Effective conductivity of periodic composites composed of two very unequal conductors

Joseph B. Keller
Departments of Mathematics and Mechanical Engineering, Stanford University, Stanford, California 94305

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The effective conductivity tensor is calculated for a periodic composite composed of alternating rectangular blocks of two very unequal conductors. The two-dimensional case of a checkerboard pattern of rectangles is also treated, and Gautesen's result for it is obtained. The checkerboard of parallelograms is treated, too. The method can be applied to alternating parallelepipeds and to certain other configurations.

I. INTRODUCTION

We consider the effective conductivity tensor $\Sigma(\sigma_a,\sigma_b)$ of certain two- and three-dimensional periodic composites composed of two materials with scalar conductivities σ_a and σ_b. Examples are the "checkerboard" patterns of rectangles or parallelograms in two dimensions (Fig. 1) and the analogous arrangement of rectangular blocks or parallelepipeds in three dimensions (Fig. 2). We shall show how to calculate Σ asymptotically as σ_a/σ_b tends to zero or to infinity. This work grew out of an attempt to obtain a simpler deriviation of one of Gautesen's[1] recent results for a rectangular "checkerboard" in two dimensions.

First we shall present our result for the three-dimensional alternating arrangment of rectangular blocks shown in Fig. 2. Let the edges of the blocks be parallel to the axes, and let h_i be the length of the edge parallel to the x_i axis. Clearly the axes are the principal directions of Σ. Our result for Σ_{11} is

$$\Sigma_{11}(\sigma_a,\sigma_b)\sim[h_1(h_2+h_3)/h_2h_3](\sigma_a\sigma_b)^{1/2} \quad \text{as } \sigma_a/\sigma_b\to 0 \text{ or } \infty. \tag{1.1}$$

For cubes this yields $2(\sigma_a\sigma_b)^{1/2}$, which was obtained before by Milton[2] and by Söderberg and Grimvall,[3] while when h_3 tends to infinity it yields Gautesen's two-dimensional result $(h_1/h_2)(\sigma_a,\sigma_b)^{1/2}$. Cyclic permutation of indices in (1.1) yields Σ_{22} and Σ_{33}.

In Sec. II we derive the result for a two-dimensional rectangular checkerboard and in Sec. III we derive the three-dimensional result (1.1). In Sec. IV we calculate the conductance σ between two highly conducting parallelograms that meet at a corner. Then in Sec. V we use σ to determine Σ for a checkerboard of parallelograms. The result (1.1) is not uniform in the h_i, so an appropriate modification of it is discussed in Sec. VI. Finally in Sec. VII we discuss these results and indicate some generalizations of them.

II. RECTANGULAR CHECKERBOARD PATTERN

We begin with the two-dimensional checkerboard of rectangles with conductivities σ_a and σ_b shown in Fig. 1(a). The principal axes of the effective conductivity tensor Σ are the x_1 and x_2 axes, so $\Sigma_{12}=\Sigma_{21}=0$. By definition Σ_{11} is just the average current density in the x_1 direction resulting from an electric field of unit strength in the x_1 direction. We suppose that $\sigma_a\gg\sigma_b$. Then the current will flow through the highly conducting regions as much as possible, and it will traverse the poorly conducting regions only at the corners where it goes from one highly conducting rectangle to a diagonally adjacent one. In Sec. IV we shall show that there is a well-defined conductance σ associated with such a corner.

We now use σ to find the current density due to a unit electric field along the x_1 axis. This field produces a voltage difference h_1 between the planes $x_1=-h_1/2$ and $x_1=h_1/2$. As a result a current $h_1\sigma$ flows across each corner, and the resulting current density Σ_{11} is this current divided by the vertical spacing h_2 between corners. Thus $\Sigma_{11}\sim h_1\sigma/h_2$, and similarly $\Sigma_{22}\sim h_2\sigma/h_1$. These results are asymptotic as $\sigma_a/\sigma_b\to\infty$ because only then can we associate all the conductance with the corners.

