XII. *A Dynamical Theory of the Electric and Luminiferous Medium.*

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1. THE object of this paper is to attempt to develope a method of evolving the dynamical properties of the æther from a single analytical basis. One advantage of such a procedure is that by building up everything *ab initio* from a consistent and definite foundation, we are certain of the congruity of the different parts of the structure, and are not liable to arrive at mutually contradictory conclusions. The data for such a treatment lie of course in the properties of the mathematical function which represents the distribution of energy in the medium, when it is disturbed. The consequences which should result from the disturbance are all deducible by dynamical analysis from the expression for this function; and it is the province of physical interpretation to endeavour to identify in them the various actual phenomena, and in so far to establish or disprove the explanation offered. A method of this kind has been employed by CLERK MAXWELL with most brilliant results in the discovery and elucidation of the laws of electricity; he has also been led by its development into the domain of optics, and has thus arrived at the electric theory of light. His expression for the energy of the active medium has been constructed from reasoning on the phenomena of electrification and electric currents; this procedure offers perhaps difficulties greater than might be, owing to the intangible character of the electric co-ordinates, and their totally undefined connexion with the co-ordinates of the material system which is the seat of the electric manifestations. In the following discussion, the order of development began with the optical problem, and was found to lead on naturally to the electric one. We shall show that an energy-function can be assigned for the æther which will give a complete account of what the æther has to do in order to satisfy the ordinary demands of Physical Optics; and it will then be our aim to examine how far the phenomena of electricity can be explained as non-vibrational manifestations of the activity of the same medium. The credit of applying with success the pure analytical method of energy to the elucidation of optical phenomena belongs to MACCULLAGH; he was however unable to discover a mechanical illustration such as would bring home to the mind by analogy the properties of his medium, and so his theory has fallen rather into neglect from supposed incompatibility

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with the ordinary manifestations of energy as exemplified in material structures. We shall find that such difficulties are now removed by aid of the mechanical example of a gyratory æther, which has been imagined by Lord KELVIN to illustrate the properties of the luminiferous and electric medium. The æther whose properties are here to be examined is not a simple gyrostatic one;* it is rather the analogue of a medium filled with magnetic molecules which are under the action, from a distance, of a magnetic system. But the same peculiarities that were supposed to fatally beset MACCULLAGH's medium and render it inconceivable, are present in an actual mechanical medium dominated by gyrostatic momentum.

2. The general dynamical principle which determines the motion of every material system is the Law of Least Action, expressible in the form that $\delta\int(\mathrm{T}-\mathrm{W})\,dt=0$, where T denotes the kinetic energy and W the potential energy of the system, each formulated in terms of any co-ordinates that are sufficient to specify the configuration and motion in accordance with its known properties and connexions; and where the variation refers to a fixed time of passage of the system from the initial to the final configuration considered. The power of this formula lies in the fact that once the energy-function is expressed in terms of any measurements of the system that are convenient and sufficient for the purpose in view, the remainder of the investigation involves only the exact processes of mathematical analysis. It is to be observed that forces which can do no work by reason of constraints of the system tacitly assumed in this specification, but which nevertheless may exist, do not enter at all into the analysis. Thus in the dynamics of an incompressible medium, the pressure in the medium will not appear in the equations, unless the absence of compression is explicitly recognised in the form of an equation of condition between co-ordinates otherwise redundant, which is combined into the variation in LAGRANGE's manner; in certain cases (*e.g.* magnetic reflexion of light, *infra*) we are in fact driven to the explicit recognition of such a pressure in order that it may be possible to satisfy all the necessary stress-conditions of the problem, while in other cases (*e.g.* ordinary reflexion of light) the pressure is not operative in the phenomena. There is also a class of cases at the other extreme—typified by a medium such as Lord KELVIN's labile æther which opposes no resistance to laminar compression,—where a certain co-ordinate does not enter into the energy-function because its alteration is not opposed and so involves no work; in these cases there is solution of a constraint which reduces by one the number of kinematic conditions to be satisfied. In intermediate cases the energy corresponding to the co-ordinate will enter into the function in the ordinary manner.

3. It is to be assumed as a general principle, that all the conditions necessary to be satisfied in any dynamical problem are those which arise from the variation of the

* A medium has however been invented by Lord KELVIN, containing gyrostatic cells composed of arrangements of Foucault gyrostats whose cases are imbedded in it, such as give precisely the rotational elasticity of the æther.

Action of the system in the manner of LAGRANGE. If these conditions appear to be too numerous, the reason must be either that the forcive which compels the observance of some constraint has not been explicitly included in the analysis, or else that the number of the constraints has been over-estimated. In each problem in which the mathematical analysis proceeds without contradiction or ambiguity to a definite result, that result is to be taken as representing the course of the dynamical phenomena in so far as they are determined by the energy as specified; a further more minute specification of the energy may however lead to the inclusion of small residual phenomena which had previously not revealed themselves.

4. The object of these remarks is to justify the division of the problem of the determination of the constitution of a partly concealed dynamical system, such as the æther, into two independent parts. The first part is the determination of some form of energy-function which will explain the recognized dynamical properties of the system, and which may be further tested by its application to the discovery of new properties. The second part is the building up in actuality or in imagination of some mechanical system which will serve as a model or illustration of a medium possessing such an energy-function. There have been cases in which, after the first part of the problem has been solved, all efforts towards the realization of the other part have resulted in failure; but it may be fairly claimed that this inability to directly construct the properties assigned to the system should not be allowed to discredit the part of the solution already achieved, but should rather be taken as indicating some unauthorized restriction of our ideas on the subject. Of course where more than one solution of the question is possible on the ascertained data, that one should be preferred which lends itself most easily to interpretation, unless some of the others should prove distinctly more fertile in the prediction of new results, or in the inclusion of other known types of phenomena within the system.

5. In illustration of some of these principles, and as a help towards the realization of the validity of some parts of the subsequent analysis, a dynamical question of sufficient complexity, which has recently occupied the attention of several mathematicians, may be briefly referred to. The problem of the deformation and vibrations of a thin open shell of elastic material has been reduced to mathematical analysis by Lord RAYLEIGH,* on the assumption that, as the shell can be easily bent but can be stretched only with great difficulty, the potential energy of stretching would not appear in the energy-function from which its vibrations in which bending plays a prominent part are to be determined,—that in fact the shell might be treated as inextensible. But a subsequent direct analysis of the problem, of a more minute character,† led to the result that the conditions at the boundary of the shell could not all be satisfied unless stretching is taken into account. The reason of the discrepancy is

* Lord RAYLEIGH, "On the Infinitesimal Bending of Surfaces of Revolution," 'Proc. Lond. Math. Soc.,' 1882.

† A. E. H. LOVE, "On the . . . Vibrations of a Thin Elastic Shell," 'Phil. Trans.,' 1888.

that, if the question is simplified by taking the shell to be inextensible, a static extensional stress ought at the same time to be recognized as distributed all along the surface of the shell, and as assisting in the satisfaction of the necessary conditions at its free edge; the stress-condition that can be adjusted in this manner may thus be left out of consideration, as taking care of itself. If we suppose the shell to be not absolutely inextensible, this tension will be propagated over the shell by extensional waves with finite but very great velocity; it will therefore still be almost instantaneously adjusted at each moment over a shell of moderate extent of surface, and the extensional waves will thus be extremely minute; such waves would have a very high period of their own, but in ordinary circumstances of vibration they would be practically unexcited. These remarks appear to be in keeping with the explanation of this matter which is now generally accepted.

6. The dynamical method as hitherto explained applies only to cases in which the forces are all derived from a potential-energy function, or are considered as explicitly applied from outside the system; in the latter case they may be, as VON HELMHOLTZ remarks, any arbitrary functions of the time. By means of the Dissipation Function introduced by Lord RAYLEIGH, the equation of Varying Action will be so modified as to include probably all the types of frictional internal forces that are of much importance in physical applications.

7. A few words may be said with respect to notation. In order to reduce as much as possible the length to which formulæ involving vector quantities extend themselves in ordinary Cartesian analysis, a vector will usually be specified by its three Cartesian components enclosed in brackets, in front of which may be placed such operators as act on the vector. Of particularly frequent occurrence is the operator which deduces the doubled rotation of an element of volume from the vector which represents the translation; this will, after MAXWELL, receive a special designation, and will here be called the vorticity or curl of that vector. If the vector represent the displacement in an incompressible medium, *i.e.*, if it has no convergence, we have $(\text{curl})^2 = -\nabla^2$, where ∇^2 is LAPLACE'S well-known scalar operator. The introduction of still more vector analysis would further shorten the formulæ, and probably in practised minds lead to clearer views; but the saving would not be very great, while as yet facility in vector methods is not a common accomplishment. In the various transformations by means of integration by parts that occur, after the manner of GREEN'S analytical theorem, it is not considered necessary to express at length the course of the analysis; so as there is no further object in indicating explicitly by a triple sign the successive steps by which a volume integration is usually effected, it will be sufficient to take the symbol $d\tau$ to represent an element of volume and cover it by a single sign of integration. In the notation of surface integrals, the ordinary usage is somewhat of this kind.*

* Various matters have been treated from rather different points of view in the abstract of this paper, 'Roy. Soc. Proc.,' vol. 54, pp. 438–461.

PART I.—PHYSICAL OPTICS.

Preliminary and Historical.

8. The development of the analytical theory of the æther which will be set forth in this paper originated in an examination of Professor G. F. FITZGERALD'S Memoir, "On the Electro-magnetic Theory of the Reflection and Refraction of Light,"* of which the earlier part is put forward by the author as being a translation of MACCULLAGH'S analysis of the problem of reflexion into the language of the electro-magnetic theory. Later on in the Memoir the author discusses the rotation of the plane of polarization of the light, which is produced by reflexion from the surface of a magnetized medium, assumed in the analysis to be transparent; but the application of MACCULLAGH'S method to this case leads him to more surface-conditions than can be satisfied by the available variables, and the rigorous solution of the problem is not attained. After satisfying myself that this contradiction is really due to the omission from consideration of the *quasi*-hydrostatic pressure which must exist in the medium and assist in satisfying the stress-conditions at an interface, though on account of the incompressible character of the medium this pressure takes no part in the play of energy on which the kinetic phenomena depend, it was natural to turn to MACCULLAGH'S optical writings,† in order to ascertain whether a similar idea had already presented itself. An examination, particularly of "An Essay towards a Dynamical Theory of Crystalline Reflexion and Refraction,"‡ led in another direction, and showed that to MACCULLAGH must be assigned the credit of one of the very first notable applications to physical problems of that dynamical method which in the hands of MAXWELL, Lord KELVIN, VON HELMHOLTZ, and others, has since been so productive, namely, the complete realization of LAGRANGE'S theory that all the phenomena of any purely dynamical system free from viscous forces are deducible from the single analytical function of its configuration and motion which expresses the value of its energy. The problem proposed to himself by MACCULLAGH was to determine the form of this function for a continuous medium,§ such as would lead to all the various laws of the propagation and reflexion of light that had been ascertained by FRESNEL, supplemented by the exact and crucial observations on the polarization produced by reflexion at the surfaces of crystals and of metallic media, which had been made by BREWSTER and

* G. F. FITZGERALD, 'Phil. Trans.,' 1880.

† 'The Collected Works of JAMES MACCULLAGH,' ed. JELLETT and HAUGHTON, 1880.

‡ MACCULLAGH, *loc. cit.*, p. 145; 'Trans. Roy. Irish Acad.,' XXI., Dec. 9, 1839.

§ The problem had already been fully analyzed by GREEN, shortly before, and unknown to MACCULLAGH, precisely on these principles, but without success owing to his restriction to elasticity of the type of an ordinary solid body; *cf.* GREEN'S "Memoir on Ordinary Refraction," 'Trans. Camb. Phil. Soc.,' Dec. 11, 1837, introduction, and his "Memoir on Crystalline Propagation," 'Trans. Camb. Phil. Soc.,' May 20, 1839.

SEEBECK. He arrived at a complete solution of this problem, and one characterized by that straightforward simplicity which is the mark of all theories that are true to Nature; but he was not able to imagine any mechanical model by which the properties of his energy-function could be realized. In another connexion, in vindicating his equations for the rotatory polarization of quartz* against a theory of CAUCHY's leading to different results, he however expresses himself on such a question, as follows.† "For though, in my Paper, I have said nothing of any mechanical investigation, yet as a matter of course, before it was read to the Academy, I made every effort to connect my equations in some way with mechanical principles; and it was because I had failed in doing so to my own satisfaction, that I chose to publish the equations without comment, as bare geometrical assumptions, and contented myself with stating orally . . . . that a mechanical account of the phenomena remained a *desideratum* which no efforts of mine had been able to supply." And again, "though for my own part I never was satisfied with that theory [of CAUCHY], which seemed to me to possess no other merit than that of following out in detail the extremely curious, but (as I thought) very imperfect analogy which had been perceived to exist between the vibrations of the luminiferous medium and those of a common elastic solid, ... still I should have been glad, in the absence of anything better, to find my equations supported by a similar theory, and their form at least countenanced by a like mechanical analogy."

9. After trying an empirical alteration of CAUCHY's equations for the stress in his medium,‡ which sufficed to satisfy BREWSTER's observations on reflexion from crystals, but did not agree with subsequent observations of a different kind by SEEBECK, MACCULLAGH was finally led to results which were in keeping with all the experiments by means of the principles§ that (i) the displacements in the incident and reflected waves, compounded as vectors, are geometrically equivalent at the interface to the displacements in the refracted waves, compounded in the same manner, and (ii) there is no loss of energy involved in the act of reflexion and refraction. This agreement was obtained, provided he took the displacement to be in the plane of polarization of the light, and the density of the æther to be the same in all media.

Shortly before, and unknown to MACCULLAGH, F. E. NEUMANN‖ had based the solution of the problem of reflexion on the very same principles; and he had as early as 1833, ascertained that his results agreed with SEEBECK's experiments, though MACCULLAGH had priority in publication. He began by applying to the problem of reflexion the equations of motion of an elastic solid, as then imperfectly understood in accordance with the prevalent theory of NAVIER and POISSON; he recognized that

* MACCULLAGH, "On the Laws of the Double Refraction of Quartz," 'Trans. Roy. Irish Acad.,' 1836; 'Collected Works,' p. 63.

† MACCULLAGH, 'Proc. Roy. Irish Acad.,' 1841; 'Collected Works,' pp. 198, 200.

‡ MACCULLAGH, "On the Laws of Reflexion from Crystallized Surfaces," 'Phil. Mag.,' vol. 8, 1835.

§ MACCULLAGH, "On the Laws of Crystalline Reflexion," Dec. 13, 1836; 'Phil. Mag.,' vol. 10, 1837.

‖ F. E. NEUMANN, 'Abhandl. der Berliner Akad.,' 1835, pp. 1–116.

there were six interfacial conditions to be satisfied, three of displacement and three of stress, while in the absence of compressional waves there were enough variables to satisfy only four of them; he cut the knot of this difficulty by assuming that the displacement must be continuous, to avoid rupture of the medium at the interface, and assuming that there is no loss of energy in the act of reflexion and refraction of the light, thus asserting the absence of waves of compression, and at the same time leaving the conditions as to continuity of stress altogether out of his account. As his displacement is in the plane of polarization, the solution arrived at by NEUMANN is formally the same as MACCULLAGH'S; but it can be shown that the reasoning by which NEUMANN arrived at it, from the basis of an elastic solid æther, is invalid, so that the solution as stated by him must be considered to be the result of a fortunate accident, the correctness of which he would have had no real ground, in the absence of comparison with observations, for anticipating; while MACCULLAGH afterwards (in 1839) placed his own empirical theory on a real dynamical foundation.

10. The hypothesis on which NEUMANN'S surface-conditions are virtually based has been expounded and amplified in more recent times by KIRCHHOFF;* and in this form it is often quoted as KIRCHHOFF'S principle. The analysis of KIRCHHOFF also amends NEUMANN'S defective energy-function by the substitution for it of the one determined by GREEN, by the condition that the displacements in two of the three types of waves that can travel unchanged in the medium are in the plane of the wave-front. About the rate of propagation of the third wave, involving compression in the medium, KIRCHHOFF makes no hypothesis, but he avails himself of the remark (originally due to MACCULLAGH) that the transverse waves involve no compression, and therefore are independent, as regards their propagation, of the term in the energy which involves compression. He assumes that in the act of reflexion and refraction no compressional waves are produced; and that this is so because extraneous forces act on the interface just in such manner as to establish the continuity of stress across it, while on account of the conservation of the energy they can do no work in the *actual motion* of the medium at the interface. The explicit recognition of such forces constitutes KIRCHHOFF'S principle; as to their origin he says that it lies in traction exerted by the matter on the æther which is unbalanced at the surface of discontinuity, and that they are somehow of the same nature as the capillary force at the interface between two liquids; as to their happening to be precisely such as will extinguish the compressional waves, he merely says that it must be so, because as a matter of fact no compressional waves are produced by the reflexion, the energy being assumed to be all in the reflected and refracted light-waves. On the other hand, the pure elastic theory has been worked out on NEUMANN'S hypothesis, for the simple case of an isotropic medium, without the assumption of these extraneous forces, by LORENZ, Lord RAYLEIGH, and others, and has been shown to lead to loss of light

* G. KIRCHHOFF, "Ueber die Reflexion und Brechung des Lichtes an der Grenze krystallinischer Mittel," 'Abh. der Berl. Akad.,' 1876; 'Ges. Abh.,' p. 367.

owing to the formation of compressional waves which carry away some of the energy, and to laws of reflexion quite irreconcilable with observation.

