Classical Motion of an Extended Charged Particle*

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The classical equations of motion of an extended particle are re-examined and methods are found which eliminate the usual difficulties with the $\frac{4}{3}$ factor, runaway solutions, and preacceleration. Point particles are not considered, but rather localized, nonsingular distributions. The $\frac{4}{3}$ factor is eliminated by using Dixon's method for deriving the macroscopic equations of motion, while runaway solutions and preacceleration are eliminated by retaining the structure-dependent terms that the Dirac equation neglects. Finally, it is shown that in the limit of slowly varying external forces, these solutions become identical to those obtained from the integral form of the Dirac equation.

I. INTRODUCTION

The equation of motion for a charged particle was originally developed by Abraham and Lorentz in its nonrelativistic form, and later, in the relativistic form by Dirac.[1] The resulting equations possessed two highly undesirable properties of which there was no analog in classical physics, namely, runaway solutions and either the specification of initial acceleration or a preacceleration. The runaway solutions and the specifications of initial acceleration can be eliminated if we allow *ad hoc* conditions to be imposed, such as bounded motion as $t\rightarrow\infty$ or the "principle of undetectablility of small charges."[2] However, when it is eliminated, the resulting equation then contains a preacceleration.

If we restrict our attention to classical physics, these results are, to say the least, very distrubing. If we would consider the particle not as a point, but rather as a highly localized collection of interacting fields, then there is a very general conservation law of classical physics which should require that no runaway solutions exist, namely, conservation of energy, which the Dirac equation appears to violate. Also, in Newtonian mechanics, the specification of initial position and velocity, along with the value of the force, is sufficient to determine the future motion, while with the Dirac equation, in addition to the above quantities, we must also specify either the initial acceleration or the *future* values of the force. And, the latter violates the usual concept of causality.

In this paper, we will show by deriving an alternative equation that the above-mentioned difficulties do not actually exist. In Sec. II, we will specify our conventions and the microscopic equations from which we will derive our results, and in Sec. III, we will present a derivation of the macroscopic equation of motion for a monopole particle. In Sec. IV, we will derive the nonrelativistic limit of the macroscopic equation which results in a linear integral equation for the acceleration. From this alternative equation, the nonrelativistic Dirac equation can be derived. Then in Sec. V, we will show that preacceleration and runaway solutions of the Dirac form do not occur. Furthermore, we will show that if the external forces are sufficiently slowly varying, the solutions of the alternative equation and of the Dirac equation are identical.

II. MICROSCOPIC EQUATIONS AND CONVENTIONS

Throughout this paper, we will never consider point particles, but rather, will only consider nonsingular, localized distributions. Since such distributions are unstable if only electromagnetic fields are present, we will assume, for stability, that there are also attractive, localized fields present. We will loosely call the collection of these fields a particle. Gravitational effects will be neglected in order to utilize the simplicity of special relativity, and the tensor notation will be used. In Cartesian coordinates, the metric $g_{\mu\nu}$ is

$$g_{\mu\nu}=\begin{bmatrix}-1&&&\\&-1&&\\&&-1&\\&&&+1\end{bmatrix}, \tag{2.1}$$

where x^k ($k=1, 2, 3$) is a space-like coordinate, x^4 is the time-like coordinate, and $c=1$. Greek letters can take the values of 1, 2, 3, or 4, while Latin letters are restricted to 1, 2, or 3.

From the nonelectromagnetic fields present in the particle we assume that a stress-energy tensor $T_m{}^{\mu\nu}$ and a current density J^ν can be formed which satisfy the following microscopic equations:

$$\partial_\nu T_m{}^{\mu\nu}+\partial_\nu T_e{}^{\mu\nu}=0\,, \tag{2.2}$$

$$T_m{}^{\mu\nu}=T_m{}^{\nu\mu}\,, \tag{2.3}$$

$$\partial_\nu F^{\mu\nu}=4\pi J^\mu\,, \tag{2.4}$$

where

$$F^{\mu\nu}=-F^{\nu\mu}\,, \tag{2.5}$$

$$T_e{}^{\mu\nu}=-\frac{1}{4\pi}\left[F^{\mu\rho}F^\nu{}_\rho-\tfrac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right], \tag{2.6}$$

$$\partial_\rho F_{\mu\nu}+\partial_\nu F_{\rho\mu}+\partial_\mu F_{\nu\rho}=0\,. \tag{2.7}$$

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[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938).
[2] F. Rohrlich, Ann. Phys. (N. Y.) **13**, 93 (1961).