For a square checkerboard we have shown[4,5] that $\Sigma_{11}=\Sigma_{22}=(\sigma_a\sigma_b)^{1/2}$. Therefore by applying our asymptotic result to this case, for which $h_1=h_2$, we find that $\sigma\sim(\sigma_a\sigma_b)^{1/2}$. By using this value of σ in the preceding for-

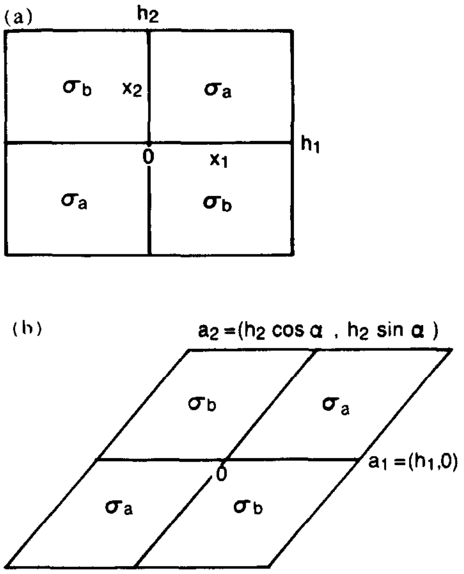

FIG. 1. (a) Part of a checkerboard pattern of rectangles with conductivities σ_a and σ_b. Edges parallel to the x_1 axis are of length h_1 and those parallel to the x_2 axis are of length h_2. (b) Part of an alternating pattern of parallelograms with conductivities σ_a and σ_b. The vertices are generated by the vectors $a_1=(h_1,0)$ and $a_2=(h_2\cos\alpha, h_2\sin\alpha)$.

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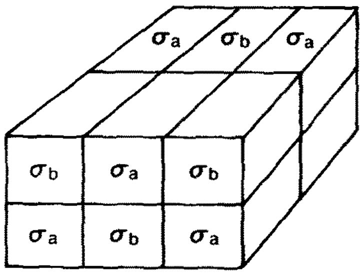

FIG. 2. Part of an alternating arrangment of rectangular parallelipipeds of conductivities σ_a and σ_b.

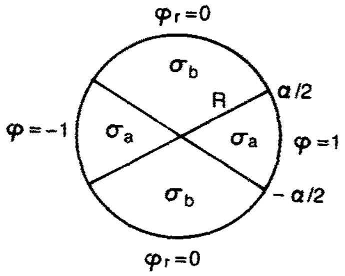

FIG. 3. A corner of the pattern in Fig. 1(b), rotated to be symmetric about the coordinate axes. The value of the potential $\varphi = \pm 1$ and its derivative $\varphi_r = 0$ are indicated on a circle of radius R centered at the vertex.

mulas, we obtain for the rectangular checkerboard

$$\Sigma_{11} \sim (h_1/h_2)(\sigma_a\sigma_b)^{1/2}, \quad \Sigma_{22} \sim (h_2/h_1)(\sigma_a\sigma_b)^{1/2} \quad \text{as } \sigma_a/\sigma_b \to \infty. \tag{2.1}$$

This is just Gautesen's result,[1] which he derived in a different way that proves it to be asymptotically correct. In Sec. IV we shall calculate σ directly for general corners, including rectangular ones, and again obtain the value $(\sigma_a\sigma_b)^{1/2}$ for the present case.

III. RECTANGULAR BLOCK PATTERN IN THREE DIMENSIONS

We shall now obtain the result (1.1) for the medium of alternating rectangular blocks shown in Fig. 2. The diagonal element Σ_{11} is, as before, the average current density in the x_1 direction due to a unit electric field in the x_1 direction. When $\sigma_a \gg \sigma_b$ the current will flow through the highly conducting blocks as much as possible. It will pass through the poorer conductors only along the edges where it goes from one highly conducting block to another. The conductance per unit length of such an edge is just σ, where σ is the two-dimensional conductance introduced in the preceding section. The voltage between the planes $x_1 = \pm h_1/2$ is just h_1. Therefore the current through each highly conducting block is $(2h_2 + 2h_3)h_1\sigma$ because $2h_2 + 2h_3$ is the length of edge between a highly conducting block and its highly conducting neighbors in the direction of increasing x_1. The current density is obtained by dividing this current by the area $2h_1h_2$, which is the cross-sectional area normal to the x_1 axis of a highly conducting block and a poorer conducting neighbor. In this way we get $\Sigma_{11} \sim h_1(h_2+h_3)\sigma/h_2h_3$. When we use the value $\sigma \sim (\sigma_a\sigma_b)^{1/2}$ in this formula, we obtain our result (1.1).

