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MODELS OF THE CLASSICAL ELECTRON AFTER A CENTURY

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The point charge and the extended charge as models of the classical electron are studied. The problems inherent to each model with respect to self-interaction are disclosed and the possible solutions are analysed. The possible relation between these models is also discussed. We argue that a totally electromagnetic electron, as well as a point electron, is beyond the scope of classical electrodynamics. This work is intended as a complement to F. Rohrlich's recent account in the sense that it points at challenges still present in the way to a deeper understanding of the electron.

Key words: classical electron, radiation reaction, electromagnetic mass, extended charge, point charge.

1. INTRODUCTION

Recently [1], Prof. Rohrlich has once more analysed fundamental problems of classical electrodynamics that go back to the end of the last century, when phycisists were trying to understand the structure and dynamics of the electron. These problems derive from the interaction of a charged body with its self-field, and take specific aspects according to the postulated model, either a finite point charge or an extended charged body in interaction with itself.

This self-interaction gives rise to the radiation reaction prob-

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lem, that for the point charge appears as self-acceleration or preacceleration, and to an extra inertia, the electromagnetic mass, whose behaviour has been thought to be in conflict with relativity theory [2-5]. These problems have been treated abundantly in the literature [6-14], usually in controversies involving great names, but somehow ambiguities and obscure points remain.

In this work, intended to be a complement to professor Rohrlich's account, we try to illuminate some of these obscure points by making as clear as possible the model of charged particle used, either a point or an extended particle, and explore the consequences of each model with respect to self-interaction. We have found that often in the literature this crucial distinction between models is not made, resulting in more confusion.

Thus, for example, in the case of the extended charge the problem of the self-stress is crucial, while for the point charge the main problem is to "isolate" the singular trajectory of the charge in a consistent way. There is also the problem of relating both models, for example by taking the point charge as an appropiate limit of an extended charge. Finally we must also explore the relation between these classical models and the models of charged particles consistent with quantum electrodynamics (QED).

2. THE EXTENDED CHARGE AND THE RADIATION REACTION

Historically the first model of the electron to be explored was the extended electron. Lorentz [2,3] and others conceived the electron as a small spherical charge and the self-force, or radiation reaction, as arising from the retarded interaction of one infinitesimal part of the electron on another. The final result of this approach in the non relativistic limit, for a charge distribution with spherical symmetry, without rotation, and neglecting nonlinear terms, is the series that represents the radiation reaction force $\mathbf{f}$ as [2,3]

$$\mathbf{f} = -\frac{2e^2}{3c^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{c^n n!} G_n \frac{d^n \mathbf{a}(t)}{dt^n}, \tag{1}$$

where

$$G_n = \int d^3x d^3x' \rho(\mathbf{x}, t) \rho(\mathbf{x}', t) \mid \mathbf{x} - \mathbf{x}' \mid^{n-1} . \tag{2}$$

The first two terms are

$$\mathbf{f}_0 = -\frac{4}{3} \frac{U}{c^2} \mathbf{a}, \tag{3}$$

U being the electrostatic energy of the charge distribution, and

$$\mathbf{f}_1 = \frac{2e^2}{3c^3}\,\dot{\mathbf{a}}; \tag{4}$$

here $\mathbf{a}$ is the acceleration and $\dot{\mathbf{a}}$ is the time derivative of acceleration. The other terms are proportional to the size of the charge and therefore go to zero for the point charge. The term proportional to the acceleration may be written as

$$\mathbf{f}_0 = -\frac{4}{3}m\mathbf{a}, \tag{5}$$

defining $m = U/c^2$. Here appears the factor 4/3, that supposedly is in conflict with relativity theory. We will argue that the conflict is rather between two different conceptions of a purely electromagnetic electron. The other term, proportional to $\dot{\mathbf{a}}$ and independent of the size of the electron, is usually but not correctly interpreted as a radiation reaction force for a point charge. This interpretation is based on the fact that from equating the power of the radiation reaction force to the radiated power,

$$\mathbf{f}\cdot\mathbf{v} = -\frac{2}{3}\frac{e^2}{c^3}a^2, \tag{6}$$

one can obtain, with the use of the identity

$$\frac{d}{dt}\,\mathbf{a}\cdot\mathbf{v} = a^2 + \mathbf{v}\cdot\dot{\mathbf{a}},$$

$$\mathbf{f} = \frac{2}{3}\frac{e^2}{c^3}\,\dot{\mathbf{a}}. \tag{7}$$

However this derivation neglects the term $d(\mathbf{a}\cdot\mathbf{v})/dt$, since after integration yields $\mathbf{a}\cdot\mathbf{v}$ evaluated at the limits of integration and may be zero in particular cases, for example in periodic motion. In addition $\dot{\mathbf{a}}$ has no definite sign, behaving as a friction only in particular cases. For these and other reasons it has been argued that this term represents a force that results from a deformation or strain of the bound field, whose associated energy may be retrieved, which is not the case for the emitted radiation energy [15].