11. Can then any justification be offered of KIRCHHOFF's doctrine of extraneous surface-forces? The parallel case which is appealed to for its support is that of capillary forces at an interface between two fluids. Now on GAUSS' theory of capillarity these forces are derived simply from the principle of energy; each fluid being in equilibrium, its intrinsic energy is distributed throughout its interior with so to speak uniform volume-density; if we imagine the surface of transition to be sharp, and each fluid to retain its properties unaltered right up to it, the total energy will be simply the sum of the two volume-energies and will not depend on the surface at all; as a matter of necessity, however, there is a gradual transition from one fluid to the other across a thin surface-layer, and the energy per unit volume in this layer alters with the change of properties; so that to the energy estimated as if the transition were sharp, there is to be made a correction which takes the form of a surface distribution of energy; and this latter term must reveal itself, according to GAUSS' well-known reasoning, in the phenomena of capillary surface-tension. The relation between the volume-densities of the energy in the two fluids is determined by the proper balance of intrinsic hydrostatic pressure across the interface. Now if we adhere at all to the principle that the play of energy, as distributed throughout the masses in the field, is the proper basis for the interpretation of physical phenomena, the extraneous surface-forces of KIRCHHOFF must also be accounted for in some such way as the above; they must arise out of the influence of a layer of gradual transition between the media. But superior limits have been obtained to the thickness of such a layer in various ways, by actual measurement; such limits are found in the thickness of the thinnest possible soap-film, as measured by REINOLD and RÜCKER, or in the thickness of the film of silvering which in QUINCKE's experiments just suffices to extinguish the influence of the glass, on which it is deposited, on the phenomena of surface-tension. The former limit is about one-fortieth of the wave-length of green light, the latter limit is well within one-tenth of the same wave-length.* The quantity with which to compare the surface-energy due to this transition is the energy contained in a wave-length of the light whose reflexion is under consideration. It is plain that such an amount of surface-energy as is here possible will not suffice to totally transform the circumstances of the reflexion, and therefore will not account for KIRCHHOFF's extraneous forces. Furthermore, a layer of transition, of thickness of the same order of magnitude as the wave-length, would introduce a change of phase into the reflexion, such as we know, from Lord RAYLEIGH's and DRUDE's experiments on reflexion from absolutely clean surfaces of transparent media, does not exist, and such as even KIRCHHOFF's own theory does not allow for. It is for these reasons that it is here considered that NEUMANN's theory of light is, on

* REINOLD and RÜCKER, 'Roy. Soc. Proc.' 1877; 'Phil. Trans.,' 1883. QUINCKE, 'Pogg. Ann.,' vol. 137, 1869. *Cf.* Lord KELVIN, "Popular Lectures and Addresses," vol. 1, p. 8.

his own dynamical basis, untenable, and leads to the correct result only by accident,—and that the credit of the solution of the fundamental dynamical problem of Physical Optics belongs essentially to MACCULLAGH.

12. To return now to the course of the development of optical doctrine in MACCULLAGH's hands, he recounts in straightforward fashion,* somewhat after the custom usual with FARADAY, the way in which after successive trials he was at last guided to the formal laws which govern the phenomena of reflexion. To his success two main elements contributed; the bent of his genius led him to apply the methods of the ancient Pure Geometry, of which he was one of the great masters, to the question, and this resulted in simple conceptions, such as the principle of equivalent vibrations already explained, which are applicable to the most general aspect of the problem; while the variety and exactness of the experiments of BREWSTER and SEEBECK on the polarization of the light reflected from a crystal gave him plenty of material by which to mould his geometrical views. The simple theorems† of the *polar plane* and of *transversals*, by which he expressed without symbols in the compass of a single sentence, and in two different ways, the complete solution of the most general problem of crystalline reflexion, contrast with the very great complexity of the analytical solutions of NEUMANN and KIRCHHOFF. Thus at the end of this paper he remarks that "several other questions might be discussed, such as the reflexion of common light at the first surface, and the internal reflexion at the second surface of a crystal;‡ but these must be reserved for a future communication. It would be easy indeed to write down the algebraical solutions resulting from our theory; but this we are not content to do, because the expressions are rather complicated, and when rightly treated will probably contract themselves into a simpler form. It is the character of all true theories that the more they are studied the more simple they appear to be." "We are obliged to confess that, with the exception of the law of *vis viva*, the hypotheses" on which the solution is founded "are nothing more than fortunate conjectures. These conjectures are very probably right, since they lead to elegant laws which are fully borne out by experiments; but that is all that we can assert respecting them. We cannot attempt to deduce them from first principles; because, in the theory of light, such principles are still to be sought for. It is certain, indeed, that light is produced by undulations, propagated, with transversal vibrations, through a highly elastic æther; but the constitution of this æther, and the laws of its connexion (if it has any connexion) with the particles

* MACCULLAGH, "On the Laws of Crystalline Reflexion and Refraction," 'Trans. R.I.A.,' XVIII., Jan. 9, 1837.

† MACCULLAGH, 'Collected Works,' pp. 97 and 176.

‡ It is interesting to observe that, in the notes appended to the paper, MACCULLAGH has actually obtained the geometrical solution of this seemingly most complicated question, by means of a very powerful and refined application of the principle of reversibility of the motion, which was afterwards employed to such good purpose by Sir S. G. STOKES.

of bodies, are utterly unknown. The peculiar mechanism of light is a secret which we have not yet been able to penetrate . . . but perhaps something might be done by pursuing a contrary course; by taking these laws for granted, and endeavouring to proceed upwards from them to higher principles . . ." He then allows himself to give a pure mechanical interpretation to his formal results, taking his displacement to be linear, and he derives the conclusion that the effective density of the æther is the same in all bodies.

13. In the notes appended to this purely formal paper MACCULLAGH "afterwards proved that the laws of reflexion at the surface of a crystal are connected, in a very singular way, with the laws of double refraction, or of propagation in its interior;" he was led to infer that "all these laws and hypotheses have a common source in other and more intimate laws that remain to be discovered; and that the next step in physical optics would probably lead to those higher and more elementary principles by which the laws of reflexion and the laws of propagation are linked together as parts of the same system." And in the following memoir* he takes this step by developing his dynamical theory. His analysis is based on the hypothesis of constant density of the æther, and on the principle of rectilinear vibrations in crystalline media, substances like quartz being excepted. "Concerning the peculiar constitution of the ether we know nothing, and shall assume nothing, except what is involved in the foregoing assumptions," and that it may be taken as homogeneous for the problem in hand.

In Section III. of this paper MACCULLAGH proceeds to determine the potential-energy function on which the transverse rectilinear vibrations propagated through the æther must depend. He observes that such vibrations involve no condensation; and as in a plane wave all the points in the medium move in parallel directions, the effective strain produced in it may be taken to be specified by the rotation of the element, which is round a line in the plane of the wave-front and at right angles to the line of the displacement, this rotation being proportional to the rate of change of the displacement in the direction of propagation. Having previously shown, probably for the first time, that the expression now interpreted as representing the elementary rotation in the displacement of a medium by strain, enjoys the invariant properties of a vector, he at once seizes upon it as the very thing he wants, as it has a meaning independent of any particular system of axes to which the motion is referred; and he makes the potential energy of the medium a quadratic function of the components of this elementary rotation. As pointed out by STOKES†, the possible forms of the effective strain and therefore of the energy-function are by no means thus restricted: in fact GREEN had a short time previously established another form, in which the

* MACCULLAGH, "An Essay towards a Dynamical Theory of Crystalline Reflexion and Refraction," 'Trans. R.I.A.,' 21, Dec. 9, 1839.

† Sir G. G. STOKES, "Report on Double Refraction," 'Brit. Assoc.,' 1862. MACCULLAGH possibly perceived this afterwards himself; *cf.* note at the end of his memoir.

energy depends on the components of the strain of the medium, as it would do if the medium possessed the properties of an elastic solid.

At any rate, MacCullagh assumes a purely rotational quadratic expression for the energy, which he reduces to its principal axes in the ordinary manner; and then he deduces from it in natural and easy sequence, without a hitch, or any forcing of constants, all the known laws of propagation and reflexion for transparent isotropic and crystalline media. In common with Neumann, he cannot understand how with Fresnel the inertia in a crystal could be different in different directions, or its elasticity isotropic; so he assumes the density of the æther to be the same in all media, but its elasticity to be variable. The laws of crystalline reflexion are then established as below, and shown to be embraced in a single theorem relating either to his transversals or to his polar plane; and the memoir ends with a remark "which may be necessary to prevent any misconception as to the nature of the foundation on which" the theory stands. "Everything depends on the form of the function V; and we have seen that, when that form is properly assigned, the laws by which crystals act upon light are included in the general equations of dynamics. This fact is fully proved by the foregoing investigations. But the reasoning which has been used to account for the form of the function is indirect, and cannot be regarded as sufficient, in a mechanical point of view. It is, however, the only kind of reasoning that we are able to employ, as the constitution of the luminiferous medium is entirely unknown."

MacCullagh's Optical Equations.

14. Let the components of the linear displacement of the primordial medium be represented by (ξ, η, ζ), and let (f, g, h) represent the curl or vorticity of this displacement, *i.e.*

$$(f, g, h) = \left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}, \frac{d\xi}{dz} - \frac{d\zeta}{dx}, \frac{d\eta}{dx} - \frac{d\xi}{dy}\right),$$

so that this vector is equal to twice the absolute rotation of the element of volume. The elasticity being purely rotational, the potential energy per unit volume of the strained medium is represented by a quadratic function U of (f, g, h), so that

$$W = \int U \, d\tau$$

where $d\tau$ denotes an element of volume. The kinetic energy is

$$T = \tfrac{1}{2} \rho \int \left(\frac{d\xi^2}{dt^2} + \frac{d\eta^2}{dt^2} + \frac{d\zeta^2}{dt^2}\right) d\tau.$$

The general variational equation of motion is

$$\delta \int (T - W)\, dt = 0,$$

for integration through any fixed period of time. Thus*

$$\int dt\left[\rho\int\left(\frac{d\xi}{dt}\frac{d\delta\xi}{dt}+\frac{d\eta}{dt}\frac{d\delta\eta}{dt}+\frac{d\zeta}{dt}\frac{d\delta\zeta}{dt}\right)d\tau \right.$$
$$\left. -\int\left\{\frac{dU}{df}\left(\frac{d\delta\zeta}{dy}-\frac{d\delta\eta}{dz}\right)+\frac{dU}{dg}\left(\frac{d\delta\xi}{dz}-\frac{d\delta\zeta}{dx}\right)+\frac{dU}{dh}\left(\frac{d\delta\eta}{dx}-\frac{d\delta\xi}{dy}\right)\right\}d\tau\right]=0.$$

On integration by parts in order to replace the differential coefficients of $\delta(\xi, \eta, \zeta)$ by these variations themselves, we obtain, leaving out terms relating to the beginning and end of the time,

$$\int dt\left[-\rho\int\left(\frac{d^2\xi}{dt^2}\delta\xi+\frac{d^2\eta}{dt^2}\delta\eta+\frac{d^2\zeta}{dt^2}\delta\zeta\right)d\tau\right.$$
$$-\int\left\{\left(\frac{d}{dy}\frac{dU}{dh}-\frac{d}{dz}\frac{dU}{dg}\right)\delta\xi+\left(\frac{d}{dz}\frac{dU}{df}-\frac{d}{dx}\frac{dU}{dh}\right)\delta\eta+\left(\frac{d}{dx}\frac{dU}{dg}-\frac{d}{dy}\frac{dU}{df}\right)\delta\zeta\right\}d\tau$$
$$\left.+\int\left\{\left(m\frac{dU}{dh}-n\frac{dU}{dg}\right)\delta\xi+\left(n\frac{dU}{df}-l\frac{dU}{dh}\right)\delta\eta+\left(l\frac{dU}{dg}-m\frac{dU}{df}\right)\delta\zeta\right\}dS\right]=0,$$

where (l, m, n) are the direction-cosines of the element of surface dS. As the displacements $\delta(\xi, \eta, \zeta)$ are as yet quite arbitrary, the equations of elastic vibration of the medium are therefore

$$\rho\frac{d^2\xi}{dt^2}+\frac{d}{dy}\frac{dU}{dh}-\frac{d}{dz}\frac{dU}{dg}=0$$
$$\rho\frac{d^2\eta}{dt^2}+\frac{d}{dz}\frac{dU}{df}-\frac{d}{dx}\frac{dU}{dh}=0$$
$$\rho\frac{d^2\zeta}{dt^2}+\frac{d}{dx}\frac{dU}{dg}-\frac{d}{dy}\frac{dU}{df}=0.$$

From them it follows that

$$\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}=0,$$

in other words, that there is no compression of the medium involved in this motion, whether we assume that it has the property of incompressibility or not.

15. In accordance with the general dynamical principle, all the conditions which it is essential to explicitly satisfy at an interface between two media are those which secure that the variation of the energy shall not involve a surface integral over this interface. To express these conditions most concisely, let us take for the moment the

* *Cf.* G. F. FitzGerald, "On the Electromagnetic Theory . . . .," 'Phil. Trans.,' 1880. In that memoir the rotation is represented by $4\pi(f, g, h)$, instead of simply (f, g, h) as above, in order to be in line with Maxwell's electrodynamic equations.

element of the interface to be parallel to the plane of yz, so that $(l, m, n) = (1, 0, 0)$; the surface integral term corresponding to one side of the interface is now

$$\int\left(-\frac{d\text{U}}{dg}\,\delta\eta + \frac{d\text{U}}{dh}\,\delta\zeta\right) d\text{S},$$

where $\delta\eta$, $\delta\zeta$ are perfectly arbitrary, subject only to being continuous across the interface. Thus to make the surface integral part of the variation vanish, we must have $d\text{U}/dg$ and $d\text{U}/dh$, the tangential components of the traction, continuous across the interface; it follows from the first of the equations of motion that the continuity of ξ is also thereby secured, provided the density is the same on both sides; and the normal traction on the interface is null. The continuity in the flow of energy across the interface is of course also necessarily involved. Of the complete set of six conditions only four are thus independent, which is the precise number required for the problem of optical reflexion between crystalline media.

It has not been necessary to assume incompressibility of the medium in order to avoid waves of longitudinal disturbance. A medium of this type, however heterogeneous in elastic quality from part to part, whether compressible or not, will transmit waves of transverse displacement in absolute independence of waves of compression, provided its density is everywhere the same; the one type of wave cannot possibly change into the other.

16. If

$$(\xi, \eta, \zeta) = \text{curl}\,(\xi_1, \eta_1, \zeta_1),$$

so that

$$(f, g, h) = -\nabla^2(\xi_1, \eta_1, \zeta_1),$$

and if the equations of propagation are referred to the principal axes of the medium so that now

$$\text{U} = \tfrac{1}{2}(a^2f^2 + b^2g^2 + c^2h^2),$$

they assume the form

$$\rho\frac{d^2}{dt^2}(\xi_1, \eta_1, \zeta_1) = \nabla^2(a^2\xi_1, b^2\eta_1, c^2\zeta_1),$$

which are precisely FRESNEL's equations of crystalline propagation.* The vector (ξ_1, η_1, ζ_1) of FRESNEL is at right angles to the plane of polarization, therefore its curl (ξ, η, ζ) which is the displacement of the medium on MACCULLAGH's theory, is in the plane of polarization.

17. In the theory of reflexion the tangential components of the displacement are continuous, and the tangential components of the stress are continuous; these conditions, or the more direct conditions of continuity of displacement and continuity

* MACCULLAGH, 'Proc. R.I.A.,' vol. II., 1841; 'Collected Works,' p. 188.

of energy, taken in conjunction with the hypothesis of effective density constant throughout space, lead immediately to FRESNEL'S equations of reflexion for isotropic media, and in MACCULLAGH'S hands give a compact geometrical solution when the media are of the most general character. A medium of this kind, however heterogeneous and æolotropic as regards elasticity, is still adapted to transmit transverse undulations without any change into the longitudinal type; and the conditions of propagation are all satisfied without setting up any normal tractions in the medium, which might if unbalanced produce motion of translation of its parts. Thus the incidence of light-waves on a body will not give rise to any mechanical forces.

Alternative Optical Theories.

18. The equations of propagation of FRESNEL above-mentioned obviously agree with those which are derivable from the variational equation

$$\delta\int dt\left[\tfrac{1}{2}\kappa\int\left(\rho a^{-2}\frac{d\xi_1^2}{dt^2}+\rho b^{-2}\frac{d\eta_1^2}{dt^2}+\rho c^{-2}\frac{d\zeta_1^2}{dt^2}\right)d\tau-\tfrac{1}{2}\kappa\int(f_1^2+g_1^2+h_1^2)\,d\tau\right]=0,$$

which belongs to a medium having æolotropic inertia of the kind first imagined by RANKINE, and having isotropic purely rotational elasticity. The coefficient of elasticity κ may be in the first instance assumed to be different in different substances. The surface-conditions for the problem of reflexion which are derived from this equation are clearly, in the light of the above analysis, continuity of tangential displacement and of tangential stress. A compression of the medium now takes part in the propagation of transverse undulations, yet the compression does not appear in this isotropic potential energy-function; hence the resistance to laminar compression must be null, the other alternative infinity being on the latter account inadmissible. The surface condition as to continuity of normal displacement need not therefore be explicitly satisfied; and the remaining surface condition of continuity of normal traction is non-existent, there being no normal traction owing to the purely rotational quality of the elasticity. Whether a medium of this type could be made to lead to the correct equations of reflexion we need not inquire. [See however § 21.]

19. It has been shown by Lord KELVIN* that a medium of elastic-solid type is possible which shall oppose no resistance to laminar compression, viz. to compression in any direction without change of dimensions sideways, and that its potential energy if elastically isotropic is of the same form as the above, with the addition of some terms which, integrated over the volume, are equivalent to a surface integral. The remaining coefficient of elasticity, that is the rigidity, must then be the same in all

* Lord KELVIN (Sir W. THOMSON) "On the reflexion and refraction of light," 'Phil. Mag.,' 1882 (2), p. 414; GLAZEBROOK, do., p. 521.

media, to avoid static instability; that condition is in fact required as below, in order that waves may be transmissible at all through a heterogeneous medium of this type.