IV. RESISTANCE OF A CORNER

In order to treat the two-dimensional medium of alternating parallelograms shown in Fig. 1(b), we shall first determine the conductance $\sigma(\alpha)$ of the corner shown in Fig. 3. The medium with the high conductivity σ_a occupies the sector $-\alpha/2 < \theta < \alpha/2$ and the opposite sector, while the other two sectors contain the medium of conductivity σ_b. The corner is surrounded by a circle of radius R which is an insulator in the σ_b regions and a perfect conductor in the σ_a regions. Its potential is $+1$ in the interval $-\alpha/2 < \theta < \alpha/2$ and -1 in the opposite sector. Then the current between these two conductors is just the potential difference multiplied by the conductance, i.e., 2σ. We shall calculate the current and thus determine σ.

In terms of polar coordinates $\hat\rho$, θ the potential φ must be a function of $\hat\rho/R$ and θ, by dimensional analysis: $\varphi = \varphi(\hat\rho/R,\theta)$. Then the current, which is equal to 2σ, is given by

$$2\sigma(\alpha) = \int_{-\alpha/2}^{\alpha/2} \sigma_a \frac{\partial\varphi}{\partial\hat\rho}\left(\frac{\hat\rho}{R},\theta\right)\Big|_{\hat\rho=R} R\, d\theta = \sigma_a \int_{-\alpha/2}^{\alpha/2} \varphi_\rho(1,\theta)d\theta. \tag{4.1}$$

Here φ_ρ is the derivative of φ with respect to its first argument $\rho = \hat\rho/R$. From (4.1) we see that σ is independent of R, the radius of the circular conductors and insulators, so it can be interpreted as a property of the corner.

To simplify (4.1) we use the symmetry of φ about $\theta = 0$ to write the integral as twice the integral from 0 to $\alpha/2$:

$$\sigma = \sigma_a \int_0^{\alpha/2} \varphi_\rho(1,\theta)d\theta. \tag{4.2}$$

Now φ must be a harmonic function satisfying the following conditions:

$$\varphi(1,\theta) = 1, \quad 0 < \theta < \alpha/2, \tag{4.3}$$

$$\varphi_\rho(1,\theta) = 0, \quad \alpha/2 < \theta < \pi/2, \tag{4.4}$$

$$\varphi_\theta(\rho,0) = \varphi(\rho,\pi/2) = 0, \quad 0 < \rho < 1, \tag{4.5}$$

$$\varphi\left(\rho,\frac{\alpha}{2}-\right) = \varphi\left(\rho,\frac{\alpha}{2}+\right), \quad \sigma_a\varphi_\theta\left(\rho,\frac{\alpha}{2}-\right) = \sigma_b\varphi_\theta\left(\rho,\frac{\alpha}{2}+\right), \quad 0 < \rho < 1. \tag{4.6}$$

Equation (4.3) follows from the specification of the potential on the conductor, (4.4) is the condition of no current flow into the insulator, (4.5) expresses the evenness of φ about $\theta = 0$ and its oddness about $\theta = \pi/2$, while (4.6) states that φ and the normal component of current are continuous at $\theta = \alpha/2$.

