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An experimental analysis of electromagnetic forces in liquids

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Abstract. Three theories are outlined to explain the observed forces in fluids. A systematical analysis, via a number of appropriate experiments with liquids, shows that only one of them is substantially correct. A new method for studying the electrostriction in liquids is presented.

1. Introduction

Although the technological importance of the electromagnetic forces in fluids seems to be rather secondary, it has been an exciting subject of controversy during the last hundred years. We must say that there is no reason to consider this as an open problem any more: a long series of experiments scattered along the years show conclusively that the actual electromagnetic force density on a neutral fluid, without any macroscopic current in it, is given, within a reasonable degree of approximation, by the concordant theories of Maxwell (1873), Einstein and Laub (1908), Livens (1918) and others:

$$\boldsymbol{f}_1=(\boldsymbol{P}.\nabla)\,\boldsymbol{E}+(\boldsymbol{M}.\nabla)\,\boldsymbol{H} \tag{1}$$

or, more precisely, by the theories of Helmholtz (1881), Jeans (1911), Debye (1925) and others:

$$\hat{\boldsymbol{f}}_1=(\boldsymbol{P}.\nabla)\,\boldsymbol{E}+(\boldsymbol{M}.\nabla)\,\boldsymbol{H}+\boldsymbol{G} \tag{2}$$

with

$$\boldsymbol{G}=\hat{\boldsymbol{f}}_1-\boldsymbol{f}_1=\left[\frac{\epsilon_0}{2}\nabla\left(E^2\tau\,\frac{\partial\epsilon_r}{\partial\tau}\right)-\tfrac{1}{2}\nabla\,(\boldsymbol{P}.\boldsymbol{E})\right]+\left[\frac{\mu_0}{2}\nabla\left(H^2\tau\,\frac{\partial\mu_r}{\partial\tau}\right)-\tfrac{1}{2}\nabla\,(\boldsymbol{H}.\boldsymbol{M})\right]. \tag{3}$$

Note that SI units have been used with the definitions

$$\boldsymbol{D}=\epsilon_0\boldsymbol{E}+\boldsymbol{P}$$

$$\boldsymbol{B}=\mu_0\boldsymbol{H}+\boldsymbol{M}$$

$$\epsilon=\epsilon_0\epsilon_r;\quad \mu=\mu_0\mu_r$$

τ = mass density.

Let us recall that, in all the experiments up to date, the difference $\hat{\boldsymbol{G}}=\hat{\boldsymbol{f}}_1-\boldsymbol{f}_1$ is experimentally undetectable in the gas case (Jaumann and Stipa 1934) as well as in all paramagnetic liquids. In the case of high permittivity liquids $\hat{\boldsymbol{G}}$ is negative when $\partial\epsilon_r/\partial\tau$ is measured along an adiabatic curve and positive when the isothermal quantity $(\partial\epsilon_r/\partial\tau)_T$ is used. Its actual value, in any case, is relatively small (Bruhat 1962), and only after

Zahn's (1962) quantitative experiments has a decision been made in favour of Debye's theory, in agreement with (2).

The old theories of Hertz (1890), Heaviside (1891), Drude (1894), Minkowski (1907), Abraham (1909) and many others, which, indefensibly, omit the electrostriction and magnetostriction terms, became obsolete after the electrostriction experiments with gases by Gans (1903) and, more clearly, in the 1920s when the semiquantitative experiments on electrostriction in liquids by Pauthenier and Bruhat (1925) appeared.

In spite of all this, some unjustified doubts and confusing opinions have been recently raised (Lee 1957); furthermore, the Minkowskian momentum-tensor (without any mention of the electrostriction term) is still discussed nowadays as valid in several theoretical papers (Pauli 1958) which seem to ignore the aforesaid experimental facts. In order to clarify the present status of this force, we have successfully performed a number of simple, though basic, experiments which, through their new technical details and originality, add further confirmation of the force densities (1) and (2) against the rest of the theories. Penfield and Haus (1967) have stressed the importance of experimental verification in dealing with these topics.