As an illustration of this somewhat abstract discussion, let us conduct the variation of the Action in this labile elastic-solid medium. The equation takes the form

$$\delta\int dt\left[\tfrac{1}{2}\int\left(\alpha^2\frac{d\xi^2}{dt^2}+\beta^2\frac{d\eta^2}{dt^2}+\gamma^2\frac{d\zeta^2}{dt^2}\right)d\tau-\tfrac{1}{2}\kappa\int\left\{\left(\frac{d\zeta}{dy}+\frac{d\eta}{dz}\right)^2+\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)^2\right.\right.$$
$$\left.\left.+\left(\frac{d\eta}{dx}+\frac{d\xi}{dy}\right)^2-4\left(\frac{d\eta}{dy}\frac{d\zeta}{dz}+\frac{d\zeta}{dz}\frac{d\xi}{dx}+\frac{d\xi}{dx}\frac{d\eta}{dy}\right)\right\}d\tau\right]=0;$$

it would be illegitimate for the present purpose to replace the potential energy by a surface part and a volume part, because then it would not be correctly located in the medium. We obtain on the left-hand side the time-integral of the expression

$$-\int\left(\alpha^2\frac{d^2\xi}{dt^2}\delta\xi+\beta^2\frac{d^2\eta}{dt^2}\delta\eta+\gamma^2\frac{d^2\zeta}{dt^2}\delta\zeta\right)d\tau$$
$$-\kappa\int\left\{\left(\frac{d\zeta}{dy}+\frac{d\eta}{dz}\right)(m\delta\zeta+n\delta\eta)+\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)(n\delta\xi+ld\zeta)\right.$$
$$+\left(\frac{d\eta}{dx}+\frac{d\xi}{dy}\right)(ld\eta+md\xi)-2\left(\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)ld\xi-2\left(\frac{d\zeta}{dz}+\frac{d\xi}{dx}\right)m\delta\eta$$
$$\left.-2\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}\right)n\delta\zeta\right\}dS$$
$$+\kappa\int\left\{\left(\delta\zeta\frac{d}{dy}+\delta\eta\frac{d}{dz}\right)\left(\frac{d\zeta}{dy}+\frac{d\eta}{dz}\right)+\left(\delta\xi\frac{d}{dz}+\delta\zeta\frac{d}{dx}\right)\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)\right.$$
$$+\left(\delta\eta\frac{d}{dx}+\delta\xi\frac{d}{dy}\right)\left(\frac{d\eta}{dx}+\frac{d\xi}{dy}\right)-2\,\delta\xi\frac{d}{dx}\left(\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)$$
$$\left.-2\,\delta\eta\frac{d}{dy}\left(\frac{d\zeta}{dz}+\frac{d\xi}{dx}\right)-2\,\delta\zeta\frac{d}{dz}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}\right)\right\}d\tau,$$

or collecting and exhibiting specimen terms only,

$$-\int\left(\alpha^2\frac{d^2\xi}{dx^2}\delta\xi+\ldots\right)d\tau$$
$$-\kappa\int\left[l\left\{\left(\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)\delta\xi+\left(\frac{d\eta}{dx}+\frac{d\xi}{dy}\right)\delta\eta+\left(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\right)\delta\zeta\right\}+\ldots\right]dS$$
$$-\kappa\int\left[\delta\xi\left\{\frac{d}{dy}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)-\frac{d}{dz}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)\right\}+\ldots\right]d\tau.$$

The equations of motion are thus

$$\alpha^2\frac{d^2\xi}{dt^2}=\frac{dh}{dy}-\frac{dg}{dz},\quad \beta^2\frac{d^2\eta}{dt^2}=\frac{df}{dz}-\frac{dh}{dx},\quad \gamma^2\frac{d^2\zeta}{dt^2}=\frac{dg}{dx}-\frac{df}{dy}$$

reducible to MacCullagh's by changing (ξ, η, ζ) into (a^2f, b^2g, c^2h), making the corresponding change for (f, g, h), and taking $(\alpha^2, \beta^2, \gamma^2) = \rho\,(a^{-2}, b^{-2}, c^{-2})$; while the surface conditions are easily seen by taking $(l, m, n) = (1, 0, 0)$ to be continuity of tangential elastic-solid tractions, and continuity of tangential displacement; both these results might of course have been foreseen from the formulæ for the tractions in an elastic solid, without special analysis. The surface condition involving normal displacement can be adjusted by the lability of the medium as regards simple elongation; and the continuity of its coefficient, that is, of the normal forcive as determined by the lateral contraction, is already secured by the other surface conditions, provided the elasticity is continuous. The mode in which lability thus affects the surface-conditions in the method of variations, is the chief point that required illustration; the addition to the energy of § 18 of terms which form a perfect differential is seen to be immaterial, provided they show no discontinuity at the interface.

20. It is of interest to observe that a geometrical transformation, specified by the equations*

$$(x, y, z) = pqr\left(\frac{x'}{p}, \frac{y'}{q}, \frac{z'}{r}\right), \quad \text{and} \quad (\xi, \eta, \zeta) = pqr\,(p\xi', q\eta', r\zeta'),$$

leads to

$$d\tau = d\tau', \quad \text{and} \quad (f, g, h) = pqr\left(\frac{f'}{p}, \frac{g'}{q}, \frac{h'}{r}\right),$$

and so leaves the elastic quality of a purely rotational medium unaltered.

Also, the variational equation of MacCullagh

$$\delta\int dt\left[\tfrac{1}{2}\rho\int\left(\frac{d\xi^2}{dt^2} + \frac{d\eta^2}{dt^2} + \frac{d\zeta^2}{dt^2}\right)d\tau - \tfrac{1}{2}\int(a^2f^2 + b^2g^2 + c^2h^2)\,d\tau\right] = 0$$

may be expressed, so far as regards vibrations of period $2\pi/n$, in the form

$$\delta\int dt\left[\tfrac{1}{2}\int\rho n^2\,(\xi^2 + \eta^2 + \zeta^2)\,d\tau - \tfrac{1}{2}\int(a^2f^2 + b^2g^2 + c^2h^2)\,d\tau\right] = 0,$$

in which the distinction between co-ordinates and velocities, between potential and kinetic energy, has been obliterated, if we regard n as simply a numerical coefficient.

If in the above transformation, (p, q, r) is taken equal to (a, b, c), this variational equation of MacCullagh is changed into the one appropriate to an æther of isotropic rotational elasticity and æolotropic effective density, as discussed above; and the wave-surface is changed into its polar reciprocal, which is also a Fresnel's surface in which a, b, c, are replaced by their reciprocals; and the geometrical relations between the two schemes may be correlated on this basis. This mode of transformation does not however extend to surface integral terms, and so cannot be applied to the problem of reflexion.

* *Cf.* 'Proc. Lond. Math. Soc.,' 1893, p. 278.

The same end might have been attained by taking (f, g, h) to denote displacement and (ξ, η, ζ) proportional to rotation in the variational equation; for $\nabla^2 (\xi, \eta, \zeta) = -\text{curl}\ (f, g, h)$, and the operator ∇^2 may be replaced by a constant so far as regards light-propagation in a single medium. This interchange, which has already been indicated in § 18, does not affect the development of the variational equation except as regards surface-integral terms; and the character of the modification of the geometrical relations of the wave surface, on passing from the one theory to the other, is now open to inspection.*

[21. (Added June 14.) The formal relations between these various mechanical theories may be very simply traced by comparing them with the electromagnetic scheme of MAXWELL. In that theory the electric and magnetic inductions, being circuital, are necessarily in the plane of the wave-front; while the electric and magnetic forces need not be in that plane. On taking the electric or the magnetic induction to represent the mechanical displacement of the medium, the electric theory coincides formally with that of FRESNEL or that of MACCULLAGH respectively; while on taking the electric or the magnetic force to represent the mechanical displacement, we obtain the equations of the correlative theories of BOUSSINESQ, Lord KELVIN, and other authors.† Thus, for example, it follows at once from this correlation that the combination of æolotropic inertia with labile isotropic elasticity will lead, not only to FRESNEL's wave surface as GLAZEBROOK has shown, but also to MACCULLAGH's theory of crystalline reflexion and refraction. If we suppose the magnetic quality of the medium to take part in the vibrations, as would probably be the case to some extent with very slow electric waves, the equations of propagation would possess features analogous to those due to an alteration of density in passing from one medium to another, on the mechanical theory here adopted. But the continuity of normal displacement of the medium could not now be satisfied in the problem of reflexion, the appropriate magnetic condition being instead continuity of induction. A homogeneous mechanical medium representing or illustrating such a case would thus have to possess suitable labile properties; in the ordinary optical circumstances in which magnetic quality is not effective, the degree of compressibility is on the other hand immaterial, and no normal wave will be started in reflexion.]

Treatment of the Problem of Reflexion by the Method of Rays.

22. We are now in a position to compare the various investigations of the problem of reflexion, by means of rays, that have been given by FRESNEL, NEUMANN, MACCULLAGH and others. It is a cardinal principle in all theories of transparent media that there is no loss of energy in the act of reflexion and refraction. Consequently there is no energy carried away by longitudinal waves in the æther;

* *Cf.* J. WILLARD GIBBS, "A comparison of the electric theory of light and Sir W. THOMSON's theory of a quasi-labile æther," 'Phil. Mag.,' 1889.

† *Cf.* DRUDE, 'Göttinger Nachrichten,' 1892.

and this must usually be either because the medium offers no resistance to laminar compression, or because it is incompressible, the case of rotational elasticity being however not thus restricted. The rays are most simply defined as the paths of the energy.

23. Let us consider the first of these hypotheses, that of null velocity of longitudinal waves. At the interface the tangential components of the displacement must be continuous, otherwise there would be very intense tangential tractions acting in the thin interfacial layer of transition, such as could not be equilibrated by the tractions outside that layer. The normal components of the displacement need not be made continuous, for the neighbourhood of this thin interfacial layer will stretch without effort as much as may be required. The tangential stresses must be continuous across the layer of transition, otherwise they would produce very great acceleration of this layer which could not be continuous with the moderate accelerations outside it. As we have thus already obtained the sufficient number of conditions the normal pressure need not also be explicitly made continuous, for the continuity of tangential displacements should secure its continuity as well; if the medium is constituted so as to regularly reflect waves at all, this must be the case, and it is clear on a moment's consideration of the formula for the pressure that it is so in a labile medium of isotropic elastic-solid type. We have thus the four conditions, continuity of tangential displacement and of tangential stress; and the one sufficient condition which will secure that they also make the normal stress continuous, *i.e.* that the medium is a possible one, is that there shall be no loss of energy in the operation of reflexion and refraction. The four conditions here specified are mathematically equivalent to those of FRESNEL's theory of reflexion; and the satisfaction of the fifth condition carries with it the justification of that theory for the type of medium which it implies. For the case worked out by FRESNEL, that of isotropic media, the constitution of his medium is thus limited to be precisely that of the labile æther of Lord KELVIN; in order to satisfy also the fifth condition, that of continuity of energy, we are constrained to take the displacement perpendicular to the plane of polarization, which gives a reason independent of experiment for FRESNEL's choice.

24. Let us next consider the second form of hypothesis, that of incompressibility. At the interface all three components of the displacement must now be continuous; and to obtain a solution, there is needed only one other condition, which may be taken to be the preservation of the energy of the motion. Here, as NEUMANN remarks, there is absolutely nothing assumed about the elastic condition of the media, which may in fact remain wholly unknown except as to their assumed incompressibility and as to the law of density, and the problem of reflexion will nevertheless be completely solved. But if we go further than this, and attempt to speculate about the elasticity of the optical medium, it must be limited to be of such nature as also to satisfy two other conditions which are involved in the continuity of the tangential stress at the interface.

Thus on the principles that the energy is propagated along the rays, that it is at any instant half potential and half kinetic, and that there is no loss of energy of the light in the act of reflexion, and on the hypothesis that the medium is incompressible, the solution of the problem of reflexion as distinct from that of the elastic constitution of the medium is immediately derived, for all media which polarize the light linearly, without the aid of further knowledge except the law of density and the form of the wave-surface. If the density is uniform and the same in all media, the solution is that of MACCULLAGH and NEUMANN, which is known to be correct in form for isotropic (and also for crystalline) media. There is nothing so far to indicate whether the vibrations are in the plane of polarization or at right angles to it, but that point is soon settled by the most cursory comparison with observation of the resulting formulæ for the two kinds of polarized light; the vibrations must be in the plane of polarization of the light. It remains in this order of procedure, to discover a form of the potential-energy function which will lead to the correct form of wave-surface in crystalline media, at the same time making the vibrations in the plane of polarization, and which also will conform to the additional surface conditions not utilized in order to obtain merely the solution of the problem of reflexion; the discovery of such a function, as a result of a precise estimation of what was really required, is MACCULLAGH's special achievement.

25. If the æther in crystalline media is of æolotropic rotational elastic quality, and of isotropic effective inertia the same in all media, all the conditions of the problem of actual optical reflexion are satisfied whatever be the degree of its compressibility. While, on the other hand, if it is of isotropic elastic-solid quality and æolotropic effective inertia, and there is no elastic discontinuity in passing from one medium to another, *i.e.* if the elasticity is the same in all media, all the conditions are satisfied when there is no resistance to laminar compression. It is somewhat remarkable that the condition of continuity of the energy assumes the same form in both these cases.

What happens under more general conditions, or in circumstances of mixed elastic-solid and rotational elasticity, or possibly yet more general types of elasticity, we shall not stop at present to inquire. [See however § 21.] For the explanation of electrical phenomena, MACCULLAGH's energy-function possesses fundamental advantages for which none of these other possible optical schemes appear to be able to offer any equivalent; it is therefore not necessary to examine whether they can survive the searching ordeal of crystalline reflexion.

Total Reflexion.

26. So long as there actually exist the full number of refracted waves, this simple mode of solution of the problem by means of rays is perfectly rigorous, and puts the matter in as clear a light as a more detailed analysis of what is going on in the media; it is not necessary to make any assumption about the character of the incident wave, except that it is propagated without change. But the case is different when

the incidence on a rarer medium is so oblique that one or both the refracted waves disappear; if we simply treat these waves as non-existent, the four surface-conditions cannot all be satisfied. The natural inference is that the solution of the problem now depends on the particular form of the wave; the fundamental simple-harmonic form is the obvious one to choose, so let the vibration be represented by

$$\mathrm{A} \exp \iota\, 2\pi\lambda^{-1} (lx + my + nz - vt),$$

real parts only being in the end retained. The satisfaction of the interfacial conditions,—which must now be chosen all linear as we are running a real and an imaginary part concurrently, and they must not get mixed up,—leads to a complex value of n for one or both of the refracted waves and of A for both of them. The interpretation is of course, in the first case purely surface waves, in the second a change of phase in the act of reflexion or refraction. With this modification the celebrated interpretation of the imaginary expression in his formulæ, by FRESNEL, becomes quite explicit, and the general problem of total or partial crystalline reflexion is solved for the type of medium virtually assumed by him, without any detailed consideration of the nature of the elasticity. The hypothesis is implied, and may be verified, that the surface waves penetrate into the medium to a depth either great, or else small, compared with the thickness of the layer of transition between the media,—a point which has not always been sufficiently noticed.

Reflexion at the Surfaces of Absorbing Media.

27. The fact that homogeneous light in passing through a film of metal does not come out a mixture of various colours, or more crucially the fact that the use of a metallic speculum in a telescope does not interfere with spectrum observations, shows that the equation of vibration of light in a metallic medium is linear, and therefore that to represent the motion of the light in the metal requires simply the introduction of an ordinary exponential coefficient of absorption. The interface being the plane of xy, the light propagated in the absorbing medium will be represented by the real part of an expression of the form $\mathrm{A} \exp \iota\, 2\pi\lambda^{-1} (lx + my + nz - vt)$, where n is now complex with its real part negative if the axis of z is towards the direction of propagation. If the opacity of the medium is so slight that the light gets down some way beyond the interfacial layer of transition without very sensible weakening, we may therefore solve the problem of reflexion by an application of the ordinary surface-conditions stated in a linear form, but with a complex coefficient of elasticity; for we may treat the layer of transition as practically indefinitely thin. This comes to the same thing as the method used first by CAUCHY, of simply treating the index of refraction as a complex quantity in the ordinary formulæ for transparent media; and it should give a satisfactory solution of the problem, provided the opacity is not excessive.

The results obtained for metallic reflexion are however found to suffer, when compared with observation, from several serious defects; the real part of the *quasi*-index of refraction becomes negative, which is sufficient to prevent any stable self-subsisting medium from acting in this manner; while on transmission through certain metallic films there is a gain of phase of the light compared with vacuum, when there ought, according to the equations, to be a loss.

Optical Dispersion in Isotropic and Crystalline Media.

28. In order to make our luminiferous medium afford an explanation of electric and magnetic phenomena, it will be necessary to assume its potential energy to be wholly rotational, therefore quite independent of compression or distortion. When bodies are displaced through it, its motion will then be precisely that of a continuous frictionless incompressible fluid, and therefore no rotational stress will be thereby produced in it.

The phenomena of optical dispersion require us to recognize a dependence of the effective elasticity of the medium on the wave-length of the light; for we are bound on this theory, in the absence of sympathetic rotational vibrations of the atoms, to take the effective density of the primordial medium to be the same throughout all space. The dependence of the elasticity on the length of the wave can only arise from the presence of a structure of some sort in the medium, representing the molecular arrangement of the matter, whose linear dimensions are comparable with the wave-length of the disturbance that is propagated through it. The actual motion will now be of a very complicated character; but the fact that a wave is propagated through without change, in certain media (those which are at all transparent), shows that for the present purpose it is formally sufficient to average the disturbance into a continuous differential analysis, and thus take it to be a simple one as if there were no molecular discreteness, but with an effective elastic modulus proper to its wave-length. The expression for the potential energy of the medium will thus have to be of a form that will vary with the wave-length, while it is still a quadratic function of differential coefficients of the displacements; therefore we must now assume it to involve differential coefficients of higher order than the first. This mode of formulating the problem is what is led up to by the transparency of dispersive media *i.e.* by the permanence of type of simple waves travelling through them, and by the rotational character of the optical elasticity which is quite distinct from that of the molecular web, and, we may assume, of a different order of magnitude. It need excite no surprise if in extreme circumstances, involving near approach to equality with free periods of vibration, it is insufficient.

29. Now if the medium is to be thoroughly and absolutely fluid as regards non-rotational motions, *i.e.* if a vortex-atom theory of matter is to be part of the theory of the æther, this potential-energy function must be such that no work is done by

any displacement which does not involve rotation, therefore such that the work done by any displacement whatever is of the form

$$\int (\mathrm{L}\delta f + \mathrm{M}\delta g + \mathrm{N}\delta h)\, d\tau$$

or

$$\int \left\{ \mathrm{L}\left(\frac{d\delta\zeta}{dy} - \frac{d\delta\eta}{dz}\right) + \mathrm{M}\left(\frac{d\delta\xi}{dz} - \frac{d\delta\zeta}{dx}\right) + \mathrm{N}\left(\frac{d\delta\eta}{dx} - \frac{d\delta\xi}{dy}\right)\right\} d\tau$$

together with possible surface-integral terms. Integration by parts leads to the expression

$$\int \left\{ \left(\frac{d\mathrm{N}}{dy} - \frac{d\mathrm{M}}{dz}\right)\delta\xi + \left(\frac{d\mathrm{L}}{dz} - \frac{d\mathrm{N}}{dx}\right)\delta\eta + \left(\frac{d\mathrm{M}}{dx} - \frac{d\mathrm{L}}{dy}\right)\delta\zeta \right\} d\tau.$$

This expression must be the same as the one derived by integration by parts in the usual manner from the variation of the potential energy $\delta\int \mathrm{W}\, d\tau$, where W is now of the second degree in spacial differential coefficients, of various orders, of (ξ, η, ζ). The result, as far as the volume integral is concerned, will be the same as if the symbols of differentiation d/dx, d/dy, d/dz were dissociated from ξ, η, ζ and treated like symbols of quantity, after the sign of each has been changed, so that for example $d\xi/dy\; d^2\eta/dx^2$ is to be taken the same as $-\, d/dy\; d^2/dx^2\; \xi\eta$; the function W may thus be replaced for this purpose by

$$\mathrm{W}' = \mathrm{A}\xi^2 + \mathrm{B}\eta^2 + \mathrm{C}\zeta^2 + 2\mathrm{D}\eta\zeta + 2\mathrm{E}\zeta\xi + 2\mathrm{F}\xi\eta,$$

where A, B, C, D, E, F are functions of d/dx, d/dy, d/dz.