Differentiation of (2.4) and (2.6) gives

$$\partial_\nu J^\nu = 0, \tag{2.8}$$

and

$$\partial_\nu T_m{}^{\mu\nu} = -F^{\mu\nu} J_\nu. \tag{2.9}$$

From these two equations, we can derive the macroscopic equations of motion for a monopole[3] particle.

III. MACROSCOPIC EQUATIONS OF MOTION

In order to consistently eliminate the commonly occurring factor of $\frac{4}{3}$, we will carry out the integration of the microscopic equations over the hyperplane which is orthogonal to the 4-velocity of the particle. This method has been discussed by Rohrlich[4] in his recent book; however, Dixon[5] has given a more comprehensive discussion for deriving the equations of motion via this method. Although Dixon's equations are in a general relativistic form, they can be quite easily specialized for special relativity. This we will do here and also give a brief outline of Dixon's method.

The basic equation we need is for the derivative with respect to proper time of an integral over a hypersurface, where the hypersurface is a function of the proper time. This is given by Dixon's equation (5.7),[5] which when specialized to special relativity is

$$\frac{d}{ds}\int A^\mu d\Sigma_\mu = \int \frac{\partial A^\mu}{\partial x^\mu} w^\rho d\Sigma_\rho + \int U^\alpha \frac{\partial A^\mu}{\partial z^\alpha} d\Sigma_\mu. \tag{3.1}$$

A^μ is any arbitrary tensor of any rank in Minkowski coordinates which is a function of two points: x^μ and z^μ. Here x^μ is an arbitrary point on the hypersurface Σ_β and z^α is the point where the world line intersects the hypersurface. In the monopole approximation, the exact position of the world line is more or less arbitrary,[5] except that it should be located somewhere inside the particle. Here we shall merely assume that it has been uniquely specified, and define s to be the arc length along it. U^α is the tangent vector of the world line and also is the 4-velocity of the particle. w^ρ is defined by $w^\rho \equiv dx^\rho/ds$. When the hyperplane is maintained orthogonal to U^α, we have

$$w^\rho = U^\rho[1-(x^\alpha - z^\alpha)a_\alpha], \tag{3.2}$$

and $a^\alpha = dU^\alpha/ds$.

Now, let A^μ in (3.1) be replaced by $T_m{}^{\mu\nu}$; then, using (2.9), we obtain

$$\frac{d}{ds}\int T_m{}^{\mu\nu} d\Sigma_\nu = -\int F^{\mu\nu} J_\nu w^\rho d\Sigma_\rho. \tag{3.3}$$

Likewise, for the quantities $(x^\alpha - z^\alpha)T_m{}^{\mu\nu}$, J^ν, and $(x^\alpha - z^\alpha)J^\nu$, we have

$$\frac{d}{ds}\int (x^\alpha - z^\alpha)T_m{}^{\mu\nu} d\Sigma_\nu = \int T_m{}^{\alpha\mu} w^\rho d\Sigma_\rho - U^\alpha \int T_m{}^{\mu\nu} d\Sigma_\nu - \int (x^\alpha - z^\alpha) F^{\mu\nu} J_\nu w^\rho d\Sigma_\rho, \tag{3.4}$$

$$\frac{d}{ds}\int J^\nu d\Sigma_\nu = 0, \tag{3.5}$$

$$\frac{d}{ds}\int (x^\alpha - z^\alpha) J^\nu d\Sigma_\nu = \int J^\alpha w^\rho d\Sigma_\rho - U^\alpha \int J^\nu d\Sigma_\nu. \tag{3.6}$$

If we define

$$p_m{}^\alpha \equiv \int T_m{}^{\alpha\nu} d\Sigma_\nu, \tag{3.7}$$

$$\epsilon \equiv \int J^\nu d\Sigma_\nu, \tag{3.8}$$

$$M^{\alpha\beta} \equiv \int T_m{}^{\alpha\beta} w^\rho d\Sigma_\rho, \tag{3.9}$$

then upon neglecting the dipole moment and angular momentum, we have

$$\frac{dp_m{}^\mu}{ds} = -\int F^{\mu\nu} J_\nu w^\rho d\Sigma_\rho, \tag{3.10}$$

$$M^{\alpha\beta} = U^\alpha p_m{}^\beta + \int (x^\alpha - z^\alpha) F^{\beta\nu} J_\nu w^\rho d\Sigma_\rho, \tag{3.11}$$

$$\int J^\alpha w^\rho d\Sigma_\rho = \epsilon U^\alpha, \tag{3.12}$$

as well as the fact that ϵ is a constant of the motion.