When the point charge limit is taken in the above series, one obtains the nonrelativistic equation of motion

$$m'\mathbf{a} - \frac{2e^2}{3c^3}\dot{\mathbf{a}} = \mathbf{F}_{\mathbf{ex}}, \tag{8}$$

where m' contains the electromagnetic mass. This equation of motion, known as the Abraham-Lorentz equation, involves difficult interpretative problems that arise from its non newtonian nature, being third order in the time derivative of position. The main complication is the existence of non physical solutions that violate conservation of energy. These self-accelerating or runaway solutions appear, for example, when $F_{\mathbf{ex}} = 0$. However, this case is analogous to taking a friction force as impelling force. One flaw in this argumentation is that the equation of powers, Eq. (6), does not take into account that the radiation reaction force also changes the kinetic energy of the radiating particle. Indeed, if the charged particle radiates energy at the cost of mechanical energy, then the balance of powers must be of the form [16,17]

$$\frac{d}{dt}\left(\frac{1}{2}mv^2 + \phi\right) = -\frac{2}{3}\frac{e^2}{c^3}a^2, \tag{9}$$

where ϕ is the potential energy, or

$$mv\frac{dv}{dt} + \nabla\phi \cdot \mathbf{v} = -\frac{2}{3}\frac{e^2}{c^3}a^2. \tag{10}$$

For a harmonic oscillator, $\phi = \frac{1}{2}kx^2$ and this equation becomes the nonlinear equation

$$mv\frac{dv}{dt} + kxv = -\frac{2}{3}\frac{e^2}{c^3}a^2,$$

or

$$v\frac{dv}{dt} + \omega_0^2 xv + \tau a^2 = 0, \tag{11}$$

where $\omega_0^2 = k/m$ and $\tau = \frac{2}{3}\, e^2/mc^3$ is a time interval of order $10^{-23}s$ for the electron.

This nonlinear equation was studied by Planck and has no divergent solutions [6], as can be seen from the homogeneous solution (with $\omega_0 = 0$, and assuming $a \neq 0$), which is of the form

$$v(t) = v_0 e^{\frac{-t}{\tau}}. \tag{12}$$

The solutions of the inhomogeneous equation are of the same form,

$$x = Ae^{-\lambda t}, \tag{13}$$

where λ is any of the roots of the equation

$$\tau\lambda^3 - \lambda^2 - \omega_0^2 = 0. \tag{14}$$

It is interesting that the linearized equation for a damped oscillating charge,

$$\tau \dot{a} - a - \omega_0^2 x = 0, \tag{15}$$

does have a divergent solution [6,18]. Thus the process of linearization may not be sound enough in this case.

These problems, however, do not arise for the extended electron. First we note that the series (1) can be summed in the case of a spherical shell of charge [19]:

$$\sum \frac{(-1)^n}{n! c^n} R^n \frac{d^n \mathbf{a}(t)}{dt^n} = \exp\left(-\frac{R}{c}\frac{d}{dt}\right) \mathbf{a}\,(t); \tag{16}$$

with $R = \mid \mathbf{x} - \mathbf{x}' \mid$.

And so the self-force can be written in the form

$$\mathbf{f} = -\frac{2e^2}{3c^3} \int\int d^3x d^3x' \; \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{R} \; \mathbf{a} \; \left(t - \frac{R}{c}\right). \tag{17}$$

Equation (17) can also be derived easily without summing the series [20,21]. The method consists in working in the Coulomb gauge, so that the self-interaction arising from the irrotational electric field that comes from the scalar potential, being a static one and with spherical symmetry, is zero. Then it is necessary to take into account only the contribution from the transverse vector potential, which satisfies the wave equation with the transverse current as source. Therefore, from

$$\mathbf{E}_{\text{self}} = -\frac{1}{c}\; \frac{\partial \mathbf{A}_{\text{tret}}}{\partial t}, \tag{18}$$

where $\mathbf{A}_{\text{tret}}$ is the retarded transverse part of the potential $\mathbf{A}$, we obtain

$$\int \rho \mathbf{E}_{\text{self}} d^3x = -\frac{1}{c} \int d^3x \; \rho(\mathbf{x}) \frac{\partial}{\partial t} \int d^3x' \frac{\mathbf{J}_{\text{tret}}(\mathbf{x}', t)}{\mid x - x' \mid}. \tag{19}$$