We shall then have

$$\delta\int \mathrm{W}\, d\tau = \int \{\ldots\ldots\}\, d\mathrm{S} + \int \left(\frac{d\mathrm{W}'}{d\xi}\delta\xi + \frac{d\mathrm{W}'}{d\eta}\delta\eta + \frac{d\mathrm{W}'}{d\zeta}\delta\zeta\right) d\tau.$$

On comparing these expressions there results

$$\left(\frac{d\mathrm{N}}{dy} - \frac{d\mathrm{M}}{dz},\; \frac{d\mathrm{L}}{dz} - \frac{d\mathrm{N}}{dx},\; \frac{d\mathrm{M}}{dx} - \frac{d\mathrm{L}}{dy}\right) = \left(\frac{d}{d\xi},\; \frac{d}{d\eta},\; \frac{d}{d\zeta}\right)\mathrm{W}'.$$

Hence

$$\left(\frac{d}{dx}\right)\frac{d\mathrm{W}'}{d\xi} + \left(\frac{d}{dy}\right)\frac{d\mathrm{W}'}{d\eta} + \left(\frac{d}{dz}\right)\frac{d\mathrm{W}'}{d\zeta} = 0$$

identically, where the differential operators in brackets are to be treated as if they were symbols of quantity. The vanishing of this expression, for all values of ξ, η, ζ, involves three conditions between A, B, . . ., one of which may be stated in the form that the quadratic expression W′ is the product of two linear factors; these are in fact the general analytical conditions that a medium shall not propagate waves of compression involving sensible amounts of energy.

30. But these conditions are not sufficient to insure that the elasticity shall be purely rotational, and in no wise distortional. For example, as may be seen from the above, the elasticities of Lord KELVIN's labile elastic-solid æther and of GREEN's incompressible æther satisfy them. What is required is that for any displacement of a given portion of the medium, the total work done by both the bodily forcive and the surface tractions shall be expressible in terms of the rotations of its elementary parts alone. In the particular case in which the medium is in internal equilibrium in a state of strain, the part of this work which is due to bodily forcive is of course null; so that the surface-tractions are then all-important.

31. Now let us examine a form of W_2, the dispersional part of the energy, which has been put forward by MACCULLAGH solely in order to explain the fact that the character of the crystalline wave-surface is not altered by the dispersional energy. He assumes that W_2 is a function of (f, g, h) and of its vorticity or curl, and of the curl of that curl, say its curl squared, and so on; and he observes that if this quadratic function only involve squares and products of the respective components of odd powers of the curl, FRESNEL's wave-surface is unaltered, while if even powers come in, the surface is modified in a simple and definite manner;* it will be clear on consideration that if an odd power of the operator is combined with an even power, in any term, rotational quality of the medium must be introduced. It will be sufficient for practical applications to attend to the dispersional terms of lowest order. Since in an incompressible medium $(\text{curl})^2 = -\nabla^2$, these terms yield two possible forms for the dispersional part of the energy,

$$f\nabla^2 f + g\nabla^2 g + h\nabla^2 h$$

and

$$(\nabla^2 \xi)^2 + (\nabla^2 \eta)^2 + (\nabla^2 \zeta)^2;$$

or in a crystalline medium we might take the corresponding forms

$$\alpha^2 f\nabla^2 f + \beta^2 g\nabla^2 g + \gamma^2 h\nabla^2 h$$

and

$$\alpha'^2 (\nabla^2 \xi)^2 + \beta'^2 (\nabla^2 \eta)^2 + \gamma'^2 (\nabla^2 \zeta)^2;$$

or we could have more generally the lineo-linear function of (f, g, h) and $\nabla^2 (f, g, h)$ and the general quadratic function of $\nabla^2 (\xi, \eta, \zeta)$, respectively, which would not be symmetrical with respect to the principal optical axes of the medium.

The first of these forms, the intermediate case being taken for brevity, yields a bodily forcive

$$\nabla^2 \left(\frac{d\gamma^2 h}{dy} - \frac{d\beta^2 g}{dz}, \quad \frac{d\alpha^2 f}{dz} - \frac{d\gamma^2 h}{dx}, \quad \frac{d\beta^2 g}{dx} - \frac{d\alpha^2 f}{dy}\right),$$

* MACCULLAGH, "On the dispersion of the Optic Axes and of the Axes of Elasticity in Biaxal Crystals," 'Phil. Mag.,' October, 1842, 'Collected Works,' pp. 221–226; "On the law of Double Refraction," 'Phil. Mag.,' 1842, 'Collected Works,' pp. 227–229.

and the second one yields a bodily forcive

$$(\alpha'^2\nabla^2\nabla^2\xi, \quad \beta'^2\nabla^2\nabla^2\eta, \quad \gamma'^2\nabla^2\nabla^2\zeta).$$

Both of these forcives satisfy the condition of being null when the medium is devoid of rotation. But, as in the motion of a train of plane waves of length λ the operator ∇^2 is replaceable by the constant $-(2\pi/\lambda)^2$, we see that the first forcive merges in the ordinary rotational forces of the medium, only altering its effective crystalline constants in a manner dependent on the wave-length; while the second forcive alters the character of the equations by adding to the right-hand sides terms proportional to ξ, η, ζ, and so modifies the wave-surface. If with MACCULLAGH we had taken the last and most general type of terms, which are not symmetrical with respect to the principal axes of optical elasticity, the observed dispersion of the optic axes of crystals would clearly have been involved in the equations. The nature of the proof of MACCULLAGH's general proposition is easily made out from the examination here given of this particular case.

32. The question has still to be settled, whether the postulate of complete fluidity as regards irrotational motion limits the form of W_2 to the one assumed by MACCULLAGH. It will I think be found that it does. For the final form of the variation of the potential energy is

$$\delta \int W \, d\tau = \int \{\ldots\} \, dS + \int (P\,\delta f + Q\,\delta g + R\,\delta h) \, d\tau,$$

where (P, Q, R) involve (f, g, h) linearly, but with differential operators of any orders. We may change it to

$$\delta \int W \, d\tau = \int \{\ldots\} \, dS - \int \text{curl}\,(P, Q, R)\,\delta\,(\xi, \eta, \zeta) \, d\tau,$$

the expression in the integral representing a scalar product; and this form shows that the bodily forcive in the medium is curl (P, Q, R). It also shows that the curl operator persists on integration by parts. Now this forcive is linear in (ξ, η, ζ), and taking for a moment the case of an isotropic medium, it must be built up of invariant differential operators. The complete list of such operators consists of curl, convergence, and shear operators, and their powers and products; and these operators are mathematically convertible with each other. Any combination of them, operating on (ξ, η, ζ), which involves curl as a factor, will limit the medium, as has been already seen, to the propagation of waves only rotational; but in order to secure perfect fluidity as regards irrotational motions it is necessary also that the surface tractions, involved in the surface-integral part of the variation of the energy, shall not depend on the shear or convergence of the medium. Now in arriving at the final form of the variational equations, by successive integrations by parts, if a convergence or shear occur in either factor of a term in W, it will emerge at some stage as an actual convergence or shear of the medium in a surface-integral term, indicating a surface traction which violates the condition of fluidity. But the only forms of W_2 for an isotropic medium, which maintain an invariantive character independent of axes of co-ordinates, and in which each factor involves only (f, g, h), appear to be made up of MACCULLAGH's forms and the form

$$\left(\frac{dh}{dy} + \frac{dg}{dz}\right)^2 + \left(\frac{df}{dz} + \frac{dh}{dx}\right)^2 + \left(\frac{dg}{dx} + \frac{df}{dy}\right)^2;$$

and if the medium is incompressible this new form is identical with the second type of MACCULLAGH. The conclusion thus follows that for isotropic media, the form of the potential energy, when we include dispersion and other secondary effects in it, is that of MACCULLAGH, the two forms given by him being in this case identical.

33. The question now presents itself, whether there is any distinction between the two types into which MACCULLAGH divides possible energy-functions of this kind, which will enable us to reject the one that modifies the form of the wave-surface. It seems fair to lay stress on the circumstance that the first of MACCULLAGH's types of dispersional energy may represent an interaction between the average strain of the medium (f, g, h) and the average disturbance of the strain due to molecular discreteness, while the other form represents the energy of some type of disturbance of the strain which combines only with itself, and is not directly operative on the average strain. It would seem natural to infer that a term of the second type would have its coefficient of a higher order of small quantities than the ones we are now investigating.

For the most general case of æolotropy, the dispersional energy W_2 must be either a quadratic function of first differential coefficients of (f, g, h), or else a lineo-linear function of (f, g, h) and its second differential coefficients. If the first alternative be rejected for the reason just given, there remains a form of which MACCULLAGH's is the special case in which the second differential coefficients group themselves into the operator ∇^2. A reason for this restriction is not obvious, unless we may take the form already determined for an isotropic medium as showing that the dispersion arises from the interaction of (f, g, h) on $\nabla^2 (f, g, h)$; such a restriction is in fact demonstrable when we bear in mind the scalar character of the energy-function.

The Influence of Dispersion on Reflexion.

34. It has been explained that on this theory the mode of formal representation of dispersion without sensible absorption, is by the inclusion of differential coefficients of the displacement, higher than the first, in the energy function. This makes the dispersion depend on change of elasticity, and not on any effective change of inertia of the primordial medium; in the neighbourhood of a dark band in the absorption spectrum of the medium, absorption plays an important part, rendering the phenomena anomalous, and we must then have recourse to some theory of the

YOUNG-SELLMEIER type, involving perhaps change of effective inertia, which will take a more complete account of the sympathetic interaction which occurs between the electric vibrations of the molecules and the vibrations of the medium, when their periods are very nearly alike.

The sum of the orders of the differential coefficients in any term of the energy must usually be even; a term in which it is odd would introduce unilateral quality into the medium, typified by such phenomena as rotatory polarization; and it is known from the facts and principles of crystalline structure that such terms can be, when existent at all, only of a very minute residual kind.

When we come to discuss the problem of reflexion, the surface-terms derived from the variation of the energy-function must be retained, and they should be adjusted so as to maintain the continuity of the manifestations of energy in crossing the interface. But the dispersional terms will introduce into the variational equation surface-integrals involving not only $\delta\xi$, $\delta\eta$, $\delta\zeta$, but also $\delta(d\xi/dx)$, $\delta(d^2\xi/dx^2)$, ... ; and we cannot even attempt to make all these independent terms continuous across the interface. We therefore cannot follow in our analysis the complete circumstances of the problem of reflexion. This is not cause for surprise, because the essence of the method of continuous analysis consists of averaging the molecular discreteness of the medium; and we are now trying to fit this analysis on to conditions at an interface where the law of the discreteness changes abruptly or rather very rapidly.

35. In a problem of this kind the procedure by the method of rays asserts a marked superiority. The interfacial layer being assumed for other reasons to be very thin compared with a wave-length, the displacement of the medium must be continuous across it. And it may be fairly assumed that there is no sensible amount of degradation of energy in this very thin superficial layer; so that the principle of continuity of energy gives the remaining interfacial condition. The result of these hypotheses will be that, so far, the law of reflexion of each homogeneous portion of the light depends on its own index, and not on the amount of the dispersion in its neighbourhood. The assumption of continuity of energy is the same thing as recognizing that the continuity of the dispersional part of the stress at the interface is maintained by surface forces of molecular character, which absorb no energy, and which need not be further specified for the present purpose,—thus forming an instance of a perfectly valid application of a surface-traction principle of the same kind as that of NEUMANN and KIRCHHOFF (§ 10).

This explanation is based on MACCULLAGH's theory of reflexion. If, merely for further illustration, we take FRESNEL's analysis of that problem, the medium is thereby assumed to be labile, and we must employ a stress condition at the interface as well as the energy condition. Now it is exactly in the insufficient specification of the stress near the surface that the trouble with respect to the dispersional terms came in; thus, if FRESNEL's theory were the tenable one, it would be a matter of some difficulty to get from it a clear view of reflexion in its relation to dispersion.

The Structural Rotational, or Helical, Quality of Certain Substances.

36. The quality of rotatory polarization, exhibited by quartz and turpentine, depends on the structure of the optical medium, and therefore must be expressed by a term in the potential-energy W. When symbols of differentiation are imagined for the moment as separable from their operands, this term must be of the third degree in $(d/dx, d/dy, d/dz)$; and it must be quadratic in (ξ, η, ζ). It can therefore only involve the rotation (f, g, h) and its curl, each of them linearly;* therefore, being a scalar, the only form it can have is that of their scalar product; thus the term we are in quest of must be

$$C\left\{f\left(\frac{dh}{dy}-\frac{dg}{dz}\right)+g\left(\frac{df}{dz}-\frac{dh}{dx}\right)+h\left(\frac{dg}{dx}-\frac{df}{dy}\right)\right\},$$

or what is the same

$$-C\left\{f\nabla^2\xi+g\nabla^2\eta+h\nabla^2\zeta\right\}.$$

This is in fact the term invented by MACCULLAGH for the purpose of explaining the rotational phenomena of liquids, and of quartz in the direction of its optic axis, and shown by him and subsequent investigators to account for the facts. In the case of a crystalline medium, we might have for this term the general function of (f, g, h) and its curl, that is linear in both; but probably in all uniaxial crystals, certainly in quartz, the principal axes of this term are the same as the principal axes of optical elasticity of the medium.

On the Elasticity of the Primordial Medium.

37. The objection raised by Sir G. G. STOKES† in 1862 against the possibility of a medium of the kind contemplated by MACCULLAGH's energy-function, and since that time generally admitted, is that an element of volume of such a medium when strained could not be in equilibrium under the elastic tractions on its boundaries, but would require the application of an extraneous couple of amount proportional to its surface, and therefore very great in proportion to its mass, in order to keep it balanced. Such a state of matters is of course in flagrant contradiction to the character of the elasticity of solid bodies, and can only occur if there is some concealed rotational phenomenon going on in the element, the kinetic reaction of which can give rise to the requisite

* [(Added June 14.) The rotatory term in the energy function cannot involve differential coefficients with respect to the time; for to obtain the structural type of rotation these would have to appear in the second degree, which would make the term, as it involves only (f, g, h), of the fourth order in differential operators; *cf.* 'Brit. Assoc. Report,' 1893, "Magnetic Action on Light," § 3. Thus MACCULLAGH'S term involves on the present theory only the one hypothesis that the medium is self-contained, and not effectively under the influence of another interpenetrating medium.]

† Sir GEORGE STOKES corroborates my impression that his criticism is expressly limited to media the elements of which are at rest and self-contained, and that it is not to be regarded as effective against a medium of gyrostatic quality or of the *quasi*-magnetic quality described below.

couple. If the medium had acquired its rotational elasticity by means of a distribution of rotating simple gyrostats, such a kinetic couple would be afforded by it so long as rotational motion of the element is going on,* and STOKES' criticism would not apply in this case. If again we imagine an ordinary elastic medium full of elementary magnets with orientations distributed according to some law or even at random, and in internal equilibrium either in its own magnetic field or in the field of some external magnetic system, then on rotational distortion a couple will be required to hold each element in equilibrium ; so that the conjugate tangential tractions on the surface of the element cannot be equal and opposite in this case either. The couple depends here on the absolute rotation of the element of volume, not on its angular velocity as in the previous illustration. The potential energy of such a medium as this will contain rotational terms of MACCULLAGH'S type, and its condition of internal equilibrium will be correctly deduced from an energy-function containing such terms by the application of the Lagrangian analysis. The origin of the elasticity purely rotational of MACCULLAGH'S medium is we may say unknown ; the first example here given shows that it cannot be simply gyrostatic, though Lord KELVIN has invented a complex gyrostatic structure that would produce it ;† and either example shows that we are not warranted in denying the possibility of such a medium because the equilibration of an element of it requires an extraneous couple. The explanation of gravitation is still outstanding, and necessitates some structure or property quite different from, and probably more fundamental than, simple rotational elasticity of the æther and simple molar elasticity of material aggregations in it; and this property may very well be also operative in the manner here required.

38. It becomes indeed clear when attention is drawn to the matter, that there is something not self-contained and therefore not fundamental, in the notion of even a gyrostatic medium and the resistance to absolute motion of rotation which it involves. For we want some fixed frame of reference outside the medium itself, with respect to which the absolute rotation may be specified ; and we also encounter the question why it is that rotatory motion reveals absolute directions in this manner. Another aspect of the question appears when we consider the statical model with its rotational property produced by small magnets interspersed throughout it, the medium being in internal equilibrium in a magnetic field when unstrained ; the unbalanced tractions on the element of volume are here supplemented by a couple due, as to sense, to magnetic action at a distance, and it is the energy of this action at a distance which constitutes the rotational part of the energy of the model. We may if we please suppose some analogous action at a distance to exist in the case of the actual æther, the ultimate explanation of which will be involved in the explanation of gravitation. Now in this magnetic analogue to our medium the equations of equilibrium and motion are clearly quite correctly determined by the analytical method of LAGRANGE. So

* *Cf.* 'Proc. Lond. Math. Soc.,' 1890.

† Lord KELVIN (Sir W. THOMSON), 'Comptes Rendus,' Sept. 16, 1889; 'Collected papers,' Vol. III., p. 467.

long as the potential energy is derived from a forcive emanating and transmitted nearly instantaneously from all parts of the medium and not merely from the contiguous elements, its location is expressed, quite sufficiently for dynamical purposes which are concerned with a finite volume of the medium and finite velocity of propagation, by attaching it to the element on which the forcive acts. The medium of MacCullagh therefore, on a saving hypothesis of this kind, appears to escape the kind of objection above mentioned.

Part II.—Electrical Theory.

39. The next stage in the development of the present theory is the application of the properties of non-vibrational types of motion of the primordial medium to the explanation of the phenomena of electricity. In accordance with the interpretation of MacCullagh's equations, on the ideas of the electro-magnetic theory of light, the electric displacement in the medium is its absolute rotation (f, g, h) at the place, and the magnetic force is the velocity of its movement $d/dt\ (\xi, \eta, \zeta)$. At the beginning, our view will be confined to rotational movements unaccompanied by translation, such namely as call into play only the elastic forces which are taken to be the cause of optical and electro-motive phenomena; but later on we shall attempt to include the electrical and optical phenomena of moving bodies.

In the ordinary electro-magnetic system of electric units we should have $4\pi(f, g, h) = \text{curl}\ (\xi, \eta, \zeta)$; but in purely theoretical discussions it is a great simplification to adopt a new unit of electric quantity such as will suppress the factor 4π, as Mr. Heaviside has advocated. Except in this respect, the quantities are all supposed to be specified in electro-magnetic units.

It may be mentioned that a scheme for expressing the equations of electro-dynamics by a minimal theorem analogous to the principle of Least Action, has recently been constructed by von Helmholtz.*

Conditions of Dielectric Equilibrium.