To solve for φ we write

$$\varphi = A_a\rho^\nu \cos \nu\theta, \quad 0 \leqslant \theta \leqslant \alpha/2, \tag{4.7}$$

$$\varphi = A_b\rho^\nu \sin \nu(\pi/2 - \theta), \quad \alpha/2 \leqslant \theta \leqslant \pi/2. \tag{4.8}$$

These functions are harmonic for any ν and they satisfy (4.5). Upon imposing (4.6) we get

$$A_a \cos\frac{\nu\alpha}{2} = A_b \sin\nu\left(\frac{\pi}{2}-\frac{\alpha}{2}\right),$$
$$A_a\sigma_a \sin\frac{\nu\alpha}{2} = A_b\sigma_b \cos\nu\left(\frac{\pi}{2}-\frac{\alpha}{2}\right). \tag{4.9}$$

Dividing the second equation in (4.9) by the first yields

$$\sigma_a \tan\frac{\nu\alpha}{2} = \sigma_b \cot\nu\left(\frac{\pi}{2}-\frac{\alpha}{2}\right). \tag{4.10}$$

When $\sigma_a/\sigma_b \gg 1$, it follows from (4.10) that the first positive root for ν is small. Therefore we expand tan and cot and solve for ν to obtain

$$\nu \sim 2(\sigma_b/\alpha(\pi-\alpha)\sigma_a)^{1/2}, \quad \text{for } \sigma_a/\sigma_b \gg 1. \tag{4.11}$$

Now (4.7) and (4.8) become

$$\varphi \sim A_a\rho^\nu, \quad 0 \leqslant \theta \leqslant \alpha/2, \tag{4.12}$$

$$\varphi \sim A_b\rho^\nu\nu(\pi/2-\theta), \quad \alpha/2 \leqslant \theta \leqslant \pi/2. \tag{4.13}$$

By using (4.12) in (4.3) we find that $A_a \sim 1$ and then the first of Eqs. (4.9) yields $A_b \sim 2/\nu(\pi-\alpha)$. We also see from (4.13) that (4.4) is satisfied to order ν. Finally we use (4.12) for φ in (4.2) with $A_a \sim 1$ to get

$$\sigma(\alpha) \sim \sigma_a\nu\alpha/2. \tag{4.14}$$

Then by substituting (4.11) for ν into (4.14) we obtain the final result

$$\sigma(\alpha) \sim (\alpha\sigma_a\sigma_b/(\pi-\alpha))^{1/2}, \quad \text{for } \sigma_a/\sigma_b \gg 1. \tag{4.15}$$

When $\alpha = \pi/2$ this reduces to the result $\sigma(\pi/2) \sim (\sigma_a\sigma_b)^{1/2}$, which we obtained in Sec. II.

V. PARALLELOGRAMS IN A CHECKERBOARD PATTERN

We shall use the result (4.15) to calculate Σ for the two-dimensional checkerboard of parallelograms shown in Fig. 1(b). First we note that the average current density I is related to the average applied field E by $I = \Sigma E$, and therefore the component of I parallel to the unit vector n is

$$n \cdot I = n \cdot \Sigma E. \tag{5.1}$$

By using this relation for three pairs of values of n and E, we shall obtain three equations from which to determine the three independent components of Σ.

First we introduce the two vectors a_1 and a_2, which generate the lattice of vertices, defined by $a_1 = h_1(1,0)$ and $a_2 = h_2(\cos\alpha, \sin\alpha)$. Here h_1 and h_2 are the lengths of the two sides of a parallelogram, and α is the angle between them. The unit normals to these sides are $b_1 = (0,1)$ and $b_2 = (\sin\alpha, -\cos\alpha)$. Now we choose $n = E = b_1$ in (5.1) to obtain

$$b_1 \cdot I = \Sigma_{22}. \tag{5.2}$$

To compute the current density on the left side of (5.2) we note that the voltage across a parallelogram in the vertical direction is $h_2 \sin\alpha$. The vertical current through one highly conducting parallelogram is the sum of currents across two corners with angles α and $\pi - \alpha$. Thus the current is $h_2 \sin\alpha[\sigma(\alpha) + \sigma(\pi-\alpha)]$. The current density is obtained by dividing this current by $2h_1$, the horizontal extent of two parallelograms. Thus (5.2) yields

$$\Sigma_{22} \sim [(h_2\sin\alpha)/2h_1][\sigma(\alpha)+\sigma(\pi-\alpha)]. \tag{5.3}$$