And considering the spherical symmetry of the distribution, we have

$$\mathbf{J}_{\text{tret}} = \frac{2}{3} \rho \mathbf{v}_{\text{ret}}, \tag{20}$$

so that finally we get

$$\mathbf{f} = -\frac{2}{3c^3} \int d^3x \int d^3x' \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{\mid \mathbf{x} - \mathbf{x}' \mid} \; \mathbf{a} \; \left(t - \frac{R}{c}\right). \tag{21}$$

By Fourier analysis, this equation can be cast into the form

$$\mathbf{f} = \int_{-\infty}^{t} dt' g(c(t-t')\ \mathbf{a}(t')), \tag{22}$$

where

$$g(t) = \frac{32\pi^2 e^2}{3c} \int_0^{\infty} dk\ k \mid \rho(k) \mid^2 \cos\ ckt \tag{23}$$

and $\rho(k)$ is the Fourier transform of the spherical charge distribution.

Then the equation of motion for a non rotating charge distribution, with spherical symmetry and in the rest frame, is

$$m\mathbf{a} = F_{\text{ext}} - \int_{-\infty}^{t} g(t')\ \mathbf{a}\ (t-t')\ dt'. \tag{24}$$

Thus Eq. (24) exhibits the nonlocality in time interpreted as a causal connection in the context of the Kramers-Kronig dispersion relations [3]. Therefore Eq. (24) does not present any conflict with respect to causality, as the Abraham-Lorentz equation does.

For a spherical shell of radius r_0 Eq. (24) reduces to

$$(m_0 - m_{\text{elm}})\mathbf{a} = \mathbf{F}_{\text{ext}} + \frac{m_{\text{elm}} c}{2r_0} \left[\mathbf{v}\ \left(t - \frac{2r_0}{c} \right) - \mathbf{v}\ (t) \right], \tag{25}$$

that, as Rohrlich mentions, has no unphysical solutions. This equation has been studied by several authors [22-26].

We have therefore that the point charge limit and the linear approximation lead to the Abraham-Lorentz equation with all its interpretative problems. Thus we may relax these restrictions and explore the consequences. One move may be considering further terms in the series Eq. (1). However, there is an interesting theorem proved by Daboul [27] in 1973.

Briefly this theorem says that adding a *finite* number of terms to the A-L equation will not solve the problem of the runaway solutions. Therefore we must consider the infinite series, which is equivalent to abandoning the point charge limit, or to take into account nonlinear terms in the velocity and its derivatives, possibility that we explore in the next section.

It is important to remark that the treatment of the extended electron has been generalized to covariant expressions by Nodvick [28] and Kaup [29], and also Dirac [30] has explored this model of the electron.

3. THE ROLE OF NONLINEAR TERMS

The most interesting point of these generalizations, that involve to leave the rest frame of the electron, is that the non relativistic limit implies a linear approximation. Indeed, Franca [31] *et al.* point out that in order to obtain the non relativistic limit, Eq. (24), "new nonlinear terms" have been dropped. Also Jackson indicates that in obtaining the series Eq. (1) "nonlinear terms of order c^{-5} have been neglected" [3].

These nonlinear terms arise from taking into account correctly the retardation effects. One of the difficult aspects of classical electrodynamics is precisely the evaluation of retarded quantities, either potentials or fields, since it is necessary to solve for t' the retardation condition,

$$t' = t - \left(\frac{1}{c}\right) \mid \mathbf{x}(t) - \mathbf{x}(t') \mid . \tag{26}$$

This can be done explicitly only in a few particular cases and therefore retarded quantities must be treated implicitly. Griffiths [32] obtained a few nonlinear terms for a charged dumbbell in particular motions. These results can be generalized. The appropriate mathematical tool for this task is a Lagrange expansion [33], rather than a Taylor expansion. When the retardation is treated correctly, it is possible to obtain a first nonlinear term

$$\frac{2e^2}{3c^5} a^2 v, \tag{27}$$

which is part of the non relativistic limit of

$$\mathbf{F}_{rr} = \frac{2}{3}\frac{e^2}{c^3}\gamma^2 \left\{ \ddot{\mathbf{v}} + 3\frac{\gamma^2}{c^2}(\mathbf{v}\cdot\dot{\mathbf{v}})\,\dot{\mathbf{v}} + \frac{\gamma^2}{c^2}\left[\mathbf{v}\cdot\ddot{\mathbf{v}} + 3\frac{\gamma^2}{c^2}(\mathbf{v}\cdot\dot{\mathbf{v}})^2\right]\,\mathbf{v} \right\}, \tag{28}$$

an equation deduced by Abraham and that can also be obtained by the method delineated in the Appendix.