40. The conditions of electro-motive equilibrium in a general æolotropic dielectric medium are to be derived from the variation of the potential-energy function

$$W = \tfrac{1}{2}\int\left\{a^2\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right)^2 + b^2\left(\frac{d\xi}{dz} - \frac{d\zeta}{dx}\right)^2 + c^2\left(\frac{d\eta}{dx} - \frac{d\xi}{dy}\right)^2\right\} d\tau.$$

On conducting this variation, we have

* H. von Helmholtz, "Das Princip der kleinsten Wirkung in der Electro-dynamik," 'Wied. Ann.,' vol. 47, 1892.

$$\delta W = \int\left\{a^2 f\left(\frac{d\delta\zeta}{dy} - \frac{d\delta\eta}{dz}\right) + b^2 g\left(\frac{d\delta\xi}{dz} - \frac{d\delta\zeta}{dx}\right) + c^2 h\left(\frac{d\delta\eta}{dx} - \frac{d\delta\xi}{dy}\right)\right\} d\tau$$

$$= \int\{a^2 f(m\delta\zeta - n\delta\eta) + b^2 g(n\delta\xi - l\delta\zeta) + c^2 h(l\delta\eta - m\delta\xi)\}\, dS$$

$$- \int\left\{a^2\left(\frac{df}{dy}\delta\zeta - \frac{df}{dz}\delta\eta\right) + b^2\left(\frac{dg}{dz}\delta\xi - \frac{dg}{dx}\delta\zeta\right) + c^2\left(\frac{dh}{dx}\delta\eta - \frac{dh}{dy}\delta\xi\right)\right\} d\tau$$

$$= \int\{(nb^2 g - mc^2 h)\,\delta\xi + (lc^2 h - na^2 f)\,\delta\eta + (ma^2 f - lb^2 g)\,\delta\zeta\}\, dS$$

$$- \int\left\{\left(\frac{dc^2 h}{dy} - \frac{db^2 g}{dz}\right)\delta\xi + \left(\frac{da^2 f}{dz} - \frac{dc^2 h}{dx}\right)\delta\eta + \left(\frac{db^2 g}{dx} - \frac{da^2 f}{dy}\right)\delta\zeta\right\} d\tau,$$

where (l, m, n) represents the direction of the normal to the element dS.

The vanishing of the volume integral in this expression for all possible types of variation of (ξ, η, ζ) requires that

$$a^2 f\, dx + b^2 g\, dy + c^2 h\, dz = -\, dV,$$

where V is some function of position, in other words that

$$(f, g, h) = -\left(\frac{1}{a^2}\frac{d}{dx}, \frac{1}{b^2}\frac{d}{dy}, \frac{1}{c^2}\frac{d}{dz}\right) V.$$

The vanishing of the surface integral requires that the vector $(a^2 f, b^2 g, c^2 h)$ shall be at each point at right angles to the surface.

It is hardly necessary to observe that in this solution V is the electric potential, from which the electric displacement (f, g, h) is here derived by the ordinary electrostatic formulæ for the general type of crystalline medium, and that the surface condition is that the electric force is at right angles to the surface, or in other words that the electric potential is constant all over it.

In deducing these conditions it has been assumed that the electrostatic energy is null inside a conductor; thus in statical questions the conductors may be considered to be regions in the medium devoid of elasticity, over the surfaces of which there is no extraneous constraint or forcive applied.

41. In this analysis it has not been explicitly assumed that the electric displacement is circuital, *i.e.* that

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0.$$

If we were to introduce explicitly this equation of constraint, we must by LAGRANGE'S method add a term

$$\tfrac{1}{2}\lambda\left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right)^2$$

to the energy function, before conducting the variation; and we must subsequently determine the function of position λ so as to satisfy the conditions of the problem. The result would now come out

$$(a^2f + \vartheta,\ b^2g + \vartheta,\ c^2h + \vartheta) = -\left(\frac{d}{dx},\ \frac{d}{dy},\ \frac{d}{dz}\right)\mathrm{V},$$

with the condition that V is constant over the surface of the conductor: where

$$\vartheta = \lambda\left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right),$$

and would represent so to speak an electromotive pressure uniform in all directions. The introduction of such a quantity would make the equations too general for the facts of electrostatics; on this ground alone we might assume ϑ to be null, and therefore V to be subject to a characteristic equation

$$\frac{d}{dx}\left(\frac{1}{a^2}\frac{d\mathrm{V}}{dx}\right) + \frac{d}{dy}\left(\frac{1}{b^2}\frac{d\mathrm{V}}{dy}\right) + \frac{d}{dz}\left(\frac{1}{c^2}\frac{d\mathrm{V}}{dz}\right) = 0.$$

This investigation may remain as an illustration of method; but it is not required, when we bear in mind the constitution of the medium. Since

$$(f,\ g,\ h) = \mathrm{curl}\ (\xi,\ \eta,\ \zeta)$$

we *must* have (f, g, h) circuital; so that the characteristic equation for V is involved in the data, without the necessity of any appeal to observation; while the introduction of the quantity ϑ would be illicit, and would have to be annulled later on.

42. If we assumed that the energy-function contained a term

$$\tfrac{1}{2}\mathrm{A}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right)^2,$$

the conditions of electromotive equilibrium would come out

$$\left(\frac{dc^2h}{dy} - \frac{db^2g}{dz},\ \frac{da^2f}{dz} - \frac{dc^2h}{dx},\ \frac{db^2g}{dx} - \frac{da^2f}{dy}\right) = -\left(\frac{d}{dx},\ \frac{d}{dy},\ \frac{d}{dz}\right)\vartheta'$$

and

$$(mc^2h - nb^2g,\ na^2f - lc^2h,\ lb^2g - ma^2f) = -(l,\ m,\ n)\,\vartheta',$$

where

$$\vartheta = \mathrm{A}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right).$$

Throughout a region devoid of elasticity this electromotive pressure ϑ' must be constant, and the electric force just outside its boundary must be along the normal; in the dielectric ϑ' must satisfy LAPLACE's equation, and so be the potential of an ideal superficial distribution of matter; but the electric force is not now derived from a potential, although its curl is derived from the potential ϑ' just specified.

The phenomena of electrostatics require that this term does not occur in the energy; and that may be either (i) because $d\xi/dx + d\eta/dy + d\zeta/dz$ is null, and the medium so to speak incompressible, or (ii) because A is null, so that the medium offers no resistance to laminar compression. But there is, apparently, nothing as yet to negative a constitution of the medium approximating extremely close to either of these two limiting states for both of which the equations of electrostatics would be exact. It has been shown already that there is absolutely nothing against such a supposition in the theory of light. But the experiments of CAVENDISH in proof of the electrostatic law of inverse squares, as repeated by MAXWELL, may be taken as showing that the ratio of any compressional effect to the rotational part of the phenomenon is at any rate excessively minute. A very small compressional term like this might possibly be of advantage in an attempt to include gravitation among the manifestations of æthereal activity, a point to be examined later on. It differs fundamentally from the compressional term introduced by VON HELMHOLTZ into the equations of electrodynamics.

43. We may also apply the variational equation of equilibrium to a volume in the interior of the dielectric medium, and therefore subject to surface tractions from the surrounding parts. It thus appears that the component surface-tractions in the æther in the directions of the axes of co-ordinates are, per unit area lying in the direction (l, m, n),

$$nb^2g - mc^2h, \quad lc^2h - na^2f, \quad ma^2f - lb^2g;$$

their resultant is tangential, *i.e.* in the plane of the element; it is equal to the component of the electric force in that plane, and is at right angles to that component. This is the specification of the æthereal stress by which static electromotive disturbance is transmitted across a dielectric medium. This stress does not at all interfere with any irrotational fluid motion which may be going on in the medium, or with the normal hydrostatic pressure which regulates such motion.

Electrostatic Attraction between Material Bodies.

44. When two charged bodies are moved relative to each other the total electrical energy of strain in the æther is altered; on the other hand, since the electrical displacement (rotation of the æther) is circuital, the charges of the bodies are maintained constant. In the absence of viscosity, this loss or gain of energy must be due to transference to some other system linked with the electric system; it reappears

in fact as mechanical energy of the charged conductors, which determines the mechanical forcive between them. It is desirable to attempt a closer examination of the nature of the action by which this transfer of energy takes place between the æther and the material of the conductors, and by which the similar transfer takes place at a transition between one dielectric substance and another.

In the displacement of a conductor through an excited dielectric there is thus an overflow of electromotive energy, and in the absence of viscous agencies and radiation it simply displays itself in ordinary mechanical forces acting on the surface of the conductor. The magnitude of these forces has been examined experimentally in different media, and has been found to correspond precisely with this account of their origin; good reason can be assigned to show that their intensity changes from point to point of the surface according to a law* ($KF^2/8\pi$, where F is electric force) which suggests that the energy is absorbed by the conductor at its surface. In a similar way, when a dielectric body is moved through the electric field the transformation of energy takes place at the interface between the two dielectrics.

The statical distribution of electromotive stress in the excited æthereal medium is definite and has just been determined: it involves on each element of interface in the dielectric æther a purely tangential traction at right angles to the tangential component of the electric force and equal to it. This is the denomination of stress that corresponds to the displacement (ξ, η, ζ), just as an ordinary force corresponds to a translation of matter or a couple to a rotation. If we have no direct knowledge of the æthereal displacement (ξ, η, ζ) we cannot actually recognize this stress; but when (ξ, η, ζ) is taken as here to be a linear displacement, this electromotive stress must be a mechanical stress in the æther such as does work in making a linear displacement.

45. The mechanical traction along the normal, which is distributed over the surfaces of two conductors separated by an excited dielectric, as for example the coatings of a charged Leyden jar, may be balanced by supports applied to the conductors; or if there is a dielectric body between them, it may be mechanically balanced by a stress in the *material* of this dielectric. This is the only kind of mechanical stress in a dielectric of which we have direct cognizance: its amount has been calculated by KIRCHHOFF† and others for some cases, and compared with experimental measures of change of volume of dielectrics under electrification. The stress in the æther itself has been here deduced by a wholly different path.

It will possibly be a true illustration of what occurs to imagine each element

* *Cf.* "On the theory of Electrodynamics, as affected by the nature of the mechanical stresses in excited dielectrics," 'Roy. Soc. Proc.,' 1892.

† G. KIRCHHOFF, "Ueber die Formänderung, die ein fester elastischer Körper erfährt, wenn er magnetisch oder diëlectrisch polarisirt wird," 'Wied. Ann.,' 24, 1885, p. 52; 25, 1885, p. 601. Such a stress, involving the square of the electric intensity instead of its first power, must of necessity be of secondary character, and cannot take direct part in wave-propagation in the electric medium.

of surface dS of the conductor to encroach by forward movement into the excited dielectric. As it proceeds, its superficial molecules somehow dissolve or loosen the strain of each little piece of the dielectric æther as they pass over it. Each fragmentary easing of strain sends a shiver through the dielectric æther, which however practically instantaneously readjusts itself into an equilibrium state. Thus the process goes on, the gradual molecular dissolution of the strain by the advance of the conductor shooting out minute wavelets of rearrangement of strain into the dielectric, which are confined to the immediate neighbourhood and are quite undiscernible directly, because on account of their great velocity of propagation the æther is always excessively near an equilibrium condition.* The pressural reaction (§ 97) of these disturbances on the conductor may be taken to be the source of the mechanical forcive experienced by it, which does work in impelling its movement and to an equal extent exhausts the energy of the dielectric.

Imagine a very thin element dS on the surface of the conductor, thick enough, however, to include this layer of intense disturbance of the æther; it will be subject to this electric reaction of the excited dielectric acting on it on the one side, and the elastic traction of the material of the solid conductor acting on it on the other side; and as its mass is very small compared with its surface, these forcives must equilibrate. For if this superficial element is displaced outwards through a very minute distance ds, the following changes of energy result; the energy of the dielectric is altered by the subtraction of that contained in a volume $dSds$ of it, while the elastic normal traction P of the conductor does work $PdSds$. These changes must compensate each other by the energy principle of equilibrium (compare § 58); hence the normal elastic traction P is equal to the energy in the dielectric per unit volume. The consideration of a tangential displacement of the element leads in the same way to the conclusion that the tangential elastic traction, required to be exerted by its material backing in order to maintain its equilibrium, is null.

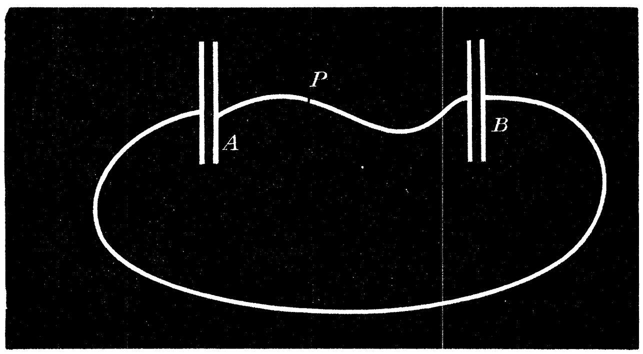
Electrodynamic Actions between Material Bodies.

46. In order to examine how far our energy-function of an æthereal medium involves an explanation of electrodynamic phenomena, we must begin with a simple case of electric currents that will avoid the introduction into the field of all complications like galvanic batteries, which could not easily be included in the energy-function. Let us therefore consider two charged condensers with their two pairs of coatings connected by thin wires as in the annexed diagram; and let us suppose the two plates of one of the condensers to be steadily moved towards each other when both pairs of coatings are thus in connexion. This will produce a steady current in the conducting wires, which will flow completely round the circuit; the only

* *Cf.* Sir G. G. STOKES, "On the Communication of Vibrations from a vibrating body to the surrounding gas," 'Phil. Trans,' 1868, p. 448; or in Lord RAYLEIGH, 'Theory of Sound,' vol. 2.

breaches of linearity of the current are at the condensers themselves, and these may be made negligible by taking the dielectric plates very thin. In this way a steady current can be realized in a conductor devoid of resistance, without the aid of any complicated electromotive source.*

47. Now we have to inquire what account the dynamical theory gives of this steady current. In the first place, the motion is very slow in comparison with the velocity of electric propagation; therefore the interior of the dielectric is at each instant sensibly in an equilibrium condition, for the same kind of reason that moving a body slowly to and fro does not start any appreciable sound waves in the atmosphere. Thus at each instant the vector (f, g, h) is derived as above from a potential function V; and at the surface of any of the conductors (supposed here of insensible resistance) it is directed along the normal, if the medium is isotropic. It is, in fact, in the more familiar electric language, at each instant the electric displacement determined by the charges which exist in a state of equilibrium on the faces of



the condensers and on the connecting wires. This electric displacement in the dielectric field is, owing to the condensing action, very small compared with the charges involved, except between the plates of the condensers and close to the thin conducting wire. Imagine a closed surface which passes between the plates of one of the condensers, and intersects the conducting wire at a place P. As the vector (f, g, h) is by its nature as a rotation circuital, its total flux through any surface must be null, if we imagine the elastic continuity of the medium inside the conductors to be restored, and such an electric displacement at the same time imparted along the wire as will leave the state of the field unaltered and thus no disturbance inside the conductors. And this flux must remain null when the plates of the condenser are slightly brought together; or rather we have to contemplate such a flow of displacement along the wire as will make it remain null. The movement of the plates will, however, very considerably alter the large flux across that portion of the

* *Cf.* "A mechanical representation of a vibrating electrical system and its radiation," 'Proc. Camb. Phil. Soc.,' 1891.

surface which lies between them; and the total flux for the other part of the surface not near the wire is as we have seen of trifling amount; therefore the alteration just mentioned must be considered to be balanced by an intense alteration of the above ideal flux in the immediate neighbourhood of the surface of the wire, in fact along its very surface if it is a perfect conductor. Immediately this change of the capacity of the condenser is over, the vector (f, g, h) will be back in its equilibrium condition in which it is, at each point of the surface of the wire, directed along the normal. As (f, g, h) represents the electric displacement in the field, the intense flux here contemplated, close to or on the surface of the wire, when the capacity is undergoing change, is the current in the wire. But all these circumstances concerning it have been made out from the dynamics alone, electric phraseology being employed only to facilitate the quotation of known analytical theorems about potential functions, and about how their distribution through space is connected with the forms of surfaces to which their fluxes are at right angles, and over which they therefore have themselves constant values.

If now while a current is flowing round the circuit, the two condensers are imagined to be instantaneously removed, and the wire made continuous, we shall be left with an ordinary circuital current, which in the absence of dissipative resistance will flow on for ever.

48. The argument in the above rests on the fact that there is circuital change of an elastic displacement $d/dt\ (f, g, h)$ distributed throughout the dielectric, while the medium is discontinuous at the surface of the perfectly conducting wire because displacement cannot be sustained inside the wire. When we for purposes of calculation imagine the elastic quality to extend across the section of the wire, and so avoid consideration of the discontinuity in the medium, we must imagine as above a flow of rotational displacement along the wire so long as the capacity of one of the condensers is being altered; and the velocity in the field will be deducible, by the ordinary formulæ for a continuous medium, from this ideal flow together with the actual changes of displacement throughout the dielectric. For a perfect conductor the circumstances will be exactly represented by confining this flow to its surface; what is required to make the analytical formulæ applicable, without modification on account of discontinuity in the medium, is simply the addition of such an ideal flow at the places of discontinuity as shall render the displacement (f, g, h) circuital throughout the field, without disturbing its actual distribution in the volume of the media.

The kinetic and potential energies of the medium may in fact either be calculated for the actual configuration, when they will involve surface integral terms extended over the surfaces of discontinuity, or they may be calculated as for a continuous medium if we take into account a flow of displacement along these surfaces, such as we would require to introduce by some agency if the medium were perfectly continuous, in order to establish the actually existing state of motion throughout it;

in estimating the energy of the medium in terms of the flow of displacement these surface sheets must be included, after the manner of vortex sheets in hydrodynamics.

In the same way, when the electric charge on a conductor is executing oscillations, a vortex sheet of changing electric displacement, such as will make the displacement in the field everywhere circuital, must be supposed to exist on the surface of the conductor.

49. There is this difference between actual electric current-systems and the permanently circulating currents, or vortex rings, in this æthereal medium, that the latter move in the medium so that their strengths remain constant throughout all time, while alteration of the strength of an electric current is produced by electrodynamic induction. In our condenser circuit, however, the strength of the current depends on the rate of movement of the plates of a condenser, that is, it is affected by changes in the rotational strain-energy of the portions of the medium which are situated in the gaps across the conducting curcuit. Motion of the condenser-plates produces a flow of displacement across any closed surface which passes between them, and therefore is to be taken as producing an equal and opposite flow where this surface intersects the connecting circuit. That ideal flow, or current, the representation of the action of the channel of discontinuity on the elastic transmission in the medium, implies on the other hand a hydrodynamical circulation of the medium round the conducting circuit, which provides the kinetic energy of the electric current. A current in a conductor has practically no elastic potential energy, because for movements of ordinary velocity the medium is always sensibly in an equilibrium condition, any beginning of an electromotive disturbance of the steady motion being instantly equalized before it has time to grow. A complete current, consisting of a flexible vortex-ring, or even circulating in a rigid core in the free æther, will thus maintain its strength unaltered, that is, the surrounding æther will move so that the electrodynamic induction in the circuit is always null; but if the current-curcuits are completed across the dielectric or through an electrolytic medium, this constraint to nullity of induction will be thereby removed, and constancy of circulation will no longer be a characteristic of such a broken vortex-ring, so to speak, in the medium.