Now, in line with the monopole approximation, we will assume that in the instantaneous rest frame, J^ν has only a fourth component. Thus, in the instantaneous rest frame, we have from (3.11) that $M^{4k} = p_m{}^k$ and also $M^{k4} = 0$. Since $M^{\alpha\beta}$ is symmetric, we then have that $p_m{}^\alpha$ must be parallel to U^α, or

$$p_m{}^\alpha = m_0 U^\alpha, \tag{3.13}$$

where m_0 is a scalar.

We will now decompose the electromagnetic field into two parts: a bound field, $F_b{}^{\mu\nu}$, and an external field, $F_{\text{ext}}{}^{\mu\nu}$, which satisfies the following equations:

$$F^{\mu\nu} = F_b{}^{\mu\nu} + F_{\text{ext}}{}^{\mu\nu}, \tag{3.14}$$

$$\partial_\nu F_b{}^{\mu\nu} = 4\pi J^\mu, \tag{3.15}$$

$$\partial_\nu F_{\text{ext}}{}^{\mu\nu} = 0. \tag{3.16}$$

In addition, we impose the boundary conditions that $F_b{}^{\mu\nu}$ be the retarded solution and that as $r \to \infty$, $F_b{}^{\mu\nu}$

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[3] We use the term monopole as referring not only to J^ν, but also to $T_m{}^{\mu\nu}$. In the monopole approximation, we neglect the dipole moment, angular momentum, spin, and higher order effects.

[4] F. Rohrlich, *Classical Charged Particles* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1965), pp. 197–207.

[5] W. G. Dixon, Nuovo Cimento 34, 317 (1964).

must vanish at least as fast as $1/r$. Then the boundary condition on $F^{\mu\nu}$ as $r\to\infty$ is that $F^{\mu\nu}\to F_{\text{ext}}{}^{\mu\nu}$.

Now, we expand $F_{\text{ext}}{}^{\mu\nu}$ in a Taylor series about the world line, and upon neglecting the dipole moment and higher order effects, we have from (3.10), (3.12), and (3.14)

$$\frac{dp_m{}^\alpha}{ds}=-\epsilon F_{\text{ext}}{}^{\alpha\beta}U_\beta-\int F_b{}^{\alpha\beta}J_\beta w^\rho d\Sigma_\rho. \tag{3.17}$$

Equation (3.17) is our macroscopic equation of motion for a monopole particle. A consequence of (3.13) and (3.17) is that the scalar m_0 is a constant of the motion when J^ν is parallel to U^ν.

IV. AN ALTERNATIVE EQUATION OF MOTION

In order to eliminate preacceleration and runaway solutions, we will retain the structure-dependent terms in the integral on the right-hand side of (3.17). Since the integral is very complicated to evaluate in general, we will only calculate it to the lowest order in velocity for the three space-like components.

Take J^ν to be

$$J^\nu(\mathbf{x},t)=U^\nu(t)\rho(|\mathbf{x}-\mathbf{z}(t)|), \tag{4.1}$$

where ρ is spherically symmetric and is the charge density. Define

$$G^\alpha=\int F_b{}^{\alpha\beta}J_\beta w^\rho d\Sigma_\rho. \tag{4.2}$$

The required solution for $F_b{}^{\alpha\beta}$ is given by

$$F_{b\alpha\beta}=\partial_\beta A_\alpha-\partial_\alpha A_\beta, \tag{4.3}$$

where

$$A^\alpha(\mathbf{x},t)=\int dt'\int d^3x' J^\alpha(\mathbf{x}',t')\frac{\delta(t'-t+|\mathbf{x}'-\mathbf{x}|)}{|\mathbf{x}'-\mathbf{x}|}. \tag{4.4}$$