2. Preliminary experiments

Quincke (1885) determined dielectric constants by measuring the level difference between the liquid in an open glass U-tube, one of the arms being placed in a known electric field which is parallel to the liquid surface. This method does not seem to have been widely used. The obvious reason is that it requires a preliminary determination of the relative impedances of the glass, liquid and surrounding air, which are difficult to ascertain: only in very good insulating liquids can this be overlooked. It is, on the contrary, a well-known practice to measure susceptibilities of paramagnetic liquids using Quincke's technique.

Three main theories can methodologically be considered in order to explain these phenomena. Their corresponding force densities are as follows:

(*a*) *Maxwell, Einstein and Laub* (1908), *and others:*

$$\boldsymbol{f}_1=(\boldsymbol{P}.\nabla)\,\boldsymbol{E}+(\boldsymbol{M}.\nabla)\,\boldsymbol{H}=\tfrac{1}{2}\,\mathrm{grad}\,(\boldsymbol{E}.\boldsymbol{P})-\frac{\epsilon_0}{2}E^2\,\mathrm{grad}\,\epsilon_\mathrm{r}+\text{magnetic terms} \tag{4}$$

(*b*) *Hertz, Heaviside, Minkowski, Abraham* (1909), *and others:*

$$\boldsymbol{f}_2=(\boldsymbol{P}.\nabla)\,\boldsymbol{E}+(\boldsymbol{M}.\nabla)\,\boldsymbol{H}-\tfrac{1}{2}\,\mathrm{grad}\,(\boldsymbol{E}.\boldsymbol{P}+\boldsymbol{M}.\boldsymbol{H})=-\frac{\epsilon_0}{2}E^2\,\mathrm{grad}\,\epsilon_\mathrm{r}-\frac{\mu_0}{2}H^2\,\mathrm{grad}\,\mu_\mathrm{r} \tag{5}$$

(*c*) '*Electret-like*' *force density:*

$$\boldsymbol{f}_3=\boldsymbol{E}\,(-\mathrm{div}\,\boldsymbol{P})+\boldsymbol{H}\,(-\mathrm{div}\,\boldsymbol{M}). \tag{6}$$

The last theory (*c*) is mentioned by Becker (1964) as a possibility, after the idea that $(-\mathrm{div}\,\boldsymbol{P})$ is the polarization charge density in dielectrics. The name 'electret-like' refers to the theory of one of us (Lahoz 1974) which claims that the force (6) must be observable on an electret material immersed inside an electric field.

Although no substantial novelties are claimed in this section, the use of more modern facilities, as well as the reasons pointed out above, seem to justify a brief description of our preliminary experiments.

The effect of a magnetic field parallel to the surface of a paramagnetic liquid can be readily observed by clamping a cylindrical test cell between two poles of an electromagnet, as in figure 1(*a*). The change in pressure is recorded by extending a tube of relatively small cross section outside the magnetic field. From the change of the liquid level and the liquid mass density, the pressure change is simply $\Delta p = \tau g \Delta h$. To obtain a sufficiently high permeability an almost totally saturated solution of $FeCl_3$ was prepared.

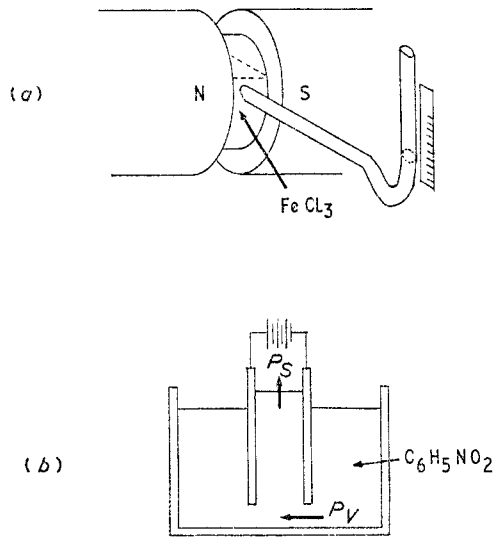

Figure 1. Field parallel to the surface: (*a*) ***B*** field, (*b*) ***E*** field.