This term is the spatial part of the four-vector

$$\Gamma^\mu = \frac{2}{3}\frac{e^2}{c^3}\left(\ddot{v}^\mu - \frac{\dot{v}^\alpha \dot{v}_\alpha}{c^2} v^\mu\right), \tag{29}$$

and is usually regarded as the real radiation reaction force, but only the second term can be taken as such a force for the reasons exposed in Sec. 2.

4. THE POINT ELECTRON AND THE RADIATION REACTION

After the advent of quantum mechanics and quantum electrodynamics, the point charge model of the electron was regarded as nearer to the true electron than the extended charge model. Thus Dirac [34] obtained in 1938 the Lorentz-Dirac equation by assuming an *ab initio* point charge, and not as a limit of an extended charge. He therefore took Maxwell's theory as valid all the way to the point-singularity that represents the electron. However, Dirac clearly points out in his work that he is proposing a phenomenological theory of the classical electron: "Our aim will be not so much to get a model of the electron as to get a simple scheme of equations which can be used to calculate all the results that can be obtained from experiment".

This covariant treatment of the point electron is based on the conservation of energy and momentum, as expressed in the covariant law

$$\partial_\mu T^{\mu\nu} = 0, \tag{30}$$

where $T^{\mu\nu}$ is the stress-energy tensor of a *closed* system.

However, the electromagnetic stress-energy tensor does not satisfy this law in the presence of a charge-current distribution, since in that case

$$\partial_\mu T^{\mu\nu}_{\text{elm}} = -\frac{1}{c}\, F^{\mu\nu} J_\mu. \tag{31}$$

That is, the four-divergence of the electromagnetic stress-energy tensor equals the covariant Lorentz force. The approach then consists in applying the conservation law *outside* the world line of the charge, that is enclosed within a tube of finite radius, but the contribution of the singular world line of the charge is then ambiguous [35,36], and Dirac [34] introduced symplifying hypotheses to arrive at his equation,

$$m\dot{v}^\mu = F^\mu_{ex} + \frac{2}{3}e^2(\ddot{v}^\mu - \dot{v}^\alpha \dot{v}_\alpha v^\mu). \tag{32}$$

In order that the contribution of the singularity behaves as a state function, Dirac proposed to cut the tube that encloses the worldline of the electron with spatial-like hypersurfaces orthogonal to the world line. This is precisely the hypothesis that Fermi [9] introduced to resolve the problem of the factor 4/3 in the electromagnetic mass.

Equation (32) has unphysical solutions for certain external forces or in absence of any. We have seen in the nonrelativistic case that the radiation reaction force has no physical meaning without an external force acting on the radiating charge. In the case of the Lorentz-Dirac equation, a line of work is the study of the conditions to be imposed on the external forces in order to have physically meaningful solutions [37].

Another way of eliminating divergent solutions is by imposing asymptotic conditions: $\mathbf{a} \to 0$ for $t \to \infty$. This approach leads to an integrodifferential equation [2],

$$ma^{\mu}(\tau) = \frac{e^{-\tau/\tau_0}}{\tau_0} \int_{\tau}^{\infty} e^{-\tau/\tau_0} K^{\mu}(\tau')d\tau', \tag{33}$$

where τ is proper time, K^{μ} is the external force, and $\tau_0 = 2/3\, e^2/mc^3$. This equation, however, presents preacceleration for time intervals of order $\tau_0 (\sim 10^{-23}$ s for an electron). But let us remember that Dirac's treatment renounces any exploration of the electron structure. Therefore the Lorentz-Dirac equation is an approximation to the true dynamics of a classical radiating charge.

5. THE PROBLEM OF ELECTROMAGNETIC MASS

As mentioned above, the Lorentz approach to the radiation reaction force gives as coefficient of the acceleration term, for a spherical shell of charge, (4/3) (U/c^2), where U is the electrostatic energy of the charge distribution. This coefficient also results when the electromagnetic mass is calculated as a coeficient of the velocity, using as definition of linear momentum of a charged particle the expression

$$\mathbf{p} = \int \mathbf{S} d^3x = \frac{4}{3}\frac{U}{c^2}\mathbf{v}, \tag{34}$$

where $\mathbf{S}$, as usual, is the Poynting vector.

This factor also appears when the "kinetic energy",

$$T = \frac{1}{8\pi}\int B^2 d^3x, \tag{35}$$

is calculated for a point charge.

The factor 4/3 seems to spoil the possibility of having a four-vector,

$$p^{\mu} = \left(\frac{U}{c}, \mathbf{p}\right), \tag{36}$$

for the electromagnetic energy-momentum of a point charge, where U is the electrostatic energy and $\mathbf{p}$ the momentum as defined in Eq. (34). Again we have different approaches to solve this problem, according to the model of the electron considered in the calculations.

