(\nabla\times\vec{E}) + \vec{H}\nabla\cdot\vec{B} - \vec{B}\times(\nabla\times\vec{H}) - \frac{\partial}{\partial t}(\vec{D}\times\vec{B}) = \rho\vec{E} + \vec{j}\times\vec{B}. \tag{52}$$

It can be shown that, for linear and quasi-static responses, [27] [28]

$$\begin{aligned} \vec{E}\nabla\cdot\vec{D} - \vec{D}\times(\nabla\times\vec{E}) &= \nabla\cdot(\vec{E}\,\vec{D}) - \frac{1}{2}\,\varepsilon\nabla\cdot\vec{E}^2\mathbf{I} \\ \vec{H}\nabla\cdot\vec{B} - \vec{B}\times(\nabla\times\vec{H}) &= \nabla\cdot(\vec{B}\vec{H}) - \frac{1}{2}\,\mu\,\nabla\cdot\vec{H}^2\mathbf{I} \end{aligned} \tag{53}$$

where $\vec{E}\vec{D}$ and $\vec{B}\vec{H}$ are tensor (dyadic) products and $\mathbf{I}$ is the unit dyadic. The last terms on the right can be expressed in the form

$$\begin{aligned} \varepsilon\nabla\cdot\vec{E}^2\mathbf{I} &= \nabla\cdot(\varepsilon\vec{E}^2\mathbf{I}) - \vec{E}^2\nabla\varepsilon \\ \mu\nabla\cdot\vec{H}^2\mathbf{I} &= \nabla\cdot(\mu\vec{H}^2\mathbf{I}) - \vec{H}^2\nabla\mu. \end{aligned} \tag{54}$$

With these results we obtain

$$\nabla\cdot(\vec{D}\vec{E} - \frac{1}{2}\,\mathbf{I}\vec{E}\cdot\vec{D} + \vec{B}\vec{H} - \frac{1}{2}\,\mathbf{I}\vec{H}\cdot\vec{B}) + \frac{1}{2}(\vec{E}^2\nabla\varepsilon + \vec{H}^2\nabla\mu) - \frac{\partial}{\partial t}(\vec{D}\times\vec{B}) = \rho_f\vec{E} + \vec{j}_f\times\vec{B} \tag{55}$$

If we define $\vec{S}_{mom} = \vec{D}\times\vec{B}$ and

$$\mathbf{T}_{ele} = \{\vec{D}\vec{E} + \vec{B}\vec{H} - \frac{1}{2}\,\mathbf{I}\,(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})\} \tag{56}$$

we can write the momentum balance as

$$\nabla\cdot\mathbf{T}_{ele} - \frac{\partial\vec{S}_{mom}}{\partial t} = \rho_f\vec{E} + \vec{j}_f\times\vec{B} - \frac{1}{2}\,(\vec{E}^2\nabla\,\varepsilon + \vec{H}^2\nabla\mu). \tag{57}$$

Note that we have introduced $\vec{S}_{mom} = \vec{D}\times\vec{B}$, different from the vector $\vec{S}_{en} = \vec{E}\times\vec{H}$ obtained in the energy balance. This difference is important, since now the combination of the energy balance and the momentum balance in the four-tensor formalism of special relativity will lack the symmetry of the microscopic case. Indeed, this asymmetry has given rise to the Abraham-Minkowski controversy, discussed in the literature since long ago.

This controversy arose after Abraham's proposal of symmetrizing the stress-energy tensor, considering the momentum density $\vec{S}_{en} = \vec{E}\times\vec{H}$ as the true expression of the field momentum density. Indeed, the momentum balance Eq. (57) can be transformed identically, adding and substracting $\frac{\partial}{\partial t}\frac{(\vec{E}\times\vec{H})}{c^2}$, to the form

$$\nabla\cdot\mathbf{T}_{ele} - \frac{1}{c^2}\frac{\partial\vec{S}}{\partial t} = \rho\,\vec{E} + \vec{j}\times\vec{B} + \frac{\partial}{\partial t}\frac{\varepsilon\mu - 1}{c^2}\,\vec{E}\times\vec{H}. \tag{58}$$

(We are considering homogeneous media for which $\nabla\varepsilon$ and $\nabla\mu$ are zero).