Finally (4.15) and (5.3) give

$$\Sigma_{22} \sim \frac{h_2}{2h_1}(\sigma_a\sigma_b)^{1/2} \times \sin\alpha\left[\left(\frac{\alpha}{\pi-\alpha}\right)^{1/2} + \left(\frac{\pi-\alpha}{\alpha}\right)^{1/2}\right]. \tag{5.4}$$

Next we take $n = b_2$ and $E = b_1$ in (5.1) to get

$$b_2 \cdot I = \sin\alpha\Sigma_{12} - \cos\alpha\Sigma_{22}. \tag{5.5}$$

The vertical voltage across a parallelogram is still $h_2 \sin\alpha$. The net current through a parallelogram in the b_2 direction is the difference between the current in at the corner of angle α and the current out at the corner of angle $\pi - \alpha$. Thus it is $h_2 \sin\alpha[\sigma(\alpha) - \sigma(\pi-\alpha)]$, and it must be divided by the width $2h_2$ of two parallelograms in the a_2 direction. Thus (5.5) becomes

$$\sin\alpha\Sigma_{12} - \cos\alpha\Sigma_{22} \sim [(\sin\alpha)/2][\sigma(\alpha)-\sigma(\pi-\alpha)]. \tag{5.6}$$

Solving for Σ_{12} in (5.6) with the aid of (5.3) yields

$$\Sigma_{12} \sim [(h_2\cos\alpha)/2h_1][\alpha(\alpha)+\sigma(\pi-\alpha)] + \tfrac{1}{2}[\sigma(\alpha)-\sigma(\pi-\alpha)]. \tag{5.7}$$

This and (4.15) for σ gives

$$\Sigma_{12} \sim \frac{h_2}{2h_1}(\sigma_a\sigma_b)^{1/2}\cos\alpha\left[\left(\frac{\alpha}{\pi-\alpha}\right)^{1/2} + \left(\frac{\pi-\alpha}{\alpha}\right)^{1/2}\right] + \frac{(\sigma_a\sigma_b)^{1/2}}{2}\left[\left(\frac{\alpha}{\pi-\alpha}\right)^{1/2} - \left(\frac{\pi-\alpha}{\alpha}\right)^{1/2}\right]. \tag{5.8}$$

As a third choice we take $n = E = b_2$ in (5.1), which becomes

$$b_2 \cdot I = \sin^2\alpha\Sigma_{11} - 2\sin\alpha\cos\alpha\Sigma_{12} + \cos^2\alpha\Sigma_{22}. \tag{5.9}$$

The voltage in the b_2 direction across one parallelogram is $b_2 \cdot a_1 = h_1 \sin\alpha$ and the current in the b_2 direction is $h_1 \sin\alpha[\sigma(\alpha)+\sigma(\pi-\alpha)]$. Dividing this current by $2h_2$ and using it in (5.9) yields

$$\sin^2\alpha\Sigma_{11} - 2\sin\alpha\cos\alpha\Sigma_{12} + \cos^2\alpha\Sigma_{22} \sim [(h_1\sin\alpha/2h_2][\sigma(\alpha)+\sigma(\pi-\alpha)]. \tag{5.10}$$

Solving for Σ_{11} leads to

$$\Sigma_{11} \sim \left(\frac{h_1}{h_2} + \frac{h_2}{h_1}\cos^2\alpha\right)\frac{1}{2\sin\alpha}[\sigma(\alpha)+\sigma(\pi-\alpha)] + \frac{\cos\alpha}{\sin\alpha}[\sigma(\alpha)-\sigma(\pi-\alpha)]. \tag{5.11}$$

When (4.15) is used in (5.11) it becomes

$$\Sigma_{11} \sim \left(\frac{h_1}{h_2} + \frac{h_2\cos^2\alpha}{h_1}\right)\frac{(\sigma_a\sigma_b)^{1/2}}{2\sin\alpha} \times \left[\left(\frac{\alpha}{\pi-\alpha}\right)^{1/2} + \left(\frac{\pi-\alpha}{\alpha}\right)^{1/2}\right] + (\sigma_a\sigma_b)^{1/2} \times \frac{\cos\alpha}{\sin\alpha}\left[\left(\frac{\alpha}{\pi-\alpha}\right)^{1/2} - \left(\frac{\pi-\alpha}{\alpha}\right)^{1/2}\right]. \tag{5.12}$$

Equations (5.4), (5.8), and (5.12) determine Σ.