Now, in the instantaneous rest frame, we have

$$G_0{}^4=0,\qquad G_0{}^k=\int F_b{}^{k4}J^4w^4d^3x_0, \tag{4.5}$$

and to the lowest order in velocity, after transforming some origins,

$$A^k(\mathbf{x},t)=\int d^3x' U^k(t-r')\frac{\rho(|\mathbf{x}+\mathbf{x}'-\mathbf{z}(t)|)}{r'}, \tag{4.6}$$

$$A^k(\mathbf{x},t)=\int d^3x'\frac{\rho(|\mathbf{x}+\mathbf{x}'-\mathbf{z}(t)|)}{r'}+\partial_n\int d^3x' \times[z^n(t)-z^n(t-r')]\frac{\rho(|\mathbf{x}+\mathbf{x}'-\mathbf{z}(t)|)}{r'}, \tag{4.7}$$

where $r'=|\mathbf{x}'|$.

Inserting (4.6) and (4.7) into (4.5) and remembering that ρ is spherically symmetric, we finally obtain

$$G_0{}^k=-\tfrac{1}{3}m_ea^k(t)+\int_0^\infty a^k(t-r)f(r)dr, \tag{4.8}$$

where m_e is defined by

$$m_e\equiv\tfrac{1}{2}\int d^3x\int d^3x'\frac{\rho(r)\rho(r')}{|\mathbf{x}-\mathbf{x}'|}, \tag{4.9}$$

and f is given by

$$f(r)=\frac{8\pi}{3}r\int d^3x'\rho(r')\rho(|\mathbf{x}'+\mathbf{x}|), \tag{4.10}$$

and $r=|\mathbf{x}|$.

If we now transform $G_0{}^\alpha$ to an arbitrary slowly moving frame, (3.17) becomes

$$m'\mathbf{a}(t)+\int_0^\infty \mathbf{a}(t-r)f(r)dr=\mathbf{F}(t), \tag{4.11}$$

where $\mathbf{F}(t)$ are the external forces and

$$m'=m_0-\tfrac{1}{3}m_e. \tag{4.12}$$

Equation (4.11) is the alternative equation which we started out to obtain. To be sure, the exact solution of (4.11) will be dependent on the exact shape of the charge distribution $\rho(r)$; however, we will show in Sec. V that if the external forces are slowly varying, the solution of (4.11) will approach that of the integral form of the Dirac equation.

The low-velocity limit of the differential Dirac equation can now be derived from (4.11); by expanding the acceleration in the power series

$$\mathbf{a}(t-r)=\mathbf{a}(t)-r\dot{\mathbf{a}}(t)+\cdots, \tag{4.13}$$

and using

$$\int_0^\infty f(r)\ dr=\tfrac{4}{3}m_e, \tag{4.14}$$

$$\int_0^\infty rf(r)\ dr=\tfrac{2}{3}\epsilon^2, \tag{4.15}$$

we obtain the Dirac equation

$$m\mathbf{a}(t)=\tfrac{2}{3}\epsilon^2\dot{\mathbf{a}}(t)+\mathbf{F}(t), \tag{4.16}$$

where $m=m_0+m_e$.

Inspection shows that the initial conditions for (4.11) and (4.16) are completely different. To specify the solution of the Dirac equation, the initial position and velocity must be specified as well as requiring that the velocity at $t=\infty$ be bounded, while for the alternative equation, specification of the past path is sufficient to determine the time development.

We are now in a position to analyze the motion given by alternative equation when the external forces are given as functions of time.

V. SOLUTIONS OF THE ALTERNATE EQUATION

A. Exponential Run-Away Solutions

We will now show that a necessary condition for (4.11) to be free of runaway solutions when the total mass is positive is for $m'>0$. This we will do by constructing an exponential runaway solution when $m'<0$. Let

$$a(t)=e^{i\omega t}, \tag{5.1}$$

with $\text{Im}(\omega)<0$. If (5.1) is to be a solution of (4.11), we must have

$$m'+M(\omega)=0, \tag{5.2}$$

where

$$M(\omega)=\int_0^\infty f(r)e^{-i\omega r}dr. \tag{5.3}$$

If we define

$$\tilde{\rho}(k)=\frac{1}{\pi}\int_0^\infty r^2\rho(r)\frac{\sin(kr)}{kr}dr, \tag{5.4}$$

so that

$$\rho(r)=\frac{i}{r}\int_{-\infty}^\infty k\tilde{\rho}(k)e^{-ikr}dk, \tag{5.5}$$

then from (4.10) and (5.5) we have

$$f(r)=-\frac{32\pi^3}{3}i\int_{-\infty}^\infty k[\tilde{\rho}(k)]^2e^{ikr}dk, \tag{5.6}$$

and from (5.2), (5.3) and (5.6),

$$m'=-\frac{32\pi^3}{3}\int_{-\infty}^\infty \frac{k^2}{k^2-\omega^2}[\tilde{\rho}(k)]^2dk. \tag{5.7}$$