The last term on the right is known as Abraham's force. It is pointless to discuss which balance is the correct one since both are mathematical theorems derived from Maxwell's equations. The question is rather about interpretation. Experimentally the interpretation is related to the radiation pressure on material media. However, experiment cannot by itself settle the controversy. It is necessary first to clarify the meaning of these expressions for the momentum balance before any coherent interpretation of the experiments can be made. This is a hard task, since the incident field excites the response of the medium by setting its charges into motion. Thus part of the energy and momentum of the field is transferred to the medium. Indeed, the momentum density $\vec{S}_{mom}/c^2$ proposed by Minkowski has been interpreted as a pseudo momentum, which is conserved in homogeneous and isotropic media [29] [30]. The presence of inhomogeneities or interfaces, however, requires the generalization of these results.

Some generalizations are based on the method of virtual work, used by Helmholtz in 1881, [28, p. 146] to derive the force density in a dielectric under electrostatic conditions. The gist of the method [8, 12, 28, 31] consists in expressing the change of the energy in time, due to a small velocity $\vec{v}$ delivered to the medium, in the general form

$$\frac{dU}{dt} = -\int dV \vec{f} \cdot \vec{v}. \tag{59}$$

In the case of a dielectric the total energy U^e is

$$U^e = \int dV \int_0^{\vec{D}} \vec{E} \cdot d\vec{D}. \tag{60}$$

If the medium is linear and isotropic, then

$$U^e = \frac{1}{2}\int \varepsilon E^2 dV. \tag{61}$$

After algebraic transformations, that include a change from $\vec{E}$ to $\vec{D}$ as the dependent variable, one obtains, for the linear initially isotropic medium,

$$\frac{dU^e}{dt} = -\int dV \left\{\rho \vec{E} \cdot \vec{v} + \frac{1}{2}\varepsilon_0 \frac{\partial \varepsilon_{ij}}{\partial t} E_i E_j\right\}. \tag{62}$$

The partial time derivative of ε_{ij} can be substituted from the convective derivative

$$\frac{\partial \varepsilon_{ij}}{\partial t} = \frac{d\varepsilon_{ij}}{dt} - v_k \frac{\partial \varepsilon_{ij}}{\partial r_k}. \tag{63}$$

On the other hand, the elastic properties of the medium are given by

$$\frac{\partial u_i}{\partial r_j} = \frac{1}{2}(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}) + \frac{1}{2}(\frac{\partial u_i}{\partial r_j} - \frac{\partial u_j}{\partial r_i}) \equiv u_{ij} + w_{ij} \tag{64}$$

where u_i is a displacement of an element, therefore $\dot{u}_i$ can be identified with v_i, u_{ij} is the symmetric part of the strain tensor, and w_{ij} is the antisymmetric part, which represent local rigid rotations. With these elements one can write

$$\frac{d\varepsilon_{ij}}{dt} = \varepsilon_{ik}\dot{w}_{ik} + \varepsilon_{kj}\dot{w}_{ik} + \alpha_{ijkl}\dot{u}_{kl} \tag{65}$$

where

$$\alpha_{ijkl} = \frac{\partial \varepsilon_{ij}}{\partial u_{kl}} \tag{66}$$

If we have a solid medium initially isotropic, $\dot{w}_{jk}$ and $\dot{w}_{ik}$ are zero. Therefore

$$\frac{\partial \varepsilon_{ik}}{\partial t} = \alpha_{ijkl}\frac{\partial v_k}{\partial r_l} - v_k \frac{\partial \varepsilon_{ij}}{\partial r_k} \tag{67}$$