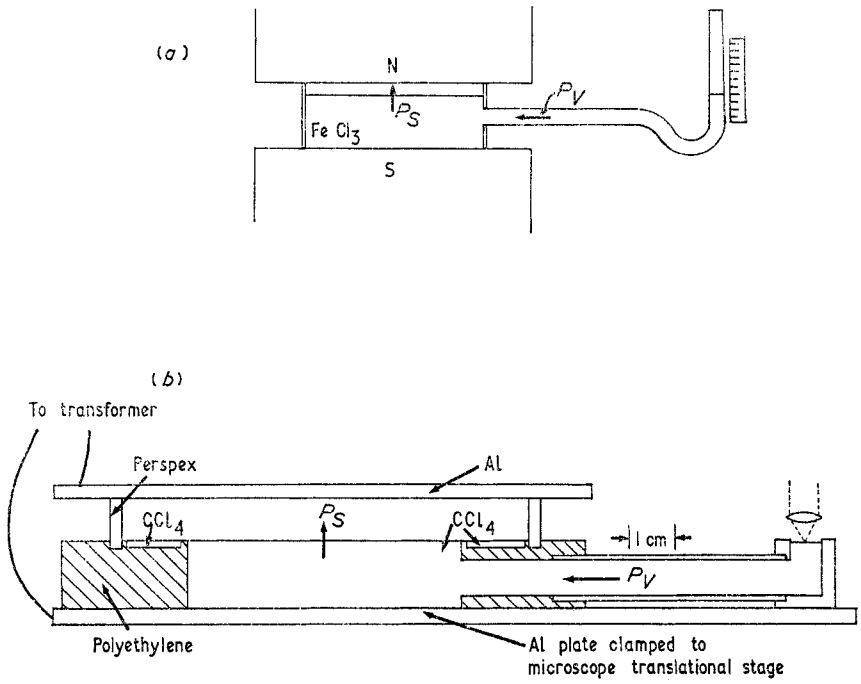

Figure 2. Field perpendicular to the surface: (*a*) ***B*** field; (*b*) ***E*** field.

For the perpendicular case using the magnetic field it was only necessary to rotate the magnetic poles to provide a vertical field. By reorientating the manometer tube the pressure was measured in precisely the same manner as before (figure 2*a*).

The parallel electric field case was also relatively easy to implement (figure 1*b*). In this instance, to create a macroscopic effect, we used nitrobenzene ($\epsilon_r \simeq 30$) since the full voltage drop is across the liquid.

However, when the electric field is perpendicular to the surface, the finite resistivity of dielectric liquids must be considered. To permit a calculably sufficient voltage drop across the liquid, the impedance of the liquid must be primarily reactive, i.e. $\omega\epsilon\rho \gg 1$. For 60 Hz operation therefore, CCl_4 was selected despite its low relative permittivity of 2·25 which results in a reduced force within the liquid. By using a precision microscope focused onto the liquid surface outside the electric field, it was possible to detect vertical liquid movements in excess of 1 μm. Figure 2(*b*) illustrates the particular geometry used, the container for the CCl_4 being machined out of polyethylene ($\epsilon_r = 2{\cdot}28$) to minimize fringing of the field at liquid interfaces. The necessary 15 kV was obtained from a high-voltage line transformer.

For all these experiments equation (2) is equivalent to (*a*) and (*b*) theories, because the striction terms, which purely compress the liquid, do not contribute any readable effect to the bulk motion of the mass centre of the liquid. A simple analysis (Lahoz 1974) shows that the implications of (*a*), (*b*) and (*c*) theories are as listed in table 1.