Now, since m' is real, we find from (5.7) that the real part of ω must vanish, and thus for $\omega=-i\sigma$, with σ real and positive, (5.7) becomes

$$m'=-\frac{32\pi^3}{3}\int_{-\infty}^\infty \frac{k^2}{k^2+\sigma^2}[\tilde{\rho}(k)]^2dk. \tag{5.8}$$

Although (5.8) is a function of σ^2, it is in general only meaningful if $\sigma>0$ because, until certain restrictions are placed on $f(r)$, $M(\omega)$ can only be defined by (5.3) in the lower half complex plane.

Since

$$m_e=8\pi^3\int_{-\infty}^\infty [\tilde{\rho}(k)]^2dk, \tag{5.9}$$

and since the integral in (5.8) is a monotonic decreasing function of σ, we have that for each value of m' within the range

$$-\tfrac{4}{3}m_e<m'<0; \tag{5.10}$$

there exists one and only one value of σ satisfying (5.8). Since for $m'<-\frac{4}{3}m_e$, m will be negative, a necessary condition for (4.11) to possess only stable solutions when $m>0$ is for $m'>0$.

B. Preacceleration

We will now use this result to show that the solution of (4.11) does not contain a preacceleration when $m'>0$. The inhomogeneous solution of (4.11) is given by

$$\mathbf{a}(t)=\frac{1}{m'}\mathbf{F}(t)+\int_{-\infty}^\infty K(s)\mathbf{F}(t-s)ds, \tag{5.11}$$

and

$$K(s)=\frac{1}{2\pi m'}\int_{-\infty}^\infty \frac{M(\omega)e^{i\omega s}}{m'+M(\omega)}d\omega, \tag{5.12}$$

and $M(\omega)$ is given by (5.3).

Now let us consider $K(s)$ for $s<0$. Since $M(\omega)$ is well defined and analytic in the lower half of the complex plane, we can evaluate $K(s)$ by means of a contour integral consisting of the real axis and an infinite semicircle which extends into the lower half plane.

From (4.10) and (5.3),

$$\omega^2M(\omega)=-f'(0)-\int_0^\infty e^{-i\omega r}f''(r)dr, \tag{5.13}$$

which (if $rf\to 0$ as $r\to\infty$) gives

$$|M(\omega)|\leq\text{const}/|\omega^2|, \tag{5.14}$$

and thus along the infiite semicircle, $M(\omega)$ is zero.

Since by (5.8), we see that the denominator of (5.12) can never vanish, there are no poles inside the contour, and consequently $K(s)$ must vanish if $s<0$.

Thus, we may rewrite (5.11) as

$$\mathbf{a}(t)=\frac{1}{m'}\mathbf{F}(t)+\int_0^\infty K(s)\mathbf{F}(t-s)ds, \tag{5.15}$$

which shows explicitly that there is no preacceleration in the solution of (4.11).

C. Stability of (5.15)

We will now show that when the charge density is localized and bounded by some reasonable function, then the solution of (5.15) cannot "runaway" after the external forces are turned off.

Let $\mathbf{F}(t)$ be zero for all $t>t_0$ and let F_m be the maximum value of $|\mathbf{F}(t)|$ in the interval $t_0\geq t>-\infty$. Then from (5.15) for $t>t_0$,

$$|\mathbf{a}(t)|\leq F_m\int_{t-t_0}^\infty |K(s)|ds. \tag{5.16}$$

Integrating (5.12) by parts, we obtain

$$s^3K(s)=-\frac{i}{2\pi m'}\int_{-\infty}^\infty e^{i\omega s}\frac{d^3}{d\omega^3}\left[\frac{M(\omega)}{m'+M(\omega)}\right]d\omega, \tag{5.17}$$

and thus, providing that $M(\omega)$ is at least differentiable to the third order, we have

$$|K(s)|\leq K_3/s^3, \qquad (5.18)$$

where K_3 is some positive constant.

