Charge density on a slender axially symmetric conducting body

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Jackson[1] recently discussed the charge density on a thin straight wire. He described Maxwell's variational result, his own asymptotic result for any axially symmetric body,[2] and the work of some other authors. We wish to point out a method for obtaining the complete asymptotic expansion of the charge density due to Handelsman and Keller.[3] They considered the electrostatic potential $\Phi(x,r^2,\varepsilon)$ around an axially symmetric conducting body in an applied axially symmetric potential $\Phi^0(x,r^2,\varepsilon)$. Here x is the distance along the body axis, with the body length as the unit of length, r is the distance from the axis, and ε is the square of the ratio of the maximum radius of the body to its length. For a body with flat or pointed ends, they assumed that $\Phi=\Phi^0$ plus the potential of a charge of density $-f(x,\varepsilon)$ along the body axis

$$\Phi(x,r^2,\varepsilon)=\Phi^0(x,r^2)-\int_0^1\frac{f(\xi,\varepsilon)d\xi}{[(\xi-x)^2+r^2]^{1/2}}. \tag{1}$$

Suppose that Φ is a constant, say $\Phi=b$, on the body surface, the equation of which is $r=\varepsilon^{1/2}R(x)$. Then on the surface, Eq. (1) becomes

$$b=\Phi^0[x,\varepsilon S(x)]-\int_0^1\frac{f(\xi,\varepsilon)d\xi}{[(\xi-x)^2+\varepsilon S(x)]^{1/2}}, \tag{2}$$

where $S(x)=R^2(x)$. They solved the integral equation (2) asymptotically for ε small, corresponding to a slender body, and obtained the asymptotic expansion of the charge density f,

$$f(x,\varepsilon)\sim\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\varepsilon^n}{(\log\varepsilon)^m}f_{nm}(x). \tag{3}$$

The expansion coefficients $f_{nm}(x)$ were determined recursively.

For a body at potential $b=1$ with no applied field ($\Phi^0\equiv 0$), the first two terms in Eq. (3) are given in Ref. 3, Eqs. (4.3) and (4.4),

$$f(x,\varepsilon)=\frac{1}{\log\varepsilon}+\frac{1}{(\log\varepsilon)^2}\log\frac{4x(1-x)}{S(x)}+\cdots. \tag{4}$$

The charge $Q(\varepsilon)$, the integral of $-f$, is the capacity of the body,

$$\begin{aligned}Q(\varepsilon)&=-\int_0^1 f(x,\varepsilon)dx\\&=\frac{-1}{\log\varepsilon}-\frac{1}{(\log\varepsilon)^2}\int_0^1[\log 4+\log x+\log(1-x)\\&\quad-\log S(x)]dx+\cdots\\&=\frac{-1}{\log\varepsilon}-\frac{1}{(\log\varepsilon)^2}\left[\log 4-2-\int_0^1\log S(x)dx\right]\\&\quad+\cdots.\end{aligned} \tag{5}$$

The ratio of the charge density $-f(x,\varepsilon)$ to the total charge $Q(\varepsilon)$ is, from Eqs. (4) and (5),

$$\begin{aligned}\frac{f(x,\varepsilon)}{Q(\varepsilon)}&=1+\frac{1}{\log\varepsilon}\left[\log\frac{4x(1-x)}{S(x)}-2\log 2+2\right.\\&\quad\left.+\int_0^1\log S(x)dx\right]+\cdots.\end{aligned} \tag{6}$$

Equation (20) of Ref. 2 is equivalent to Eq. (6), and includes the next term. For a thin circular cylinder or wire, $S(x)=1$, so the integral in Eq. (6) equals zero. Then Eq. (6) reduces to Eq. (3) of Ref. 1 when we set $x=(\zeta+1)/2$, $f(x,\varepsilon)=2c\lambda(\zeta)$, and $\Lambda=-\log\varepsilon$. When the body has smooth ends, the charge distribution $-f(x,\varepsilon)$ in (1) does not extend up to the ends. For example, for a prolate ellipsoid of revolution, $f(x,\varepsilon)$ is nonzero between the two foci of the ellipsoid, and f is zero outside them. It is an interesting problem to derive this result, and to find f, by considering the exact solution for the ellipsoid and continuing it analytically into the interior of the ellipsoid. The continuation is singular on the axis between the foci and regular outside them. The strength of the singularity at the point x on the axis determines $f(x,\varepsilon)$.

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