Let us note that we are trying to construe a four-vector with purely electrostatic energy and purely electromagnetic momentum as defined by Eq. (34).

A. The Extended Electron

This model has the problem of introducing non electromagnetic forces that supposedly hold stable the charge distribution of the classical electron. Since the Compton wavelenght of the electron is about one hundred times its classical radius, obviously we are dealing with a quantum system. However, we will see below that in the quantum level it is also necessary to introduce non electromagnetic fields in order to obtain physically meaningful results. Poincaré was able to prove that by assuming the existence of these forces, known as Poincaré stresses, the factor 4/3 can be explained, as follows.

Since the Poynting vector S is part of the electromagnetic stress-energy tensor,

$$T^{\mu\nu} = \begin{pmatrix} u & \frac{\mathbf{S}}{c^2} \\ \frac{\mathbf{S}}{c^2} & T^{ij} \end{pmatrix}, \tag{37}$$

where u is the energy density and T^{ij} is the stress tensor introduced by Maxwell, then the four-momentum can be defined as

$$p^{\mu} = \int T^{\mu o} d^3x. \tag{38}$$

In the rest frame, indicated by subindex (o),

$$p^0_{(o)} = \int T^{00} d^3x = U_0, \tag{39}$$

where U_0 is the electrostatic energy, and

$$p^i_{(o)} = \int T^{io} d^3x = \int \frac{S}{c^2} d^3x = 0. \tag{40}$$

If we apply a Lorentz transformation to p^{μ}, we find, for relative motion along z axis and with signature (- 2),

$$\mathbf{p} = \gamma \frac{\mathbf{v}}{c^2} \int (T^{oo}_{(o)} + T^{33}_{(o)}) d^3x_{(o)} \tag{41}$$

$$U = \gamma \int (T^{oo}_{(o)} + \frac{v^2}{c^2}\, T^{33}_{(o)}) d^3x_{(o)}. \tag{42}$$

Therefore we see that a four-vector for the energy and momentum can be obtained only if the electromagnetic stress is annulled by another, non electromagnetic, stress; that is, only if the charge distribution is stable. This is the content of von Laue's theorem[38]. It is worthwhile to emphasize that the condition $\partial_\mu T^{\mu\nu} = 0$ is a necessary but not sufficient condition for energy and momentum to constitute a four-vector $(U/c, \mathbf{p})$, while von Laue's theorem provides a necessary and sufficient condition to have such a four-vector for a static system.

Indeed, the condition $\partial_\mu T^{\mu\nu} = 0$ only gives us a condition of conservation of energy and momentum that is *independent* of any space-like hypersurface. Then we require that a *total* stress-energy tensor, electromagnetic plus non electromagnetic, satisfies a condition of stability given by

$$\mathbf{p} = \gamma \frac{\mathbf{v}}{c^2} \int \left[\left(T^{oo}_{\mathrm{elm(o)}} + T^{oo}_{\mathrm{nonelm(o)}} \right) + \left(T^{33}_{\mathrm{elm(o)}} + T^{33}_{\mathrm{nonelm(o)}} \right) \right] d^3 x_{(0)}, \tag{43}$$

where

$$\gamma \frac{\mathbf{v}}{c^2} \int \left(T^{33}_{\mathrm{elm(o)}} + T^{33}_{\mathrm{nonelm(o)}} \right) d^3 x_{(0)} = 0. \tag{44}$$

Poincaré used a perfect fluid as model of the non electromagnetic part of the electron and introduced as stabilizing non electromagnetic stress, for a shell model, a pressure

$$P^c_0 = \frac{-e^2}{8\pi a^4}, \tag{45}$$

which gives a non electromagnetic energy,

$$P^c_0 V = -\frac{U}{3} = \int T^{oo}_{\mathrm{nonelm(o)}} d^3 x_{(0)}. \tag{46}$$

We have that a perfect fluid is described by a tensor

$$T'^{\mu\nu} = (\rho_0 + p) v^\mu v^\nu - p \eta^{\mu\nu}, \tag{47}$$

where ρ_0 is the mass-energy density, p is the pressure and $\eta^{\mu\nu}$ is the metric tensor. Thus for the Poincaré model $p = p^c_0$.