Table 1. Forces in fluids according different theories

		(*a*)	(*b*)	(*c*)
	P_S	0	$\frac{1}{2}MH$	0
Case 1	P_V	$\frac{1}{2}MH$	0	$\frac{1}{2}MH$
(figure 1)	P_{total}	$\frac{1}{2}MH$	$\frac{1}{2}MH$	$\frac{1}{2}MH$
	P_S	$\frac{M^2}{2\mu_0}$	$\frac{1}{2}MH+\frac{M^2}{2\mu_0}$	$MH+\frac{M^2}{2\mu_0}$
Case 2	P_V	$\frac{1}{2}MH$	0	$\frac{1}{2}MH$
(figure 2)	P_{total}	$\frac{1}{2}MH+\frac{M^2}{2\mu_0}$	$\frac{1}{2}MH+\frac{M^2}{2\mu_0}$	$\frac{3}{2}MH+\frac{M^2}{2\mu_0}$

To obtain the corresponding pressures using an electric field one needs simply to replace M and H by P and E, respectively, in table 1. P_S and P_V (see figures 1 and 2) represent, respectively, the pressures at the surface and at the liquid volume where the gradient of the field takes place. When using a magnetic field the forces arising from the term

$$\frac{M^2}{2\mu_0} = \frac{\mu_0}{2}(\mu_r - 1)^2 H^2 \tag{7}$$

are obviously negligible for paramagnetic materials and the measurable hydrostatic pressure is

$$\tau g \Delta h = \tfrac{1}{2} MH = \tfrac{1}{2} \chi_m \frac{B^2}{{\mu_0}^2}. \tag{8}$$

In our case for a nearly saturated solution of $FeCl_3$ the magnetic susceptibility in SI units was:

$$\chi_m = \frac{M}{H} = \mu_0(\mu_r - 1) = 0{\cdot}431\,\mu_0{}^2\tau \tag{9}$$

where τ is the mass density of the liquid. Thus, for $B=1{\cdot}45$ Wb m^{-2}, the expected height difference of the liquid was $\Delta h=0{\cdot}046$ m. Our result for the geometry of figure 1 very consistently agreed with this value. This was as expected, since the susceptibilities are commonly obtained using this method. The significant point is, however, that in the case 2 (field perpendicular to the surface) we definitively found the same result within the readability error limits, 2%. Consequently theory (*c*) is ruled out for paramagnetic liquids.

In order to demonstrate this fact in the case of more homogeneous liquids, we will quote the results for the electric field in case 2 in some detail. Incidentally it may be mentioned that experiments were conducted with nitrobenzene for the case 1 geometry: with some kilovolts of DC across the plates and a liquid gap in the range of 2 mm, the results were found to be in agreement with the three theories. The actual predicted level difference of the liquid (from 0·2 cm up to 2 cm) as well as its quadratic dependence on the electric field were verified: in spite of small disturbances due to ionic currents and capillary action nothing of the linearity between height and electric field, as reported by Middendorf and Brown (1958), was found.

In the geometry of case 2, with CCl_4 as dielectric liquid, the predicted net pressure $\langle \Delta p \rangle$ according to theories (*a*) and (*b*) is

$$\langle \Delta p \rangle = \tau g \langle \Delta h \rangle = \frac{\epsilon_0}{2} (\epsilon_r - 1) \langle E^2 \rangle \tag{10}$$

and referring to figure 2(*b*)

$$\langle E^2 \rangle = \tfrac{1}{2} \times 533{\cdot}6 \times V_{max}{}^2 \tag{11}$$

where V_{max} is the amplitude of the 60 Hz applied voltage. The results are summarized in table 2.

Table 2. Results concerning the experiment of figure 2(*b*)

V_{max}(kV)	$\Delta h^{(a)\,(b)}_{calc.}$ (μm)	$\Delta h^{(c)}_{calc.}$ (μm)	$\Delta h_{meas.}$ (μm)
15 (±1)	47·8 (±4)	90·3 (±8)	47·5±2
7·5 (±0·5)	11·95 (±1)	22·6 (±2)	12·7±2

Therefore theory (*c*) is clearly invalid in the fluid case, and theories (*a*) and (*b*) explain very adequately the experimental results. However, as already mentioned, these experiments are not able to differentiate between theories (*a*) and (*b*).