VI. NONUNIFORMITY

The result (2.1) is not valid when h_2/h_1 tends to zero or to infinity. To obtain a result that is uniformly valid we must take account of the conductivity of the material away from the corner. We can do this roughly by replacing σ in the expression $\Sigma_{11} \sim h_1\sigma/h_2$ by the series-parallel conductance

$$\frac{1}{\sigma^{-1} + 2h_1/h_2\sigma_a} + \frac{h_2}{h_1/2\sigma_a + h_1/2\sigma_b} \sim \frac{1}{\sigma^{-1} + 2h_1/h_2\sigma_a} + \frac{2h_2\sigma_b}{h_1}. \quad (6.1)$$

The first term accounts for the fact that the corner is in series with the resistance of half the rectangle of material σ_a, and this resistance tends to $2h_1/h_2\sigma_a$ as h_1/h_2 becomes large. The second term represents the conductance directly across the rectangles, which tends to $2h_2\sigma_b/h_1$ as $\sigma_a/\sigma_b \to \infty$. Then Σ_{11} becomes, with $\sigma = (\sigma_a\sigma_b)^{1/2}$ in (6.1),

$$\Sigma_{11} \sim \frac{h_1}{h_2}(\sigma_a\sigma_b)^{1/2}\left[1 + \frac{2h_1}{h_2}\left(\frac{\sigma_a}{\sigma_b}\right)^{1/2}\right]^{-1} + 2\sigma_b \quad \text{as } \frac{\sigma_a}{\sigma_b} \to \infty. \quad (6.2)$$

By interchanging h_1 and h_2 in (6.2) we get Σ_{22}.

From (6.2) we find that

$$\Sigma_{11} \sim 2\sigma_b, \quad \text{for } \frac{h_1}{h_2} \ll \left(\frac{\sigma_b}{\sigma_a}\right)^{1/2}, \quad (6.3)$$

$$\Sigma_{11} \sim \frac{\sigma_a}{2}, \quad \text{for } \frac{h_1}{h_2} \gg \left(\frac{\sigma_a}{\sigma_b}\right)^{1/2}, \quad (6.4)$$

$$\Sigma_{11} \sim \frac{h_1}{h_2}(\sigma_a\sigma_b)^{1/2}, \quad \text{for } \left(\frac{\sigma_b}{\sigma_a}\right)^{1/2} \ll \frac{h_1}{h_2} \ll \left(\frac{\sigma_a}{\sigma_b}\right)^{1/2}. \quad (6.5)$$

The conditions for validity of (2.1) are thus those in (6.5).

In the same way, we can modify (1.1) for rectangular blocks to obtain

$$\Sigma_{11} \sim \frac{h_1}{2h_2h_3}\left[\left(\frac{(\sigma_a\sigma_b)^{-1/2}}{2h_2 + 2h_3} + \frac{h_1}{h_2h_3\sigma_a}\right)^{-1} + 2h_2h_3\left(\frac{h_1}{2\sigma_a} + \frac{h_1}{2\sigma_b}\right)^{-1}\right]$$
$$\sim \frac{h_1(h_2+h_3)}{h_2h_3}(\sigma_a\sigma_b)^{1/2} \times \left[1 + \frac{2h_1(h_2+h_3)}{h_2h_3}\left(\frac{\sigma_b}{\sigma_a}\right)^{1/2}\right]^{-1} + 2\sigma_b. \quad (6.6)$$