Since we have spherical symmetry and the electromagnetic stress-energy tensor is traceless,

$$T^{11} = T^{22} = T^{33} = \frac{1}{3} u; \tag{48}$$

then Eq. (41) and Eq. (42) become

$$\mathbf{p}_{\text{elm}} = \gamma \frac{4}{3} \frac{U_0}{c^2}\, \mathbf{v}, \tag{49}$$

and

$$U_{\text{elm}} = \gamma U_0 \left(1 + \frac{v^2}{3c^2}\right), \tag{50}$$

where U_0 is the electrostatic energy of the spherical electron.

The non-electromagnetic part is transformed as

$$U_{\text{coh}} = \gamma \int (\rho_0 + \beta^2 P_c^o)\gamma d^3x = \gamma \left(U_0^c - \frac{v^2}{c^2}\frac{U_0}{3}\right) \tag{51}$$

and

$$\mathbf{p}_{\text{coh}} = \gamma \mathbf{v} \int (\rho_0 + P_c^o)\gamma d^3x = \gamma \mathbf{v} \left(U_0^c - \frac{U_0}{3}\right). \tag{52}$$

Now, two conceptions of a "purely electromagnetic" electron have been proposed. Either we postulate that the momentum must be totally electromagnetic and then $\mathbf{P}_{\text{coh}} = \mathbf{0}$, or we postulate that the energy of the electron at rest must be only the electrostatic energy, U_0; that is, $U_0^c = 0$.

The first option, proposed by Lorentz and others, according to the "electromagnetic world-view," gives

$$U_0^c = \frac{U_0}{3}, \tag{53}$$

and then we obtain for the total energy and momentum

$$\begin{aligned} U_{\text{tot}} &= U_{\text{elm}} + U_{\text{coh}} \\ &= \gamma U_0 \left(1 + \frac{v^2}{3c^2}\right) + \gamma \left(\frac{U_0}{3} - \frac{v^2}{c^2}\frac{U_0}{3}\right) = \gamma \frac{4}{3} U_0, \end{aligned} \tag{54}$$

$$\mathbf{p}_{\text{tot}} = \mathbf{p}_{\text{elm}} = \gamma \frac{4}{3}\frac{U_0}{c^2}\, \mathbf{v}. \tag{55}$$

The second option, $U_0^c = 0$, gives

$$U_{\text{coh}} = -\gamma \frac{v^2}{c^2}\frac{U_0}{3}, \tag{56}$$

and therefore

$$U_{\text{tot}} = U_{\text{elm}} + U_{\text{coh}} = \gamma U_0 \left(1 + \frac{v^2}{3c^2}\right) - \gamma \frac{v^2}{c^2} \frac{U_0}{3} = \gamma U_0, \qquad (57)$$

and

$$\mathbf{p}_{\text{tot}} = \mathbf{p}_{\text{elm}} + \mathbf{p}_{\text{coh}} = \gamma \frac{4}{3} \frac{U_0}{c^2}\, \mathbf{v} - \gamma \frac{U_0}{3c^2}\, \mathbf{v} = \gamma \frac{U_0}{c^2}\, \mathbf{v}. \qquad (58)$$

Physically we can say that the first option implies that we add to the electrostatic energy the necessary non electromagnetic energy that appears as an extra electromagnetic energy, associated with the magnetic field, for the observer that sees the electron moving with velocity v. In other words, there is a kinematic relativistic transfer of energy from the non electomagnetic part to the electromagnetic part of the electron. Something analogous happens in the case of a confined field, as in the explanation of the experiment of Trouton and Noble. We find then that while in non relativistic mechanics constraint forces do no work in any inertial frame, in relativistic mechanics they *may* do work with respect to some inertial frames. This point has been exhibited by Boyer [10] as a direct consequence of the relativity of simultaneity. The second option means merely that we substract from the moving electron the non electromagnetic part, taking as total energy just the electrostatic energy of the charge. We can see therefore that there is no conflict with relativity theory if we accept that the electron has a non electromagnetic part. Therefore the conflict with the 4/3 factor arises when we mix both conceptions of a "purely electromagnetic electron".

B. The Point Electron

Since QED seems to imply a point electron, in 1938 Dirac published [34] a classical phenomenological theory of the point electron.

As mentioned above, one important step in his treatment is to cut the tube enclosing the world line of the electron with spatial-like hyperplanes orthogonal to the world line. With this choice he was able to obtain an energy-momentum four-vector for a purely electromagnetic electron, in the sense of considering covariantly only its electrostatic energy.

This approach, also proposed by Fermi and others and rediscovered by Rohrlich [39], is equivalent to modifying the definition of electromagnetic momentum given by the space integral of the Poynting vector. The new definition is

$$p^\mu = \int_{(\sigma)} T^{\mu\nu} d\tau_\nu = \int_{(\sigma)} T^{\mu\nu} \frac{v_\nu}{c} d^3 x, \qquad (59)$$

where

$$d\tau_\nu = \frac{v_\nu}{c} d^3x \tag{60}$$

is an element of the spatial-like hyperplane orthogonal to the world line of the electron, and the subscript σ in the integrals indicates that the volume of the electron is to be excluded from the integration. This volume will be a sphere in the rest frame, or its Lorentz transform in any other frame.