50. The above mode of representing the surface-terms in the kinetic energy of course supposes that the intensities of the vortex sheets have been somehow already determined, or else that they are to be included in the scheme of variables of the problem. When the conductors are of narrow section, then as regards their action at a distance all that is wanted is the aggregate amount of flow across the section, that is, the electric current in the wire in the ordinary sense; and the introduction into the energy of terms calculated with reference only to these aggregates of flow is sufficient as regards the effect at distances from the conductors that are great compared with the dimensions of their cross sections. But if the details of the distribution round the section are required, the term in the energy must be more minutely specified as a surface-integral due to the interaction of the different elementary fila-

ments of the flow which are situated round the periphery of the section, much as the energy of a vortex sheet is introduced in the theory of discontinuous fluid motion; and its variation will now lead to electro-dynamic equations of continuous electric flow in the ordinary manner. There is no difficulty in extending this view to cases in which the breach of circuital character of the displacement-current $d/dt\ (f, g, h)$ may have to be made up by an ideal distribution of flow throughout the volume, that is, by a volume instead of a surface distribution of electric currents, as in an actual conductor of finite resistance.

[51. (Added June 14.) The velocity of a fluid is derivable in hydrodynamics, by kinematic formulæ, from the vorticity of its flow, provided we suppose the vorticity to include the proper vortex sheets spread over the surfaces of discontinuity of flow, if such exist; in the same way the magnetic force is derivable as above from the displacement-current, provided this current includes the proper current-sheets over the surfaces of the conductors or other surfaces of discontinuity of the magnetic field.

Let us consider an isolated uncharged conductor, and imagine an electric charge imparted to it. This charge is measured by the integral of the electric displacement (f, g, h) taken over any closed surface surrounding the conductor. Now if this rotational displacement were produced by continuous motion in the surrounding medium, its surface integral over any open sheet would be equal to the line integral of the linear displacement of the medium taken round the edge of the sheet. In a closed sheet the surface-integral would therefore be null; thus a charge cannot be imparted to a conductor without some discontinuous motion, or slip, or breach of rotational elasticity, in the medium surrounding it. If we imagine the charge to be imparted by means of a wire, the integral of electric displacement over any open surface surrounding the conductor and terminated by the wire is equal to the line-integral of the linear displacement of the medium round the edge of this surface where it abuts on the wire. If the wire is thin, this line integral is therefore the same at all sections of it, and thus involves a constant circulatory displacement of the medium around it. If the wire is a perfect conductor, there is no elasticity and therefore no rotational displacement of the æther inside its surface; thus there is slip in the medium at the surface of the wire; and if we desire to retain the formulæ of continuous analysis, we must contemplate a very rapid transition by means of a vortex sheet at the surface, in place of this discontinuity. This vortex sheet is in the present example continuous with *rotational* motion in the outside medium; the tubes of changing vorticity, *i.e.* of electric current, are completed and rendered circuital by displacement currents in the surrounding dielectric. But in the case of the condenser-circuit above considered, the alteration of the density of the vortical lines between a pair of plates, which is produced by separating them, involves a translational circulatory movement around the edge of the condenser and throughout the medium outside, which is almost entirely of *irrotational* type, except at the surface of the conducting wire where a vortex sheet has to be located in order to avoid discontinuity. The irrotational motion in the surrounding medium, which is thus continuous with the vortex sheet, and therefore determined by it, represents the magnetic field of the current flowing in the wire. On the other hand, in the illustration of this section, the motion in the medium is not irrotational, for it represents the field determined by the displacement currents in the medium and the conduction current in the wire, taken together.]

52. To return to our condenser illustration; it does not follow from the superficial character of the current $d/dt\,(f, g, h)$ that the velocity-vector $d/dt\,(\xi, \eta, \zeta)$ is also very small throughout the field except at the very surface of the wire. We have in fact $(f, g, h) = \text{curl}\,(\xi, \eta, \zeta)$, therefore

$$\nabla^2(\xi, \eta, \zeta) - \left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) = -\,\text{curl}\,(f, g, h);$$

so that, the compression $d\xi/dx + d\eta/dy + d\zeta/dz$ being null, $d/dt\,(\xi, \eta, \zeta)$ are the potentials of certain ideal mass-distributions close to the surface of the wire; therefore they are of sensible magnitude throughout the surrounding field.

It appears from the surface character of the disturbance of the electric displacement (f, g, h) which is thus introduced for current-systems flowing in complete circuits, that if we transform the kinetic-energy function

$$T = \tfrac{1}{2}\int\left(\frac{d\xi^2}{dt^2} + \frac{d\eta^2}{dt^2} + \frac{d\zeta^2}{dt^2}\right)d\tau,$$

in which it is convenient to take the density to be unity, so that it shall be expressed in terms of the current $d/dt\,(f, g, h)$, at the same time treating the rotational displacement of the medium as continuous, we shall have practically reduced it to a surface integral along the wire. To effect this, let (F, G, H) be the potentials, throughout the region, of ideal mass-distributions of densities $d/dt\,(f, g, h)$: so that

$$(\mathrm{F}, \mathrm{G}, \mathrm{H}) = \int\frac{d\tau'}{r'}\frac{d}{dt}(f', g', h'),$$

where r' is the distance from the element of volume $d\tau$ to the point considered; then

$$\frac{d\mathrm{G}}{dx} - \frac{d\mathrm{F}}{dy} = -\frac{d}{dt}\int\left\{\nabla^2\zeta - \frac{d}{dz}\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right)\right\}\frac{d\tau}{r}$$
$$= 4\pi\frac{d\zeta}{dt}, \text{ as } \frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} \text{ is null.}$$

Thus

$$\mathrm{T} = \frac{1}{8\pi}\int\left\{\frac{d\xi}{dt}\left(\frac{d\mathrm{H}}{dy} - \frac{d\mathrm{G}}{dz}\right) + \frac{d\eta}{dt}\left(\frac{d\mathrm{F}}{dz} - \frac{d\mathrm{H}}{dx}\right) + \frac{d\zeta}{dt}\left(\frac{d\mathrm{G}}{dx} - \frac{d\mathrm{F}}{dy}\right)\right\}d\tau$$
$$= \frac{1}{8\pi}\int\left\{\mathrm{F}\frac{d}{dt}\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right) + \mathrm{G}\frac{d}{dt}\left(\frac{d\xi}{dz} - \frac{d\zeta}{dx}\right) + \mathrm{H}\frac{d}{dt}\left(\frac{d\eta}{dx} - \frac{d\xi}{dy}\right)\right\}d\tau$$

on integrating by parts. The medium is supposed here to be mathematically continuous as above, thus avoiding separate consideration of the conducting channels,—though its structure may change with very great rapidity in crossing certain interfaces; and it is taken to extend through all space, so that the surface-integral terms may be omitted, no active parts of the system being supposed to be at an infinite distance. Thus

$$\mathrm{T} = \frac{1}{8\pi}\int\left(\mathrm{F}\,\frac{df}{dt} + \mathrm{G}\,\frac{dg}{dt} + \mathrm{H}\,\frac{dh}{dt}\right)d\tau$$

$$= \frac{1}{8\pi}\iint\frac{1}{r}\left(\frac{df}{dt}\,\frac{df'}{dt} + \frac{dg}{dt}\,\frac{dg'}{dt} + \frac{dh}{dt}\,\frac{dh'}{dt}\right)d\tau\,d\tau',$$

which is the form required, expressed as a double integral throughout space.

For a network of complete circuits carrying currents $\iota_1, \iota_2, \ldots$ we may express this formula more simply as

$$4\pi\mathrm{T} = \tfrac{1}{2}\,\iota_1^2\iint\frac{\cos\epsilon}{r_1}\,ds_1\,ds_1 + \ldots \qquad + \iota_1\iota_2\iint\frac{\cos\epsilon}{r_{12}}\,ds_1\,ds_2 + \ldots,$$

where ϵ is the angle between the directions of the two elements of arc; which is NEUMANN'S well-known form of the mechanical energy of a system of linear currents. The currents are here simply mathematical terms for such flows of electric displacement along each wire as would be required to make the displacement throughout the field perfectly circuital, if the effective elasticity were continuous in accordance with the explanation above.

53. Now if two wire circuits carry steady currents, generated from condensers in this manner, and are displaced relatively to each other with velocities not considerable compared with the velocity of propagation of electromotive disturbances, the electric energy of the medium is thereby altered. There is supposed to be no viscous resistance in the system, and no sensible amount of radiation; therefore the energy that is lost by the medium must be transferred to the matter. This transfer is accomplished by the mechanical work that is required to be done to alter the configuration of the wires against the action of electrodynamic forces operating between them; for these mechanical changes have usually a purely statical aspect compared with the extremely rapid electric disturbances. The expression T, with its sign changed, is thus the potential energy of mechanical electrodynamic forces acting between the material conductors which carry the currents.

Furthermore, as above observed, the electro-kinetic energy and the electrodynamic forces at which we have arrived are expressed in terms of the total current flowing across any section of the wire supposed thin, and do not involve the distribution of the current round the contour of the section to the neighbourhood of which it is confined, nor the area or form of the section itself. It therefore does not concern us whether the wire is a perfect conductor or not; the previous argument from the circuital character of the rotation (f, g, h) shows that the total current is still the same across all sections of the wire, and that the energy relations are expressed in the same manner as before in terms of the total current.

The electrodynamic forces between linear current-systems are thus fully involved in the kinetic-energy function of the æthereal medium. The only point into which we cannot at present penetrate is the precise nature of the surface-action by which the energy is transferred (just as in § 45) from the electric medium to the matter of the perfect conductor; all the forces of the field are in fact derived from their appropriate energy-functions, so that it is not necessary, though it is desirable, to know the details of the interaction between æther and matter, at the surface of a conductor.

Mathematical Analysis of Electro-Kinetic Forces and their reaction on the Material Medium.

54. We have shown that the electro-kinetic energy of a system of linear electric currents may be expressed in the form

$$4\pi T = \tfrac{1}{2}\Sigma\iota_1^2 \iint \frac{\cos\epsilon}{r_1}\, ds_1\, ds_1 + \Sigma\iota_1\iota_2 \iint \frac{\cos\epsilon}{r_{12}}\, ds_1\, ds_2,$$

the velocity-system which they involve being sufficiently described by the set of velocity co-ordinates $\iota_1, \iota_2, \ldots$ combined with the kinetic constraints derived from the constitution of the æther. To mark that these quantities are dynamically velocities, let us denote $\iota_1, \iota_2, \ldots$ by $de_1/dt, de_2/dt, \ldots$ so that $e_1, e_2, \ldots$ will be taken as electric co-ordinates of position. The general variational equation of motion may be expressed in the form

$$\delta \int T\, dt = \int \delta W_1\, dt + \int (E_1\, \delta e_1 + E_2\, \delta e_2 + \ldots)\, dt,$$

where E_1 is by definition such that $E_1\, \delta e_1$ is the work done in the system during a displacement δe_1, so that in electric phraseology E_1 with sign changed is the electric force integrated round the circuit 1, or the electromotive force in that circuit. Also W_1 is any other potential energy the system may possess; the energy of electric strain throughout the medium being now very small, as there are no static electrifications, and the motions are supposed slow compared with the velocity of radiation. Thus, adopting the notation of coefficients of electrodynamic induction, so that

$$T = \tfrac{1}{2}L_1 \frac{de_1^2}{dt^2} + \tfrac{1}{2}L_2 \frac{de_2^2}{dt^2} + \ldots + M_{12} \frac{de_1}{dt}\frac{de_2}{dt} + \ldots,$$

$L_1, L_2, \ldots, M_{12}, \ldots$ depending on the configurations of the circuits, we have

$$\delta \int T\, dt = \Sigma \int \left(L_1 \frac{de_1}{dt} + M_{12} \frac{de_2}{dt} + \ldots \right) \frac{d\, \delta e_1}{dt}\, dt + \int \delta_1 T\, dt,$$

where in the last term $\delta_1 T$ refers to the change of T due to change of material configuration only. Hence

$$\delta \int T\, dt = \left| \Sigma (L_1 e_1 + M_{12} e_2 + \ldots) \delta e_1 \right|$$
$$- \Sigma \int \frac{d}{dt} \left(L_1 \frac{de_1}{dt} + M_{12} \frac{de_2}{dt} + \ldots \right) \delta e_1 \, dt + \int \delta_1 T \, dt,$$

the terms in $| \ldots |$ referring to the beginning and end of the time.

Thus we derive, and that in MAXWELL'S manner but rather more rigorously, FARADAY'S law of the induced electromotive force $(- E_1)$ under the form

$$- E_1 = - \frac{d}{dt} (L_1 \iota_1 + M_{12} \iota_2 + \ldots) = - \frac{d}{dt} \frac{dT}{d\iota_1}.$$

55. As already mentioned, for currents flowing round complete conducting circuits devoid of viscosity, the values of ι_1, ι_2, ... are constant, by a sort of constraint or rather by the constitution of the medium, throughout all time; and the electromotive forces E_1, E_2, ... here determined have no activity. But if, as in actual electric currents, the strengths are capable of change owing to the circuits being completed by displacement currents in the dielectric or across a voltaic battery thus constituting gaps through which additional displacement can so to speak flow into the conductors, or owing to viscous effects in the conductors carrying them which must also involve such discontinuity, then the forces E_1, E_2, ... here deduced from the energy-function will have an active existence, and the phenomena of electrodynamic induction will occur. Alteration of the strength of a current implies essentially incompleteness of the inelastic circuit round which it travels, and may be produced either by change of displacement across a dielectric portion of the circuit, or through the successive breaches of the effective elasticity of the æther which are involved in electric transmission across an electrolyte, and also probably in transmission through ordinary media which are not ideal perfect conductors. In short, the existence of electrodynamic induction leads to the conclusion that currents of conduction always flow in open circuits; if the circuit were complete, there would be no means available for the medium to get a hold on the current circulating in it. On this view the Amperean current circulating in a vortex atom is constant throughout all time, and unaffected by electrodynamic induction, so that there is apparently no room for WEBER'S explanation of diamagnetisn.

56. The vorticity in a circuit, that is, the current flowing round it, can thus be changed only by an alteration of the displacement across a break in the conducting quality of the circuit, or by the transfer of electric charge across an electrolyte, in which case it is elastic rupture of the medium that is operative. Such an alteration of current will be evidenced by, and its amount will be derivable from, the change in the energy-function of the dielectric medium, in the manner above described. When

there is no break in the conducting circuit, the current in it is restricted by the constitution of the medium to remain constant; and therefore an electromotive force E round the circuit, of the kind here determined, can do no work; it is not operative in the phenomena. The induction of a current on itself, due to change of form of its circuit, is bound up with the continued maintenance of the current by feed from batteries or other sources included in the circuit, in opposition to dissipation in the conductors which is connected with a sort of transfer by discharge from molecule to molecule within their substance: in an ideal perfectly conducting circuit there would be no such induction. A case which strikingly illustrates these remarks is the maintenance of a continuous current by a dynamo without any source other than mechanical work. The very essence of this action consists in the rhythmical make and break of the two circuits of the dynamo in synchronism with their changes of form, so that they are interlocked during one portion of the cycle and unlocked during the remainder. Such lockings and unlockings of the circuits may of course be produced by sliding contacts, but these are equivalent for the present purpose to breaches in the continuity of the conductors. The original apparatus of FARADAY's rotations (MAXWELL, "Treatise," Vol. II., § 486), which was the first electromotor ever constructed, and which driven backwards would act also as a dynamo, illustrates this point in its simplest form. Without some arrangement which allows the two circuits to cut across each other in this manner, there could be no induction of a continuous current, but only electric oscillations in the dielectric field, which could however be guided along conducting wires, as in alternate-current dynamos. The phenomena of electric currents in ordinary conducting circuits are thus more general than the phenomena of vortex-rings in hydrodynamics, or of atomic electric currents, in that the strengths of the currents in them are not constrained to remain constant; an additional displacement current can, so to speak, flow into a conductor at any of its breaches of continuity. The variables of the problem are thus more numerous, and the energy-function leads to more equations connecting them.

57. We might now attempt to proceed, by including the mechanical energy of the material conductors in the same function as the electro-kinetic energy, thus deducing that the energy gained by altering the co-ordinate ϕ_1 is $(d\mathrm{T}/d\phi_1)\,\delta\phi_1$, in other words that the displacement $\delta\phi_1$ is *opposed* by a force equal to $d\mathrm{T}/d\phi_1$. This would make currents flowing in the same direction along parallel wires repel each other, and in fact generally the force thus indicated is just the opposite to the reality.

The expression T represents completely the energy of the system so far as electromotive disturbances are concerned, as has been proved above. But we have no right to assume that the energy of the system, so far as to include movements of the conductors and mechanical forces, can be completely expressed by this formula with only the electric co-ordinates and the sensible co-ordinates of the matter involved in it; for the mechanism that links them together is too complicated to be treated otherwise than statistically. We may however proceed as in the electrostatic problem; a displacement increases T by δT; this increase must come from some source; as there is supposed to be no dissipation it must come ultimately from the energy of the material system. During the displacement the electromotive system is at each moment sensibly in an equilibrium condition, so that there is practically no interaction between the kinetic energies of the electromotive and the material systems such as would arise from mixed terms in the energy-function involving both their velocities,—a fact verified experimentally by MAXWELL.* Thus somehow by means of unknown connecting actions, the displacement alters the mechanical energy of the system by an amount $-\delta$T, and of this, considered as potential energy, the mechanical forces are the result. The mechanical force acting to *increase* the co-ordinate ϕ_1 is therefore $d\mathrm{T}/d\phi_1$. In fact, instead of considering the material system to be represented by the co-ordinates $\phi_1, \phi_2, \ldots$ which enter into the electro-kinetic energy, we must consider it to be an independent system linked on to the electro-kinetic system by an unknown mechanism, which however is of a statical character, so that energy passes over from the electro-kinetic system to the other one as mere statical work, without any complication arising from the effects of mixed kinetic reactions. In the discussion in MAXWELL's "Treatise," § 570, this idea of action and reaction between *two* interlocked systems, the electromotive one and the mechanical one, has in the end to be introduced to obtain the proper sign for the mechanical force. The energy T is electro-kinetic solely; no energy of the material system is included in it.