Thus

$$\Sigma_{11} \sim 2\sigma_b, \quad \text{for } \frac{h_1(h_2+h_3)}{h_2h_3} \ll \left(\frac{\sigma_b}{\sigma_a}\right)^{1/2}, \quad (6.7)$$

$$\Sigma_{11} \sim \frac{\sigma_a}{2}, \quad \text{for } \frac{h_1(h_2+h_3)}{h_2h_3} \gg \left(\frac{\sigma_a}{\sigma_b}\right)^{1/2}, \quad (6.8)$$

$$\Sigma_{11} \sim \frac{h_1(h_2+h_3)}{h_2h_3}(\sigma_a\sigma_b)^{1/2}, \quad \text{for } \left(\frac{\sigma_b}{\sigma_a}\right)^{1/2} \ll \frac{h_1(h_2+h_3)}{h_2h_3} \ll \left(\frac{\sigma_a}{\sigma_b}\right)^{1/2}. \quad (6.9)$$

VII. DISCUSSION

The method of Sec. V can be applied to a three-dimensional alternating configuration of parallelepipeds, using the value of σ given by (4.15). Furthermore all of our results remain valid if the squares, parallelograms, rectangular blocks, or parallelepipeds are distorted, provided that their shapes near the corners in two dimensions, and near the edges in three dimensions, are unchanged. In addition the method can be applied to three-dimensional periodic media with curved edges and a variable angle $\alpha(s)$ along each edge. Then we must integrate $\sigma[\alpha(s)]$ along each edge to find its conductance.

The concept of corner conductance can be extended to other kinds of "corners" besides those treated in Sec. IV. For example, suppose that the two highly conducting sectors in Fig. 3 did not meet, but were separated by a small gap filled with the low conductance material. Then the conductance between the two highly conducting sectors could still be defined, and the same method could employed. The results of Sec. II, III, V would still apply with the appropriate value of σ.

The possibility of analyzing a continuous problem by replacing it with a network of lumped elements is a consequence of the asymptotic behavior of the solution with respect to some parameter. In the present case the parameter is the conductivity ratio σ_a/σ_b, which tends to zero or to infinity. In other cases it is a geometrical ratio. The analytical basis for the procedure is provided by the method of matched asymptotic expansions. In the present case, for example, the construction in Sec. IV provides the leading term in the inner expansion valid near each corner of the rectangles or parallelograms. The leading term in the outer expansion within each highly conducting rectangle or parallelogram is a harmonic function. It has current sources at two vertices and curent sinks at the other two, and a vanishing normal derivative on the boundaries. The magnitudes of the currents are determined by matching the inner and outer expansions. By constructing these expansions we could obtain further terms in the asymptotic expansion of Σ.

We have used similar ideas before to treat periodic configurations of perfectly conducting cylinders or spheres, or nonconducting cylinders, in a finitely conducting matrix.[6] Batchelor and O'Brien[7] carried it over to highly conducting bodies, and Buchal and Keller[8] extended it to time harmonic problems.

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[1]A. Gautesen, "The effective conductivity of a rectangular checkerboard composite," preprint, Ames Research Laboratory, Iowa State University.
[2]G. W. Milton, "Theoretical studies of the transport properties of inhomogeneous media," Report Physics IV, Department of Theoretical Physics, University of Sydney, 1979.
[3]M. Söderberg and G. Grimvall, "Current distributions for a two-phase material with chequer-board geometry," J. Phys. C: Solid State Phys. **16**, 1085 (1983).
[4]J. B. Keller, "A theorem on the conductivity of a composite medium," J. Math. Phys. **5**, 548 (1964).
[5]J. Nevard and J. B. Keller, "Reciprocal relations for effective conductivities of anisotropic media," J. Math. Phys. **26**, 2761 (1985).
[6]J. B. Keller, "Conductivity of a medium containing a dense array of perfectly conducting spheres or cylinders, or nonconducting cylinders," J. Appl. Phys. **34**, 991 (1963).
[7]G. K. Batchelor and R. W. O'Brien, "Thermal or electrical conduction through a granular material," Proc. R. Soc. London Ser. A **355**, 313 (1977).
[8]R. Buchal and J. B. Keller, "Impedance between perfect conductors in a finitely conducting medium with application to composite media," J. Appl. Phys. **34**, 3414 (1963).