The Lorentz transformation of this four-vector gives

$$p^o = U = \int_{(\sigma)} \gamma \left(u - \frac{\mathbf{v}}{c^2} \cdot \mathbf{S} \right) d^3x = \gamma U \tag{61}$$

and

$$\mathbf{p} = \gamma \int_{(\sigma)} \left(\mathbf{S} + \frac{\mathbf{v} \cdot \mathbf{T}}{c^2} \right) d^3x = \gamma \frac{\mathbf{v}}{c^2}\, U. \tag{62}$$

In this way the factor $\frac{4}{3}$ in the electromagnetic mass does not appear, and the problem of the stability and structure of the electron can be put aside. This geometrical approach can be seen as a kind of projection of the "instantaneous" rigidity of the point electron in its rest frame to all the hyperplanes orthogonal to the world line. The construction of this four-vector represents a satisfactory way of eluding the problem that poses the non electromagnetic part of the electron.

However, this definition of the electromagnetic energy-momentum holds only for these hyperplanes. Therefore a fully relativistic treatment according to the principle of relativity, which postulates that "The laws of nature are to be formulated in a way that is independent of the choice of space-like hypersurfaces" [40], must consider a closed system. If a static and stable model is studied then also von Laue's theorem must be taken into account.

Another aspect of the problem is the possible relation between the extended electron and the point electron. Here we find that the point electron of Dirac is not a limit of the extended electron of Lorentz, in spite of the name: Lorentz-Dirac equation. Briefly, what is at stake is the notion of a relativistic rigid body. On the one hand, the extended electron of Lorentz and Poincaré is a static and stable object, with vanishing self-stress in every reference frame, while on the other hand the Dirac electron is the point limit of the instantaneous rigid body Lorentz transformed to other reference frames. This notion of rigid body can be traced to Born, notion criticized by Sommerfeld, Ehrenfest and von Laue, who proposed as definition of a rigid body one for which the space integral of the self-stress is zero. The essence of the problem is that a relativistic extended body, moving with respect to a reference frame, will have different accelerations in different points, and thus a relativistic kinematical approach is insufficient: a dynamical approach is necessary.

8. THE ELECTRON IN QED

At present QED is considered a theory of a deeper level of explanation than classical electrodynamics. As a consequence, the point electron is rather a formal limit in CED, while in QED is a fundamental entity. However, the point electron in QED is subject to quantum fluctuations that smear it over the finite volumes amenable to experimental exploration. In this way the point electron acquires and effective finite charge distribution. This makes the structure of the radiation reaction problem for the quantum point electron similar to that of a classical extended charge. Here we also find that a purely electromagnetic mass for the quantum electron is not possible [41]. While satisfactory advances have been made in understanding the behaviour of the electron, as indicated by Rohrlich [1], there is a particular point that deserves attention.

If the electron in QED is to satisfy von Laue's theorem, it must be shown that the self-stress vanishes. Here we find that there is a calculation by Pais [42] that gives the result

$$S(o) = \int T_{11}(x) d^3x = -(\frac{\alpha}{2\pi})\, m\psi\bar{\psi}, \tag{63}$$

where α is the fine structure constant, m is the mass, and ψ and $\bar{\psi}$ are the spinor field and its adjoint that represent the electron field. Rohrlich [43], however demonstrated that by using a relativistic cutoff the self-stress is zero. This approach implies the introduction of an additional neutral vector meson field, and therefore also in QED the electron appears as a not totally electromagnetic system.

Therefore we find that neither in CED nor in QED the electron is a self-sufficient entity: non electromagnetic stresses or compensating fields must be introduced in order to make the existence of the electron possible.

9. CONCLUSIONS

Both, CED and QED give satisfactory accounts of experimental facts in their respective domains of application. However, neither CED nor QED can account for the existence of electrons, in the sense that electrons are not purely electromagnetic systems. As Pauli [44], Einstein and others put it: "The electron is itself a stranger in the Maxwell-Lorentz theory as well as in present-day quantum theory".

On the other hand, the Lorentz force law gives the action of an external electromagnetic field (i.e., produced by other charges and currents) on a test charge. Thus we find two types of problems that can be dealt with in electrodynamics: either the charge-current distribution is given and then we can obtain from the theory the evolution of the electromagnetic field, or a field is given in a region of space-time and then we can obtain the trajectories of the electron in that region, through the Lorentz force law.

