A Theorem on the Conductivity of a Composite Medium*

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A composite medium consisting of a rectangular lattice of identical parallel cylinders of arbitrary cross section is considered. The cylinders have conductivity σ_2 and are imbedded in a medium of conductivity σ_1. Simple properties of the conductivity tensor of the composite medium are deduced from the theory of harmonic functions.

LET us consider a composite medium consisting of a rectangular lattice of identical parallel cylinders of any cross section, having electrical conductivity σ_2 imbedded in a medium of conductivity σ_1. Let the x and y axes lie along axes of the lattice and let $2X$ and $2Y$ be the lattice spacings in the x and y directions. When a static electric field of average field strength E_x is applied to the medium parallel to the x axis, the resulting current density will be a periodic function of x and y. If this current density is averaged with respect to y, the average current density j_x must be in the x direction and must be independent of x by conservation of charge. In terms of j_x and E_x we define the effective conductivity $\Sigma_x(\sigma_1, \sigma_2)$ in the x direction to be $\Sigma_x(\sigma_1, \sigma_2) = j_x/E_x$. The first argument of Σ_x denotes the conductivity of the medium surrounding the cylinders and the second argument denotes that of the medium constituting the cylinders. In a similar way we define Σ_y. The object of this note is to prove the following theorem:

Theorem. Let a medium contain a rectangular lattice of identical parallel cylinders, each of which is symmetric in the x and y axes, which are the lattice axes. Then the effective conductivities Σ_x and Σ_y of the composite medium in the x and y directions are related by

$$\sigma_1/\Sigma_x(\sigma_1, \sigma_2) = \Sigma_y(\sigma_2, \sigma_1)/\sigma_2. \tag{1}$$

The first argument of Σ_x or Σ_y denotes the conductivity of the medium surrounding the cylinders and the second argument that of the medium constituting the cylinders. The theorem also applies to thermal and other conductivities when the corresponding potentials are harmonic functions.

An important corollary of this theorem follows when the lattice is square and when each cylinder is symmetric in the line $x = y$. In this case the x and y directions are equivalent so $\Sigma_x(\sigma_1, \sigma_2) = \Sigma_y(\sigma_1, \sigma_2)$. Then we may omit the subscript and write $\Sigma = \Sigma_x = \Sigma_y$. Therefore the theorem yields the corollary

Corollary 1. When the lattice is square and each cylinder is symmetric in the line $x = y$,

$$\sigma_1/\Sigma(\sigma_1, \sigma_2) = \Sigma(\sigma_2, \sigma_1)/\sigma_2. \tag{2}$$

Another corollary of the theorem results from the fact that Σ_x and Σ_y have the dimensions of conductivity, so they are homogeneous of degree one in σ_1 and σ_2. Therefore $\Sigma_x(\sigma_1, \sigma_2)/\sigma_1 = \Sigma_x(1, \sigma_2/\sigma_1)$ and $\Sigma_y(\sigma_2, \sigma_1)/\sigma_2 = \Sigma_y(1, \sigma_1/\sigma_2)$. Now the theorem yields the further corollary

Corollary 2.

$$1/\Sigma_x(1, \sigma_2/\sigma_1) = \Sigma_y(1, \sigma_1/\sigma_2). \tag{3}$$

When the hypothesis of Corollary 1 is satisfied, Corollary 2 yields

$$1/\Sigma(1, \sigma_2/\sigma_1) = \Sigma(1, \sigma_1/\sigma_2). \tag{4}$$

The special cases of (3) and (4) in which $\sigma_2/\sigma_1 = \infty$ and $\sigma_1/\sigma_2 = 0$ were proved previously.[1] It is interesting to note that the approximate expression for Σ derived by Rayleigh[2] for a square lattice of circular cylinders satisfies (4).

The results (2) and (4) also apply to the average conductivity of a statistically homogeneous isotropic random distribution of cylinders of one medium in another medium. This can be proved by appropriately adapting the following proof.

To prove the theorem we first consider the harmonic function $\varphi(x, y)$ which is the potential corresponding to the applied field of average strength unity in the x direction. By symmetry, the lines $x = 0$ and $x = X$ are equipotential lines while $y = 0$ and $y = Y$ are field lines. If we let

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[1] J. B. Keller, J. Appl. Phys. 34, 991 (1963).
[2] Lord Rayleigh, Phil. Mag. Ser. 5 34, 481 (1892).

$\varphi(0, y) = 0$ then $\varphi(X, y) = X$ while $\varphi_y(x, 0) = \varphi_y(x, Y) = 0$. On the interface S between the two media, continuity of potential and of the normal component of current yield

$$\varphi^+ = \varphi^- \quad \text{on} \quad S, \tag{5}$$

$$\sigma_1\, \partial\varphi^+/\partial n = \sigma_2\, \partial\varphi^-/\partial n \quad \text{on} \quad S. \tag{6}$$

Here φ^+ and φ^- denote the values of φ outside S and inside S, respectively, while $\partial/\partial n$ denotes the derivative along the outward normal to S and σ_1 is the conductivity outside S. The average current across the line $x = X$ is

$$j_x = Y^{-1} \int_0^Y \sigma_1 \varphi_x^+(X, y)\, dy = \sigma_1 Y^{-1} \int_0^Y \psi_y(X, y)\, dy = \sigma_1 Y^{-1} \psi_0. \tag{7}$$

Here ψ is the harmonic function conjugate to φ which has the value zero on the field line $y = 0$ and the unknown value ψ_0 on the field line $y = Y$.

From (5) it follows that $\partial\psi^+/\partial n = \partial\psi^-/\partial n$ on S and from (6) that $\sigma_1\partial\psi^+/\partial s = \sigma_2\partial\psi^-/\partial s$ on S where $\partial/\partial s$ denotes differentiation along S. Upon integrating the last relation and noting that $\psi^+ = \psi^-$ at $y = 0$, it follows that $\sigma_1\psi^+ = \sigma_2\psi^-$. We also note that $\psi_x(0, y) = \psi_x(X, y) = 0$. Therefore we define Φ by the relations

$$\Phi^+ = Y\psi^+/\psi_0, \tag{8}$$

$$\Phi^- = \sigma_2 Y\psi^-/\sigma_1\psi_0. \tag{9}$$

Then Φ is a harmonic function satisfying the conditions $\Phi_x(0, y) = \Phi_x(X, t) = 0$; $\Phi(x, 0) = 0$, $\Phi(x, Y) = Y$, $\Phi^+ = \Phi^-$ on S, and $\sigma_2\partial\Phi^+/\partial n = \sigma_1\partial\Phi^-/\partial n$ on S. Thus Φ is the potential corresponding to an applied field of average strength unity in the y direction when the conductivity outside S is σ_2 and that inside S is σ_1. The average current in the y direction is then

$$j_y = X^{-1} \int_0^X \sigma_2 \Phi_y(x, Y)\, dx = \sigma_2 X^{-1} Y \psi_0^{-1} \times \int_0^X \psi_y(x, Y)\, dx = \sigma_2 X^{-1} Y \psi_0^{-1} \times \int_0^X \varphi_x(x, Y)\, dx = \sigma_2 Y \psi_0^{-1}. \tag{10}$$

Since $E_x = E_y = 1$, it follows that $\Sigma_x = j_x$ and $\Sigma_y = j_y$. Then from (7) and (10) the result (1) of the theorem follows.