58. This deduction of the electrostatic and the electrodynamic mechanical forcive may now be re-stated in a compact form, which is also noteworthy from the circumstance that it embodies perhaps the simplest method of treatment of the energy-function in all such cases. Let us consider the dynamical system under discussion to be the purely electric one, that is, to consist of the dielectric medium only, so that it has boundaries just inside the surfaces of the conductors, which are supposed to be perfectly inelastic. The energy function T + W remains as above stated, for all the energy is located in the dielectric; the electro-kinetic part T arises from motion of the medium, and the electrostatic part W from its rotational strain. But in the equation of Least Action we must also take account of tractions which may be exerted by the matter of the conductors on the boundary of this dielectric system. If $\delta w\, d\mathrm{S}$ denote the work done on the dielectric by these tractions extended over the element dS of the surface, the equation of Action will be

$$\delta\int(\mathrm{T}-\mathrm{W})\,dt - \int dt \int \delta w\, d\mathrm{S} = 0,$$

the time of passage from initial to final position being unvaried. When the disturbances considered are, as usually taken, too slow to generate sensible waves in the dielectric, and even when this restriction is not imposed, it equally follows that the

* MAXWELL, 'Treatise,' Part IV., "Electromagnetism," Chap. VI. The apparatus was constructed as early as 1861.

tractions of the conductors on the dielectric system are derived from a potential energy function T − W, only in the latter case the value of this function is more difficult to determine; hence the tractions of the dielectric on the conductors are derived from a potential energy function − (T − W). Of this potential function the first part gives the electrodynamic forces acting on the conductors, the second part the electrostatic forces. This mode of treatment is clearly perfectly general, and applies, for instance, with the appropriate modification of statement, to the determination of the electrodynamic forces of an element of a continuous non-linear current flowing through a conducting medium; it will be shown presently that the electric dissipation-function can contribute nothing to the ponderomotive forcive.

That the part of the forcive which is due to the variation of this potential energy W is correctly expressible by means of the electrostatic traction $KF^2/8\pi$ on the surfaces of the conductors, may be verified as follows. Suppose an element of surface dS of the conductor to encroach on the dielectric by a normal distance dn; the energy that was in the element of volume $dS\,dn$ of the dielectric has been absorbed; and in addition the energy of the mass of the remaining dielectric has been altered by the slight change of form of the surface of the conductor in the neighbourhood of the element dS. Now the dielectric is in internal equilibrium, therefore its internal energy in any given volume is a minimum; therefore the change produced in that energy by any small alteration of constraint, such as the one just described, is of the second order of small quantities. Hence the encroachment of the element dS of the conductor diminishes the total energy W simply by the amount contained in the volume $dS\,dn$; and therefore that encroachment is assisted somehow by a mechanical traction equal to the energy per unit volume of the dielectric at the place, that is, of intensity $KF^2/8\pi$.

Electrodynamic effect of motion of a charged Body.

59. When a charged body moves relatively to the surrounding æther, with a velocity small compared with the velocity of electric propagation, it practically carries its electric displacement-system (f, g, h) along with it in an equilibrium configuration. Thus the displacement at any point fixed in the æther will change, and we shall virtually have the field filled with electric currents which are completed in the lines of motion of the charged elements of the body, so long as that motion continues. On this view, MAXWELL'S convection-current is not differentiated from conduction-current in any manner whatever, if we except the fact that viscous decay usually accompanies the latter.

A metallically coated glass disc, rotating in its own plane without altering its position in space, would on this theory produce no convection-current at all; but if the coating of the disc is divided into isolated parts by scratches, as in ROWLAND and

HUTCHINSON'S experiments,* or even if there is a single line of division, each portion will carry its field of electric displacement along with it, the field preserving its statical configuration under all realizable speeds of rotation. If the scratches did not run up to the centre of the disc, the field of displacement due to the central parts would be quiescent, and the displacement-currents would be altered in character.† The dielectric displacement in the experiments above-mentioned, with two parallel rotating gilt glass condenser-discs having radial scratches, is across the field from one disc to the other, and is steady throughout the motion; so that the convection-currents are completely represented by the simple convection of the electric charges on the discs, and are not spread over the dielectric field.

60. The motion of a dielectric body through a field of electric force ought also to carry its system of electric displacement along with it. It appears that RÖNTGEN‡ has detected an effect of convection-currents when a circular dielectric disc is spun between the two plates of a charged horizontal condenser. In this case, however, the displacement-system in the field maintains its configuration in space absolutely unchanged; and according to the present view no effect of the kind should exist unless it be really caused by convection of an actual charge on the rotating dielectric plate (unless we find in it a proof of the convection of actual paired ions, of which the material dielectric is constituted. See § 125.)

On Vortex Atoms and their Magnetism.

61. Suppose, in the condenser-system described above, that a current is started round the circuit by a change of capacity of one of the condensers, and that then the two condensers are instantaneously taken out and the wire made continuous; the current, in the absence of resistance in the wire, will now be permanent. A permanent magnetic element will thus be represented by a circuital cavity or channel in the elastic æther, along the surface of which there is a distribution of vorticity; it will in short be a vortex-ring with a vacuum (or else a portion of the fluid devoid of rotational elasticity) for its core. An arrangement like this must be supposed, in accordance with AMPÈRE'S theory,§ to be a part of the constitution of a molecule in iron and other magnetic

* H. A. ROWLAND and C. T. HUTCHINSON, "On the electro-magnetic effect of Convection-currents." 'Phil. Mag.,' June, 1889, p. 445.

† [The statement in the text is certainly true if we can regard the disc as a perfect conductor; on the other hand if it is an insulator, the charge will be carried along with it. It has been suggested that it is open to question whether the conductivity of a coating of gold-leaf is great enough to practically come under the first of these types. But if we are to adhere to the ordinary idea that the free oscillations of an electric charge on such a conductor are absolutely unresisted by any superficial viscosity, as they are certainly independent of ohmic resistance, we must, it would seem, regard a metallic disc as practically equivalent for the present purpose to a perfect conductor. This view would also suggest an explanation of the circumstance that some experimenters have not been able to verify the existence of the ROWLAND effect.]

‡ ROWLAND, *loc. cit.*, p. 446; RÖNTGEN, "Wied. Ann.," 35, 1888.

§ MAXWELL, 'Treatise,' vol. 2, chap. 22.

metals. As a fundamental structure like the present can hardly be supposed to be broken up at the temperature at which iron becomes non-magnetic, to appear again on lowering the temperature, we must postulate that a permanent electric current of this kind is involved in the constitution of the atom ; that in iron the atoms group themselves into aggregates with their atomic currents directed in such a way as not absolutely to oppose each other's action ; while at the temperature of recalescence these groups are broken up and replaced by other atomic groups, for each of which the actions at a distance of the different atomic currents are mutually destructive. In a material devoid of striking magnetic properties, we may imagine the atoms as combined into molecules in this latter way.

62. If we imagine a vortex-ring theory of atoms, in which the velocity of the primeval fluid represents magnetic force, and the atoms are ordinary coreless vortices, we shall have made a step towards a consistent representation of physical phenomena. In such a fluid the vortices will join themselves together into molecules and molecular groups ; the vortices of each group will however tend to aggregate in the same way as elementary magnets, so that instead of neutralizing each other's magnetic effects, they will reinforce one another ; on this view substances ought to be about equally magnetic at all temperatures, instead of showing as iron does a sudden loss of the quality. We must therefore find some other bond for the atoms of a molecule, in addition to the hydrodynamic one and at least of the same order of magnitude. This is afforded by the attractions of the electric charges of the atoms, which are required by the theory of electrolysis. But even now about half of the molecules would be made up so that the atoms in them assist each other's magnetic effects, unless we suppose each molecule to contain more than two atoms, arranged in some sort of symmetry. There is however no course open but to take all matter to be magnetic in the same way, the only difference being in some very special circumstance in the aggregation of the molecules of iron compared with other molecules. The small magnetic moment of molecules of most substances may in fact be explained more fully on the same lines as their small electric moment (§ 64). The vortices will be quite permanent as regards both atomic charge and electric intensity, so that the explanation of diamagnetic polarity given by WEBER, on the basis of currents induced in the atomic conducting circuits, cannot now stand.*

* [Added June 14.—It has been suggested that the atomic electric charge might circulate round the ring under the influence of induction. It would appear however that such a circulation could have no physical meaning, for it would not at all alter the configuration of strain in the surrounding medium, which is the really essential thing.

It is otherwise with the motion of translation of a small charged body: the intrinsic twist of the surrounding medium is carried on with it, and the effect of the movement is thus to impose an additional twist or rotation round the line of motion (§ 59). Thus if we imagine an endless chain of discrete electrified particles, which circulate round and round, each particle of it will carry on independently its state of strain and so be subject separately to forcive; and we shall have the dynamical phenomena illustrated by a current of purely convective character, involving no electric displacement in the dielectric, and no generator.]

We have hitherto chosen to take the vortex-atoms with vacuous cores, so that the currents must be represented by the vortex sheets on their surfaces; and this was in order to have an exact representation of the circumstances of perfect conductors. If we assigned a rotating fluid core, devoid of elasticity, to the vortex-atom, not many essential differences would be introduced. The circumstances of an ordinary electric current flowing steadily round a channel which is not an ideal perfect conductor are somewhat more closely represented by supposing the channel to be the core of the ring, filled with fluid whose rotation is uniform across each section; this uniform distribution of the current across the channel is however primarily an effect of viscous retardation, due to the succession of discharges across intermolecular æther by which the propagation is effected.

Electrostatic Induction between Aggregates of Vortex-atoms.

63. When a piece of matter is electrified, say by means of a current conducted to it by a wire, what actually happens according to dynamical analysis on the basis of our energy-function, is that an elastic rotational displacement is set up in the æther surrounding it, the absolute rotation at each point representing the electric displacement of MAXWELL. If there is no viscosity, *i.e.* if the matter and the wire are supposed to be perfect conductors, this result is a logical consequence of the assumed constitution of the æthereal medium; and of course the circumstances of the final equilibrium condition are independent of any frictional resistance which may have opposed its development, so that the conclusion is quite general

We may now construct a representation of the phenomena of electrostatic induction. A charged body exists in the field, causing a rotational strain in the æther all round it; consider the portion of the æther inside another surface, which we may suppose traced in the field, to lose its rotational elasticity as the result of instability due to the presence of molecules of matter; the strain of the æther all round that surface must readjust itself to a new condition of equilibrium; the vortical lines of the strain will be altered so as to strike the new conductor at right angles,—and everything will go as in the electrostatic phenomenon. But there will be no aggregate electric charge on the new conductor; for the electric displacement (f, g, h) is a circuital vector, and therefore its flux into any surface drawn, wholly in the æther, to surround the new conductor, cannot alter its value from null which it was before. Now suppose a thin filament of æther, connecting the two conductors, to lose its rotational elasticity; the conditions of equilibrium will again be broken, and the effect throughout the medium of this sudden loss of elasticity will be the same as if a wave of alternating vorticity were rolling along the surface of this filament from the one conductor to the other, with an oscillation backwards and forwards along it which will persist unless it is damped by radiation or viscous action. The final result, after the decay of the oscillations, will be a new state of equilibrium, with charges on both the conductors, precisely as under electrostatic circumstances.

64. The phenomenon of specific inductive capacity has been explained or illustrated at different times by FARADAY, MOSSOTTI, Lord KELVIN, and MAXWELL, by the behaviour of a medium composed of small polar elements which partially orientate themselves under the action of the electric force; and these *quasi*-magnetic elements have been identified with the molecules, each composed of a positive and a negative ion. Another illustration* which leads to the same mathematical consequences supposes the dielectric field to be filled with small conducting bodies, in each of which electric induction occurs, thus making it a polar element so long as it is under the influence of the electric force. The *quasi*-magnetic theory is adopted by VON HELMHOLTZ in his generalization, on the notions of action at a distance, of MAXWELL'S theory of electrodynamics; and it is shown by him that such a hypothesis destroys the circuital character of the electric current, a conclusion which may also be arrived at by elementary reasoning.† The molecules must therefore on such a theory be arranged with their positive and negative elements in some form of symmetry so that they shall have no appreciable resultant electric moments;‡ and the specific inductive capacity must be wholly due to diminution of the effective elasticity of the medium. The hexagonal structure imagined for quartz molecules by J. and P. CURIE, and independently by Lord KELVIN,§ in order to explain piezo-electricity, or any other symmetrical grouping, exactly satisfies this condition; the molecule in the state of equilibrium has no resultant electric moment; but under the influence of pressure or of change of temperature a deformation of the molecule occurs, which just introduces the observed piezo-electric or pyro-electric polarity.

[(Added June 14.) On the present view however there is absolutely no room for VON HELMHOLTZ'S more general theory of non-circuital currents. The displacement of an electric charge constitutes a rotation in the medium round the line of the displacement, but the electric field which causes the displacement is here also itself a rotation round an axis in the same direction; whereas in VON HELMHOLTZ'S theory the inducing electric force is not considered to have any intrinsic electric displacement of its own. When both parts are taken into account, the electric displacement becomes circuital throughout the field. There is thus nothing in the postulate of circuital currents that would require us to make the electric moment of a molecule indefinitely small; so that specific inductive capacity might still, if necessary, be explained or illustrated in the manner of FARADAY and MOSSOTTI.]

* Employed by MAXWELL, "Dynamical Theory," § 11, 'Phil. Trans.,' 1864.

† "On the theory of Electrodynamics," 'Roy. Soc. Proc.,' 1890.

‡ The term electric moment is employed, after Lord KELVIN, as the precise analogue of magnetic moment.

§ Lord KELVIN, "On the piezo-electric quality of Quartz," 'Phil. Mag.,' Oct., 1893, Nov., 1893.

Cohesive, Chemical, and Radiant Forces.

65. If we consider a system of these vortex atoms, each of them will be subject to pulsations or vibrations, some comparatively slow, under the hydrodynamic influences of its neighbours in its own molecule; and each molecule will be subject to still slower vibrations under the influence of disturbances from the neighbouring molecules. In the former class we may possibly see the type of chemical forces, while the latter will have to represent phenomena of material cohesion and elasticity. But in addition to these purely hydrodynamical vibrations due to the inertia simply of the æther, there will be the types which will involve rotational distortion of the medium; that is, there will be the electrical vibrations of the atoms owing to the permanently strained state of the æther surrounding them which is the manifestation of their electric charges; the vibrations of this type will send out radiations through the æther and will represent the mechanism of light and other radiant energy. The excitation of these electric vibrations will naturally be very difficult; it will usually be the accompaniment of intense chemical action, involving the tearing asunder and re-arrangement of the atoms in the molecules. It is well-known that the vibrations of an electrostatic charge on a single rigid atom, if unsustained by some source of vibratory energy, would be radiated so rapidly as to be almost dead-beat, and so would be incompetent to produce the persistent and sharply-marked periods which are characteristic of the lines of the spectrum. But this objection may be to some extent obviated by considering that all the vibrational energy due to any very rapid type of molecular disturbance must finally be transformed into energy of electric strain and in this form radiated away.*

Voltaic Phenomena.

66. According to this theory a transfer of electricity can take place across a dielectric by rupture of the elastic structure of the medium, and only in that way; and this is quite in keeping with ordinary notions. Further, an electrolyte is generally transparent to light, or if not, to some kind of non-luminous radiation, so that such a substance has the power of sustaining electric stress; it follows therefore that transfer of electricity across the electrolyte in a voltameter, between a plate and the polarized atoms in front of it, can only occur along lines of effective rupture (such as may be produced by convection of an ion) of its æthereal elastic structure.

When two solid dielectrics are in contact along a surface, the superficial molecular aggregates will be within range of each other's influence, and will exert a stress which is transmitted by the medium between them. The transmission will be partly by an intrinsic hydrostatic pressure, as in LAPLACE's theory of capillarity, and partly by tangential elastic tractions produced by rotation of the elements of

* I understand that a suggestion of this nature has already been made by G. F. FITZ GERALD.

the medium. This rotation is the representative of electric force, or rather its effect electric displacement, in the medium; and, in so far as it is not along the interface, its line integral from one body to the other will account for a difference of electric potential between them. The electric force must be very intense, as in fact are all molecular forces, in order to give rise to a finite difference of potential in so short a range. If the bodies in contact are conductors, instead of dielectrics, similar considerations apply, but now the internal equilibrium of each conductor requires that the potential shall be uniform throughout it; therefore the surface stress must so adjust itself that the difference of potentials between the conductors is the same at each point of the interface.

The contact phenomena between a solid and a liquid are different from those between two solids; for the mobility of the liquid allows, after a sufficient lapse of time, an adjustment of charged dissociated ions along its surface so as to ease off the internal stress; and thus the boundary of the liquid becomes completely and somewhat permanently polarized. If we consider for example blocks of two metals, copper and zinc, separated by a layer of water, the electric stress in the interior of the water becomes null, and the difference of potential between the two metals is the difference of the potential-differences between them and water. That will not be the same as their difference of potential when in direct contact; but according to Lord KELVIN'S experiment it is sensibly the same as the difference between them and air,—owing in MAXWELL'S opinion to similarity in the chemical actions of air and water. In this experiment the electric stress is not transmitted through either of the metals; its seat is the surrounding æther, and the function of the metals is so to direct it, owing to the absence of æthereal elasticity inside them, that the axis of the rotation of the æther shall be, at all points of their surfaces, along the normal.

67. Let us imagine a VOLTA'S chain of different metals, forming a complete circuit, to be in electric equilibrium, as it must be, in the absence of chemical action and differences of temperature, by the principles of Thermodynamics. There is no electric stress transmitted through any metallic link of the chain; the stress is transmitted through the portion of the æther surrounding each metal, consisting in part of the interfacial layers separating it from the neighbouring metal, and in part of the atmosphere which surrounds its sides. In the equilibrium condition the potential in the æther all round the surface of the same metal is uniform; and this uniformity applies to each link in the chain. Therefore the sum of the very rapid changes of potential which occur in crossing the different interfaces, is, when taken all round the chain, strictly null: and we are thus led to VOLTA'S law of potential-differences for metallic conductors. Now suppose some cause disturbs this equilibrium, say the introduction of a layer of an electrolyte at an interface; this will introduce a store of chemical potential energy which can be used up electrically, and so equilibrium need no longer subsist at all. The uniformity of potential in the dielectric all round the surface of each metal will be disturbed, and a change of the electric displacement,

i.e. of the absolute rotation in the æther, will be set in action in the surrounding medium. If the metals are perfect conductors the effective flow of displacement will be confined to the surface, and will involve simply a vortex-sheet along the surface of each metal; but if the conducting power is imperfect the disturbance will diffuse itself into the metals, and the final steady condition will be one in which it is uniformly distributed throughout them, forming an ordinary electric current obeying OHM's law.

68. On the present theory, high specific inductive power in a substance is equivalent to low electric elasticity of the æther; it in fact stands to reason that an elastic medium whose continuity is broken by the inelastic and mobile portions which represent the cores of vortex-atoms may from this cause alone have its effective elasticity very considerably diminished.