However, in the radiation reaction problem the motion of the electron produces a field that influences the motion of the electron. That is, the field and the charge-current distribution are unknown. Hence the difficulty of the problem, that at present can be approached only approximately, given the smallness of the radiation reaction force. This difficulty has provoked two radical approaches: either to eliminate the field and construe a relativistic action at a distance theory, or to modify Maxwell's theory at very small distances.

We also see that the extended charge model may be the way for further exploration of the electron. Indeed, Bohm and Weinstein [20], and Dirac [30], going beyond the extended rigid electron, have found self-oscillations that after quantization give masses on the meson range. These results, together with Rohrlich's work [43], point to a connection between meson theory and electron theory. Thus, after a century, the search for a deeper understanding of the electron continues.

APPENDIX

The retardation is usually expressed with a Taylor series in the form

$$[\ ]_{ret} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{R}{c}\right)^n \frac{d^n}{dt^n}(\ ). \tag{A1}$$

However, R is also a function of t' through the dependence on $\mathbf{x}(t')$. Therefore the retardation condition is

$$t' = t - \frac{R(t')}{c}. \tag{A2}$$

And this dependence of R on t' precludes the direct use of a Taylor series. However, it is possible an expansion in terms of a Lagrange series [33]; this series is of the type

$$g(\eta) = g(\eta_1) + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \left\{ \frac{d^{n-1}}{d\eta^{n-1}} \left(\frac{dg}{d\eta} S^n(\eta) \right) \right\}_{\eta=\eta_1}, \tag{A3}$$

where the relation

$$\eta = \eta_1 + \alpha s(\eta) \tag{A4}$$

is satisfied.

If we take $\eta = t', \eta_1 = t, \alpha = -\frac{1}{c}$ and $s(\eta) = R(t')$, we obtain the retardation condition. Also, from $g(\eta) = \eta = t'$ we can obtain a series expansion for the retardation condition in terms of t. However, it is necessary to introduce a small parameter in order to be able to cut it to the desired approximation. In the extended charge the size of the distribution will play this role. In the point charge the role of this parameter is played by the radius of the tube that encloses the world-line of the electron. Therefore it is necessary to introduce a series of the form

$$d = \sum_{n=1}^{\infty} A_n (t' - t)^n, \tag{A5}$$

where the coefficients A_n will depend on the particular form of the distribution.

Then it is necessary to invert this series to obtain $t' - t$ as a power series in d, which substituted in the Lagrange expansion can give an expression for retarded quantities (the fields or potentials), as power series in d. Dirac did this [34], in his paper of 1938, by ingenious manipulations of a few terms. However, by using the theorem of the argument in complex variable theory, this series can be inverted systematically, giving [33]

$$g(t') = g(t) + \sum_{n=1}^{\infty} \frac{d^n}{n!} \left\{ \frac{d^{n-1}}{d\eta^{n-1}} \frac{g'(\eta)}{\left(\sum_{\ell=1}^{\infty} A_\ell (\eta - t)^{\ell - 1}\right)^n} \right\}_{\eta = t}. \tag{A6}$$

With these results it is possible to obtain the nonlinear terms mentioned by Jackson [3], that are the same as those obtained by Franca et al. [31] In this way one finally can get the Lorentz-Dirac equation in its non covariant expression. However, the method can also be used in an explicitly covariant way [45], arriving at the covariant Lorentz-Dirac equation. Also, in the point charge limit one obtains the radiation reaction force [33a]

$$\mathbf{F}_{rr} = \frac{2}{3}\frac{e^2}{c^3}\gamma^2 \left\{ \ddot{\mathbf{v}} + 3\frac{\gamma^2}{c^2}(\mathbf{v}\cdot\dot{\mathbf{v}})\,\dot{\mathbf{v}} + \frac{\gamma^2}{c^2}\left[\mathbf{v}\cdot\ddot{\mathbf{v}} + 3\frac{\gamma^2}{c^2}(\mathbf{v}\cdot\dot{\mathbf{v}})^2\right]\,\mathbf{v} \right\}, \tag{A7}$$

that is just the spatial part of the Lorentz-Dirac force, derived by Abraham before the advent of relativity theory. As Rohrlich mentions, this non covariant equation has as covariant generalization the

Lorentz-Dirac equation, that in the non relativistic limit contains the non linear term

$$\frac{2e^2}{3c^5}a^2v, \tag{A8}$$

which presently is interpreted as the true radiation reaction force, since it always oposes the motion, as a true friction force. It is also possible to obtain directly the non covariant limit of the Lorentz-Dirac equation by taking into account the retardation effects on the energy balance [46].

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