3. A new method for studying electrostriction in liquids

For the elimination of theory (*b*) a qualitative electrostriction experiment is sufficient. In fact, for a liquid with no free surface in the electric field region one has $-\frac{1}{2} E^2 \text{ grad } \epsilon \simeq 0$ and a glance at equations (4) and (5) shows that only the former contains qualitatively (Bruhat 1962) the electrostriction term, $\frac{1}{2} \text{ grad } E^2 \, \tau (\partial \epsilon / \partial \tau) \approx \frac{1}{2} \text{ grad } (\boldsymbol{E}.\boldsymbol{P})$.

A common technique for producing electrostriction in liquids consists of the excitation of resonant modes with the two plates of the electrostriction condenser in direct contact with the liquid (Goetz 1955). In our case a cleaner excitation of the longitudinal modes of a cylindrical liquid column, confined in a glass U-tube, was produced as follows.

Two steel plates 15 cm square, separated by 1·4 cm nylon spacers and mounted to permit precise vertical translation without touching the glass U-tube, formed the conventional electrostriction condenser. The 1·1 cm diameter U-tube was centred between the plates and clamped to a fixed support. The thickness of the plates, 1·2 cm, and the existence of the gap, prevented the coupling to the tube of acoustic waves resulting from movement of the plates. A metre length of surgical tubing led from one arm of the U-tube, through a 15 cm thick wall, to a detector situated in an adjoining room and mounted within a chamber constructed of alternate layers of brick and foam to isolate the detector from ambient noise (figure 3). The detector itself was a commercial ceramic cartridge suitably adapted to respond to acoustic pressure waves. The detected electrical signal of about 10 μV was displayed on a Tektronix 551 oscilloscope fitted with a 1A7 plug-in unit incorporating high gain and selectable filtering.

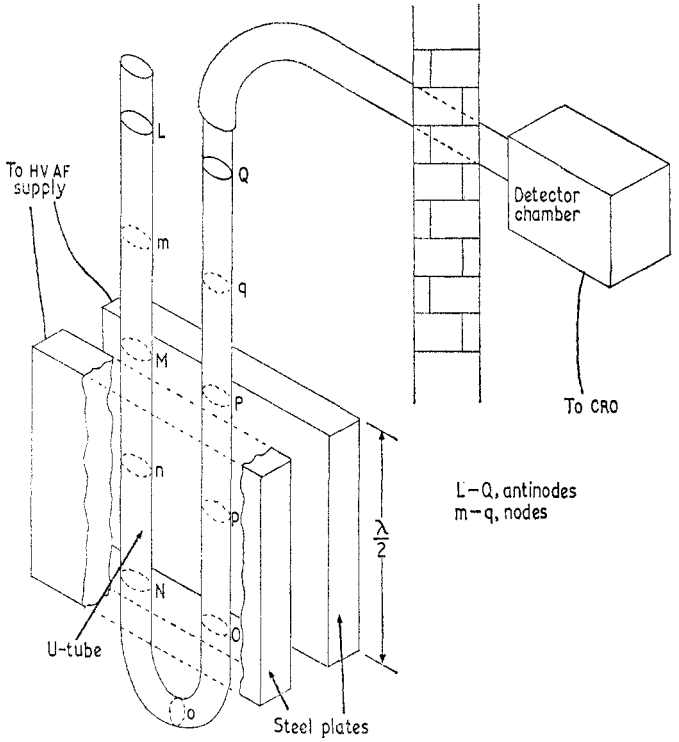

Figure 3. Electrostrictive excitation of the modes of a liquid column.

The choice of frequency was dictated by the response of the detector and the use of nitrobenzene as the dielectric: an analysis showed that in the range 3–20 kHz the impedance of the nitrobenzene was primarily reactive.

The electronic portion of the experimental system comprised a variable frequency signal source and a three-stage amplifier capable of supplying audiofrequency voltages in excess of 30 kV p/p necessary for detectable excitation of the liquid. The output push–pull stage was tuned for optimum gain in a particular frequency range by a 15 H coil in parallel with the plate capacitance of the tubes. The anodes of the output tubes were connected to the plates of the test condenser.