Then in this case

$$\frac{1}{2}\varepsilon_0 E_i E_j \frac{\partial}{\partial t}\varepsilon_{ij} = \frac{1}{2}\varepsilon_0\{\varepsilon_{kj}E_iE_j - \varepsilon_{ij}E_jE_k\}\frac{\partial v_i}{\partial r_k} + \frac{1}{2}\varepsilon_0\alpha_{ijkl}E_kE_l\frac{\partial v_i}{\partial r_j}$$

$$-\frac{1}{2}\varepsilon_0 v_i E_j E_k \tag{68}$$

Again, for a solid initially isotropic

$$\frac{1}{2}\varepsilon_0 E^2 \frac{\partial \varepsilon}{\partial t} = \frac{1}{2}\varepsilon_0\alpha_{ijkl}E_kE_l\frac{\partial v_i}{\partial r_j} - \frac{1}{2}\varepsilon_0 E^2 v_l \frac{\partial \varepsilon}{\partial r_l}. \tag{69}$$

Therefore the time derivative of the total energy is

$$\frac{dU^e}{dt} = -\int dV\{\rho\vec{E}\cdot\vec{v} + \frac{1}{2}\varepsilon_0\alpha_{ijkl}E_kE_l\frac{\partial v_i}{\partial r_j}$$

$$-\frac{1}{2}\varepsilon_o E^2 v_l \frac{\partial \varepsilon}{\partial r_l}\} \tag{70}$$

Integrating by parts the second and third terms results in

$$\frac{dU^e}{dt} = -\int dV\, v_i\{\rho E_i - \frac{1}{2}\varepsilon_0\alpha_{ijkl}\frac{\partial}{\partial r_j}(E_kE_l) - \frac{1}{2}\varepsilon_0 E^2\frac{\partial \varepsilon}{\partial r_i}\} \tag{71}$$

Then we can identify the electric force density

$$f_i^e = \rho E_i - \frac{1}{2}\varepsilon_o E^2 \frac{\partial \varepsilon}{\partial r_i} + \frac{\partial}{\partial r_j}T_{ij}^{est}, \tag{72}$$

where

$$T_{ij}^{est} = \frac{1}{2}\varepsilon_0\alpha_{klij}E_kE_l \tag{73}$$

is the electrostriction tensor.

Analogously, for the magnetic force density one obtains

$$f_i^m = (\vec{j} \times \vec{B})_i - \frac{1}{2}\mu_o H^2 \frac{\partial \mu}{\partial r_i} + \frac{\partial}{\partial r_j} T_{ij}^{mstr} \tag{74}$$

where

$$T_{ij}^{mstr} = -\frac{1}{2}\beta_{ijkl} H_k H_l \tag{75}$$

is the magnetostriction tensor, and

$$\beta_{ijkl} = \frac{\partial \mu_{ij}}{\partial u_{kl}} \tag{76}$$

Finally, adding the electric force density and the magnetic force density, one gets

$$f_i = \rho E_i + (\vec{j} \times \vec{B})_i - \frac{1}{2}\varepsilon_0 E^2 \frac{\partial \varepsilon}{\partial r_i} - \frac{1}{2}\mu_0 H^2 \frac{\partial \mu}{\partial r_i} - \frac{\partial}{\partial r_j}\left(\frac{\varepsilon_0}{2}\alpha_{ijkl} E_k E_l + \frac{\mu_0}{2}\beta_{ijkl} H_k H_l\right) \tag{77}$$

As we can see, this expression differs from the one obtained from Maxwell's equations in two terms. The term $\frac{\partial}{\partial t}(\vec{D} \times \vec{B})$ does not appear in Helmholtz's deduction, while the term

$$f_i = \frac{\partial}{\partial r_j}\left\{\frac{\varepsilon_0}{2}\alpha_{ijkl} E_k E_l + \frac{\mu_0}{2}\beta_{ijkl} H_k H_l\right\} \tag{78}$$

does not appear in the force density derived from Maxwell's equations.