Then from (5.16) and (5.18),

$$|\mathbf{a}(t)|\leq K_3F_m/2(t-t_0)^2, \qquad (5.19)$$

which shows that the velocity as well as the acceleration remains bounded. A sufficient condition for the third derivative of $M(\omega)$ to exist can be obtained by considering (5.3). Upon differentiating,

$$\frac{d^3M(\omega)}{d\omega^3}=-i\int_0^\infty r^3f(r)e^{-i\omega r}dr, \qquad (5.20)$$

and thus if $r^4f(r)$ vanishes as $r\to\infty$, (5.20) is defined for all ω, and consequently, the third differential of $M(\omega)$ exists.

In terms of the charge density, we have from the definition of f [Eq. (4.10)] that if ρ is sufficiently localized such that

$$|\rho(r)|\leq A/(\alpha^2+r^2)^4, \qquad (5.21)$$

then we are guaranteed that $r^4f(r)$ will vanish and, consequently, the solution of (5.15) cannot "runaway."

D. Comparison with Dirac Equation

We will now compare the solution of the alternative equation with that of the Dirac equation, when $\mathbf{F}(t)$ is sufficiently slowly varying, so that the second and all higher time derivatives can be neglected.

Let

$$\mathbf{F}(t-s)=\mathbf{F}(t)-s\dot{\mathbf{F}}(t)+\cdots. \qquad (5.22)$$

Then from (5.11) we have

$$m\mathbf{a}(t)=\mathbf{F}(t)+\tau\dot{\mathbf{F}}(t)+\cdots, \qquad (5.23)$$

where $\tau=2e^2/3m$.

If we had used the integral form of the Dirac equation, which contains the boundary condition at $t=\infty$, we would have

$$m\mathbf{a}(t)=\frac{1}{\tau}\int_0^\infty \mathbf{F}(t+s)e^{-s/\tau}ds, \qquad (5.24)$$

and if $\mathbf{F}$ is given by (5.22), we then find that the result of (5.24) is identical with (5.23).

Had we taken (5.22) to one order higher, the result of (5.24) would no longer agree with that of (5.11); however, the difference would be a term proportional to the radius of the particle, which the Dirac equation has ignored.

VI. SUMMARY

We have derived the monopole equation of motion for a charged particle (3.17 and 3.13) which retains the self-interaction and the structure-dependent terms. Upon taking the nonrelativistic limit, we obtained a linear integral equation for $\mathbf{a}(t)$ [Eq. (4.11)] from which the nonrelativistic Dirac equation can be derived.

In Sec. V, we have shown that the undesirable features of the Dirac equation are not present in the alternative equation. From Secs. VA and VB, we have that exponential runaway solutions and preacceleration do not occur if $m'>0$. Although we have not shown that all solutions of (4.11) are stable, we have shown that they are stable under the following conditions. If $\mathbf{a}(t)$ and external forces are zero for a period of time extending into the past such that K, as given by (5.12), vanishes, then the solution of (4.11) is given by (5.15), since the homogeneous solutions of (4.11) are absent. Then as shown in Sec. VC, if ρ is bounded by (5.21), the solution of (4.11) will never "runaway" when arbitrary forces are applied.

This is a property which the Dirac equation (4.16) does not share, because one can apply a force which will "excite" the runaway solution. Although it can be eliminated by imposing boundary conditions at $t=\infty$, when that is done, a preacceleration is introduced.

The condition of $m'>0$ is automatically satisfied by those distributions where $T_m{}^{44}$ is positive-definite in all Lorentz frames, and thus it is not an unreasonable condition. This can be seen by integrating $T_m{}^{44}$ in an arbitrary frame and then requiring that the integral of the internal stresses vanish in the rest frame of the particle.

Although the alternative and the Dirac equation are quite different, we have shown in Sec. VD that when the external forces are sufficiently slowly varying, the solutions of the integral form of the Dirac equation agree with those of (5.11), up to a term proportional to the radius of the particle, which the Dirac equtaion has ignored. Although the exact solution of (5.11) depends on the shape of the charge density, in the limit of slowly varying forces, the solution is relatively insensitive to the exact shape and is dependent only on the total charge.

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