With the AC voltage, of amplitude V_0, across the condenser an electric field, $E = E_0 \sin(2\pi f_0 t)$, where $E_0 \propto V_0$, is set up within the liquid. The corresponding electrostrictive

pressure appears at points M, N, O, P (figure 3) and is given by

$$\Delta p = \left(\tfrac{1}{2}E^2 \tau \frac{\partial \epsilon}{\partial \tau}\right)_m^n = bE^2 = bE_0^2 \left(\frac{1+\cos 2\pi (2f_0)\, t}{2}\right). \tag{12}$$

Consequently, besides the average strictive pressure $bE_0^2/2$, a sonic wave of frequency $2f_0$ is generated in the liquid column and transmitted to the detector via the air column confined in the surgical tubing.

With the aid of a pipette, liquid was added or removed from the U-tube to achieve the resonance, as depicted in figure 3. In this situation the wavelength and resonant frequency are determined by the height of the plates, $15\ \text{cm} = \lambda/2 = v/4f_0$, where $v = 1463\ \text{m s}^{-1}$ is the velocity of the longitudinal waves in nitrobenzene at 25 °C. Thus $f_0 = v/0{\cdot}6 \simeq 2{\cdot}44$ kHz: by setting the signal generator to 2·5 kHz a definite peak of about 15 μV (at 5 kHz) could be detected when the edges of the plates were appropriately placed with respect to the liquid column. The Q for the vibrating system was over 100. Slightly increasing the temperature of the liquid had the effect of lowering the resonant frequency as expected. The total length of the liquid column, 73·6 cm ($=5\ \lambda/2$), was very critical: a change of the order of 1 mm reduced the resonant amplitude significantly. The amplitude of the detected acoustic signal when the plates were systematically moved up and down along the U-tube is summarized in table 3, where D represents the distance from the upper edge of the plates to the liquid surface in the situations of maximum and minimum detected signal.

Table 3. Results concerning the experiment of figure 3

$D_{\text{expect.}}$ (cm)	$D_{\text{obs.}}$ (cm)	Detected signal (μV)
0 (max.)	0 (max.)	12
	4·2 (min.)	0
7·5 (min.)		5
	11·4 (max.)	15
15 (max.)		5
	17 (min.)	0
22·5 (min.)		5
	25 (max.)	12
30 (max.)		—

From a theoretical point of view one tends to expect that the maximums and minimums of detection should occur when the edges of the plates coincide with the location of the antinodes and nodes, respectively, as quoted in table 3. This is on the basis that for a resonant molecule (with very high Q) the force f and particle displacement s are in quadrature:

$$f = f_0 \sin \omega_0 t$$

$$s = -s_0 \cos \omega_0 t \sin kx.$$

If one assumes that the gradient of the E field is zero everywhere except in the edges of the condenser (neglecting fringe effects), then the energy given per cycle to the N molecules located at the edges of the condenser becomes

$$W = N \int_0^{2\pi/\omega_0} f \frac{\mathrm{d}s}{\mathrm{d}t}\, \mathrm{d}t = N\pi f_0 s_0 \sin kx.$$

This is a maximum at antinodes, $|\sin kx| = 1$, and equal to the energy dissipated in the liquid, which in turn is proportional to the amplitude of the oscillation. Table 3 shows that the observed sequence of max. and min. correspond to the expected ones. It is interesting, nevertheless, to remark that all these observed points are definitely shifted upwards with respect to their expected position. This was also confirmed in a second system with 5 cm high plates, implying a detected resonant frequency around 14 kHz. We do not wish to go into the rather complex theory of the method here described and of the observed anomaly in particular.

4. Conclusions

There has been expressed the need for a recent, comprehensive and ordered review of electromagnetic forces in liquids. With the aim of partly correcting this deficiency, three different theories have been outlined and systematically analysed through a set of relevant experiments. These conclusively confirm the theory presented by Einstein and Laub (1908) as the substantially correct one.

In the course of this general study a method, new to our knowledge, for generating sonic electrostrictive waves in dielectric liquids has been devised. The most important feature of this method appears in the total confinement of the liquid inside a glass tube. This avoids the obvious disadvantages of other known methods due to direct contact between liquid and condenser plates.

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