The point is that the fields in matter and the free charge and current densities do not constitute in general a closed system. We have seen that even in the simplest case of a small shell of charge in uniform motion the constraint forces play a role in the energy and momentum balances. Therefore the first and most important condition to obtain a true balance of energy and momentum is to identify clearly a closed system, and the subsystems we are interested in. Hence we cannot give general arguments in favor or against Minkowski's and Abraham's views. Indeed, other expressions for the momentum density can be derived that yield other vector fields of the type $\vec{S}_{en}$ or $\vec{S}_{mom}$. For example Peierls [30] found that for refraction the momentum density in the medium is a kind of average of $\vec{S}_{en}$ and $\vec{S}_{mom}$. On the

other hand, Lai et. al. [33] showed that this expression holds only for certain characteristic times giving at the same time other expressions for different characteristic times.

Also, other expressions for the force density can be derived from Maxwell's equations. Sometimes these expressions are discarded as mere indentities, [31] they are equivalent to Maxwell's equations.

We refer the reader to the work of de Groot and Mazur [32] who propose another form of the force density derived from Maxwell's equations and the conservation of mass. These points deserve clarification, but we do not pursue these issues further here. The details of such discussion are interesting but subtle and lie beyond the scope of this work.

The complexity of most materials takes us beyond the electromagnetic theory, and a complete energy balance requires considering these systems as mechanical, thermodynamic, and electromagnetic systems. Therefore we expect that transformations of mechanical, thermodynamic and electromagnetic energies may occur in several ways. For example, polarization, magnetization and conduction are usually accompanied by a heat exchange. We also have phenomena like piezoelectricity and pyroelectricity in which stress or heating produce electric effects. Therefore we have phenomena subject not only to the laws of electrodynamics, but also to the laws of continuum mechanics and thermodynamics, if we treat them at a phenomenological level. Furthermore, we have thermoelectric phenomena that must be dealt with the methods of irreversible thermodynamics, since these phenomena appear in systems out of thermodynamic equilibrium. In these cases the energy balance is insufficient and the entropy production must be taken into account.

5. Conclusions

We have tried to present classical electrodynamics as a living theory that still offers challenging problems. At the microscopic level the interaction between a point, finite charge and the total field presents the problem of self-interaction. In its most general form, the problem is that the field is a (tensor) function of the charge-current distribution, while at the same time the charge-current distribution is a function of the field. In this way we have that the exchange of energy between a point charge and the total field takes the form $\vec{j}(\vec{E}) \cdot \vec{E}(\vec{j})$. At present we have only approximations to this general problem. One of these is to assume that $\vec{E}$ is independent of $\vec{j}$. This is the usual assumption when we calculate the power of the Lorentz force,

$$P = \vec{f} \cdot \vec{v} = q(\vec{E}^{ex} + \vec{v} \times \vec{B}^{ex}) \cdot \vec{v} = q\vec{E}^{ex} \cdot \vec{v}.$$

However, this same formula has been extended to the total field by Lorentz, who in this way obtained an approximation to the self-interaction of a point charge and its self-field in the form of an expression for the radiation reaction force. The mathematical expression of this self-interaction leads to paradoxical interpretations, since some of these imply violation of energy conservation, while others imply violation of the principle of antecedence, or causality, as it is more commonly known [18].

At the macroscopic level classical electrodynamics also presents interesting problems. Here the challenge is the complexity of materials. This complexity involves the consistent application of classical electrodynamics, continuum mechanics, and thermodynamics. This combination of theories poses important problems of interpretation, as in the Abraham-Minkowski controversy, as well as problems of identification of the different subsystems that constitute a closed system. This identification is necessary to follow the different transformations of energy that may occur. An additional problem that appears in macroscopic CED is its relativistic generalization, since we do not have generally accepted relativistic generalizations of continuum mechanics and thermodynamics.

As a final remark we note that energy and momentum conservation hold only in closed systems and no electrodynamical system is a closed system; other (non–electromagnetic) interactions must operate for the system to be stable.

6. References

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