Moreover it has been ascertained that, in electrolytic liquids, the specific inductive capacity attains very great values; the æther in these media interposes a proportionally small resistance to rotation, and the mobility or some other property of the vortex-molecules in it has brought it so much the nearer to instability; it is thus the easier to see why such media break down under comparatively slight electric stress. Such a medium also frees itself, as described below, from electric stress, without elastic rupture, in a time short compared with ordinary standards, but in most instances long compared with the periods of light-vibrations; while in metallic media the period of decay of stress is at least of the same order of smallness as the periods of light-waves.

69. An atom, as above specified, would be mathematically a singular point inthe fluid medium of rotational elastic quality. Such a point may be a centre of fluid circulation, and may have elastic twist converging on it, but it cannot have any other special property besides these; in other words this conception of an atom is not an additional assumption, but is the unique conception that is necessarily involved in the hypothesis of a simple rotationally elastic æther.

The attraction of a positively-charged atom for a negatively-charged one, according to the law of inverse squares, has already been elucidated. If the two atoms are moved towards each other so slowly that no kinetic energy of the medium is thereby generated, the potential energy of the rotational strain between them is diminished; and this diminution can be accounted for, in the absence of dissipation, only by mechanical work performed by the atoms or stored up in them in their approach. It has been observed by VON HELMHOLTZ that the phenomena of reversible polarization in voltameters involve no sensible consumption of energy, but that it is the actions which effect the transformation of the electrically charged ions into the electrically neutral molecules that demand the expenditure of motive power; and he draws the conclusion that energy of chemical decomposition is chiefly of electrical origin. In the explanation here outlined, the chemical (hydrodynamic) forces between the component atoms of the molecule are required to be, in the equilibrium position, of the same order of intensity as the electrical forces (elastic stress); but then they are of much smaller range of action as their intensity depends on the inverse fourth power of the distance, so that the work done by them during the formation of the molecule will probably be very small compared with the work done by the electric forces.

[70. (Added June 14.) The charged atoms will tend to aggregate into molecules, and when this combination is thoroughly complete, the rotational strain of each molecule will be self-contained, in the sense that the lines of twist proceeding from one atom will end on some other atom of the same molecule. If this is not the case, the chemical combination will be incomplete, and there will still be unsatisfied bonds of electrical attraction between the different molecules. A molecule of the complete and stable type will thus be electrically neutral; and if any cause pulls it asunder into two ions, these ions will possess equal and opposite electric charges.

In the theory as hitherto considered, electric discharge has been represented as produced by disruption of the elastic quality of æther along the path of the discharge; and this is perhaps the most unnatural feature of the present scheme. If, however, we examine the point, it will be seen that the phenomena of electric flow need involve only convection of the atomic charges without any discharge across the æther, with the single exception of electrolysis. An attempt may be made (as in 'Proceedings,' p. 454) to account for the uniformity of the atomic charges thus gained or lost, from the point of view of the establishment of a path of disruptive discharge from one atom to another. But it seems preferable to adopt a more fundamental view.

The most remarkable fact about the distribution of matter throughout the universe is that, though it is aggregated in sensible amounts only in excessively widely separated spots, yet wherever it occurs, it is most probably always made up of the same limited number of elements. It would seem that we are almost driven to explain this by supposing the atoms of all the chemical elements to be built up of combinations of a single type of primordial atom, which itself may represent or be evolved from some homogeneous structural property of the æther.* It is, again, difficult to imagine how the chemical elements should be invariably connected, through all their combinations, with the same constant of gravitation, unless they have somehow a common underlying origin, and are not merely independent self-subsisting systems. We may assume that it is these ultimate atoms, or let us say monads, that form the simple singular points in the æther; and the chemical atoms will be points of higher singularity formed by combinations of them. These monads must be taken to be all quantitatively alike, except that some have positive and others negative electrifications, the one set being, in their dynamical features, simply perversions or optical images of the other set. On such a view, electric transfer from ion to ion would arise from interchange of monads by convection without any breaking down of the continuity of the æther.

* *Cf.* THOMAS GRAHAM'S "Chemical and Physical Researches," Introduction, and p. 299.

But a difficulty now presents itself as to why the molecule say of hydrochloric acid is always H + Cl −, and not sometimes H − Cl +. This difficulty would however seem to equally beset any dynamical theory whatever of chemical combination which makes the difference between a positive and a negative atomic change representable wholly by a difference of algebraic sign.]

The Connexion between Æther and Moving Matter.

71. A mode of representation of the kind developed in this paper must be expected to be in accord with what is known on the subject of the connexion between æther and matter, both from the phenomena of the astronomical aberration of light, and from recent experimental researches* on the motion of the æther relative to the Earth, and relative to transparent moving bodies.

Let us consider a wave of light propagated through the free æther with its own specific velocity, and let it be simultaneously carried onward by a motion in bulk of the æther which is its seat. That motion will produce two effects on a wave; the component along the wave-normal of the velocity of the æther will be added on to the specific velocity of the wave; while the wave-front will be turned round owing to the rotational motion of the medium. The second of these effects will result in the ray being turned out of its natural path; in order that the motion of the medium may not affect the natural path of the ray, it must therefore be of irrotational character. This will be the case as regards all motions of the free æther so long as we consider it to be hydrodynamically a frictionless fluid; and the phenomenon of astronomical aberration is, after Sir GEORGE STOKES, explained, so far as it may depend on motion of the external æther.

72. The motion of the Earth through space may however be imagined as the transference of a vortex-aggregate through the quiescent æther surrounding it and permeating it; the velocity of translation of the æther will then be null, and consequently in the comparatively free æther of the atmosphere the velocity of the light will be unaffected, to the first order of approximation. But what should happen in transparent material media it is apparently not easy to infer. On the present view of Optics, the density of the æther is constant throughout space, the mere presence of mobile electrified vortices in it not affecting the density though the effective elasticity is thereby altered. The nature of the further slight alteration of this elasticity produced by a motion of the matter as a whole, there appears to be no easy means of directly determining [see § 124]; but the experiments may be taken as verifying FRESNEL's hypothesis that its effect is to add on to the velocity of propagation of the light the fraction $1 - \mu^{-2}$ of the velocity of the matter through which it is moving, where μ represents the index of refraction.

* A. A. MICHELSON and E. W. MORLEY, 'American Journal of Science,' 1881 and 1886, also 'Phil. Mag.,' Dec., 1887; O. J. LODGE, 'Phil. Trans.,' A, 1893.

This formula of FRESNEL,* for the change of the velocity of propagation in a moving ponderable medium, was specially constructed so as to insure that the laws of reflexion and refraction of the *rays* shall be the same as if the media were at rest, a circumstance which must be intimately connected with the dynamical reason for its validity. The laws of reflexion and refraction of rays can be deduced from the theory of exchanges of radiation, on the single hypothesis that a condition of equilibrium of exchanges is possible in an enclosure containing transparent non-radiating bodies. One interpretation of FRESNEL'S principle is therefore that the exchange of radiation between the walls of an enclosure containing transparent bodies is not affected by any motion imparted to these bodies, a conclusion which may be connected with the law of entropy.

73. On the present theory, magnetic force or rather magnetic induction consists in a permeation or flow of the primordial medium through the vortex-aggregate which constitutes the matter; apparently it has not been tried (see however § 81) whether light-waves are carried on by this motion of the medium and their effective velocity is thereby altered, as we would be led to expect. It has been shown, however, by WILBERFORCE† that the velocity of light is not sensibly altered by motion along a field of electric displacement, so far negativing any theory that would connect electric displacement with considerable bodily velocity of the æther; and it has also been verified, by Lord RAYLEIGH, that the transfer of an electric current across an electrolyte does not affect the velocity of light in it.

As motion of the æther represents magnetic force, the fact that the magnetic permeability is almost the same in all sensibly non-magnetic bodies as in a vacuum must be taken to indicate that the æther flows with practically its full velocity in all such media, so that there is very little obstruction interposed by the matter; it follows that, in the motion of a body through the æther, the outside æther remains at rest instead of flowing round its sides. The æther we thus assume to be at rest in any region, except it be a field of magnetic force, even though masses are moving through the region; so that the coefficient of FRESNEL, which is null for free æther and very small for but slightly ponderable media, would represent simply a change of velocity due to slight unilateral change of effective elasticity somehow produced by the motion through the quiescent medium of the vortices constituting the matter.

74. The notion of illustrating magnetic induction by the permeation of a fluid through a porous medium containing obstacles to its motion has been shown by Lord KELVIN‡ to lead to a complete formal representation of the facts of diamagnetism;

* A. FRESNEL, letter to ARAGO, Annales de Chimie,' ix., 1818.

† L. R. WILBERFORCE, 'Trans. Cambridge Phil. Soc.,' vol. 14, 1887, p. 170.

‡ Lord KELVIN (Sir W. THOMSON), "Hydrokinetic Analogy for the magnetic influence of an ideal extreme diamagnetic," 'Proc. R.S. Edin.,' 1870, 'Papers on Electrostatics and Magnetism,' pp. 572–83: "General hydrokinetic analogy for Induced Magnetism," 'Papers on Electrostatics and Magnetism,' 1872, pp. 584–92.

and such an idea of very slightly obstructed flow might possibly be made to serve as a substitute for WEBER's theory, if we are unable to retain it. [See § 114.]

75. The motion of a material body through the æther must, in any case, either carry the æther with it, or else set up a backward drift of the æther through its substance, so that the vortex cores (which might be vacuous and therefore merely forms of motion) would be carried on, while the body of the æther remained at rest. On the first view, the motion of the body must produce a field of irrotational flow in the surrounding æther, in other words a magnetic field. Whether this would be powerful enough to be directly detected depends on the order of magnitude of the æthereal velocities which represent ordinary magnetic forces, and thus ultimately on the value of the density of the æthereal medium. But if the density were small, the square of the velocity would be large in proportion, and the influence of magnetization on the velocity of light should be the greater; so that on this account also the first of the above views must, on the present theory, be rejected. We should however expect an actual magnetic field like the Earth's to affect very slightly both the velocity of propagation and the law of reflexion.

76. The second view is, as we have stated, the one formulated by FRESNEL, and it would be strongly confirmed if the velocity of light-waves were quite unaffected by passing near a moving body, so shaped that it would on the other hypothesis cause a current in the perfectly fluid æther; but it is sometimes held (see however § 80) to be against the evidence of the null result of MICHELSON's experiments on the effect of the Earth's motion on the velocity of transmission of light through air.

There is also the fact noticed by LORENTZ that an irrotational disturbance of the surrounding æther, caused by the motion of an impermeable body through it, would necessarily involve slip along the surface, which could not exist in our fluid medium; this would at first sight compel us to recognize that the surrounding æther, instead of flowing round a moving body, must be taken to flow through it, or rather into it, at any rate to such an extent as will be necessary in order to make the remaining motion outside it irrotational, without discontinuity at the surface.

It has been shown however by W. M. HICKS that a solitary *hollow* vortex in an ordinary liquid carries along with it a disc-shaped mass of fluid and not a ring-shaped mass, unless its section is very minute; thus it is possible that the vortex-aggregate constituting a moving solid may completely shed off the surrounding fluid without allowing any permeation through its substance, and without any such discontinuity at the surface as would be produced by the motion of an ordinary solid through liquid. How far the electric charges on the vortex atoms, or their combination into molecules, would negative such a hypothesis seems a difficult inquiry. But however that may be, a *consensus* of various grounds seems to require the æther to be stationary on the present theory. Thus if the motion of solids moved the surrounding æther, two moving solids would act on each other with a hydrodynamic forcive, which would be of large amount if we are compelled to assume a considerable density for the æther.

Again, such a view would disturb the explanation, as above, of the fact that the forcive on a charged conductor in an electric field is a surface-traction equal at each point of the surface to the energy in the medium per unit volume. There is in any case nothing contradictory in the hypothesis of a stationary æther; if the fluid is not allowed to stream through the circuits of the atoms, we have only to make the ordinary supposition that the molecules are at distances from each other considerable compared with their linear dimensions, and it can stream past between them.

77. Let us test a simple case of motion of a body through the æther, with respect to the theory of radiation. Consider a horizontal slab of transparent non-radiating material, down through which light passes in a vertical direction; the equilibrium of exchanges of radiation would be vitiated if the amount of light transmitted by the slab when in motion downwards with velocity v were different from the amount transmitted when it is at rest. Let V be the velocity of the light outside the slab, and $\mathrm{V}/\mu + v - v'$ the velocity in the moving slab. For an incident beam, of amplitude of vibration which we may take as unity, let r be the amplitude of the reflected beam, and R of the transmitted beam. The conditions governing the reflexion are continuity of displacement at the surface, and continuity of energy, estimated in MacCullagh's manner as proportional to the square of the amplitude; thus the conditions at the first incidence are

$$1 + r = \mathrm{R}$$
$$\mathrm{V} - v - (\mathrm{V} + v)\, r^2 = (\mathrm{V}/\mu - v')\, \mathrm{R}^2.$$

On neglecting squares of v/V and v'/V, these equations lead to

$$\mathrm{R} = \frac{2\mu}{\mu + 1}\left\{1 - \frac{v}{\mathrm{V}}\left(\frac{\mu}{\mu + 1} + \frac{\mu - 1}{2\mu}\right) + \frac{v'}{\mathrm{V}}\frac{\mu}{\mu + 1}\right\}.$$

The ratio R′, in which the amplitude is changed by transmission at the lower surface of the slab, is derived from the above by replacing V by V/μ, and μ by $1/\mu$, and interchanging v and v'; thus

$$\mathrm{R}' = \frac{2}{\mu + 1}\left\{1 - \frac{\mu v'}{\mathrm{V}}\left(\frac{1}{\mu + 1} - \frac{\mu - 1}{2}\right) + \frac{\mu v}{\mathrm{V}}\frac{1}{\mu + 1}\right\}.$$

Hence

$$\mathrm{RR}' = \frac{4\mu}{(\mu + 1)^2}\left\{1 - \frac{v}{\mathrm{V}}\frac{\mu - 1}{2\mu} + \frac{\mu v'}{\mathrm{V}}\frac{\mu - 1}{2}\right\}.$$

That the amount of the light transmitted should not be altered by the motion of the slab requires that $v' = v/\mu^2$, which is Fresnel's law; it has been assumed in the analysis that the light is propagated down to the slab as if the æther were at rest, in accordance with Fresnel's hypothesis. It will be observed that the amplitudes of the refracted and reflected light, at either surface separately, are disturbed by the movement of the slab, though there is no loss of energy; thus, on direct refraction into a slab moving away from the light with velocity v,

$$R = \frac{2\mu}{\mu + 1}\left\{1 - \tfrac{3}{2}\frac{v}{V}\frac{\mu - 1}{\mu}\right\}, \qquad r = \frac{\mu - 1}{\mu + 1}\left(1 + \tfrac{3}{2}\frac{v}{V}\right).$$

If therefore FRESNEL'S law is not fulfilled, it would apparently be possible to concentrate the radiation from the walls of an enclosure of uniform temperature by a self-acting arrangement of moving screens and transparent bodies inside the enclosure; and this would be in contradiction to the Second Law of Thermodyamics.*

78. The whole theory of rays is derived from the existence of the Hamiltonian characteristic function U, the path of a ray from one point to another in an isotropic medium being the course which makes δU or $\delta\int\mu ds$ null, where μ is a function of position which is equal to the reciprocal of the effective velocity of the light. The general law of illumination may be shown to follow from this, that if two elements of surface A and B are radiating to each other across any transparent media, the amount of the radiation from A that is received by B is equal to the amount of radiation from B that is received by A; with the proviso, when different media are just in front of A and B, that the radiation of a body is *cæteris paribus* to be taken as proportional to the square of the refractive index of the medium into which it radiates. Now if that part $\mathfrak{v}$ of the velocity of the light, which is produced by motion through the medium of the bodies contained in it, make an angle θ with the element of path ds, this equation will assume, after H. A. LORENTZ and O. J. LODGE,† the form

$$\delta\int(V + \mathfrak{v}\cos\theta)^{-1}\,ds = 0,$$

which is to a first approximation

$$\delta\int V^{-1}\,ds + \delta\int V^{-2}(u\,dx + v\,dy + w\,dz) = 0,$$

where V is the ordinary velocity of the light, and (u, v, w) are the components of $\mathfrak{v}$. In order that the paths of the rays in a homogeneous isotropic moving medium may remain the same as when the medium is at rest, the additional terms in the characteristic function must depend only on the limits of the integral, and therefore $u\,dx + v\,dy + w\,dz$ must be an exact differential; that is, the part thus added to the velocity of the light must be of irrotational character. If this part of the velocity were rotational, the law of illumination would not hold, as the type of the characteristic equation of the rays would thereby be changed. Thus the equilibrium of exchanges of radiation which would subsist in an enclosure with the free æther in it

* *Cf.* CLAUSIUS, "On the Concentration of Rays of Light and Heat, and on the Limits of its Action," 'Papers on the Mechanical Theory of Heat,' translated by W. R. BROWNE, pp. 295–331.

† O. J. LODGE, "Aberration Problems," 'Phil. Trans.,' A, 1893, pp. 748–753.

at rest, would be violated were the æther put into a state of rotational motion. Now any modification of the laws of emission and absorption would be conditioned only by the motion of the æther close to the radiating surface; and the motion at the surface by no means determines the motion throughout the enclosure, unless it is confined to be irrotational. Hence the theory of exchanges seems to require that any bodily motion that can be set up in the free æther must be of the irrotational kind.

79. This modified characteristic equation of the rays also shows that in a heterogeneous isotropic medium containing moving bodies, the paths of the rays will be unaltered to a first approximation provided $\mu^2 (u\, dx + v\, dy + w\, dz)$ is everywhere continuous and an exact differential; and this condition virtually implies (LODGE, *loc. cit.*) FRESNEL'S hypothesis. The interchange of radiation now depends partly on the reflexion and refraction at the different interfaces in the medium, as in the simple case calculated above; but we may take advantage of a device which has been employed in other connexions by Lord RAYLEIGH, and suppose the transitions to be gradual, that is to be each spread over a few wave-lengths; the reflexions will then be insensible, and the rays will thus be propagated with undiminished energy. We thus attain a general demonstration that the theory of exchanges of radiation demands FRESNEL'S law of connexion between the velocity of the matter through the field of stationary æther and the alteration in the velocity of the light that is produced by it; while it also requires that any motion of the æther itself, such as occurs in a field of magnetic force, must be of irrotational type.

80. This theory has been developed up to and including the first order of small quantities; it seems plain therefore that the experiments of MICHELSON on the effect produced by the motion of the earth on transmission through air are not in contradiction with it, for these experiments relate to terms of the second order of small quantities. To explain the remarkable, because precisely negative, result arrived at by MICHELSON would require the elaboration of a theory including the second order of small quantities. For example, when light is reflected, as in those experiments, at the surface of a body which is moving towards it through the stationary æther, the wave-length of the reflected light is diminished so as just to make up, to the first order of approximation, for the acceleration of phase caused by the reflector moving up to meet it. The mechanism involved in this alteration of wave-length is not known, nor what is going on at the surface of the advancing reflector; and it seems to be a very uncertain step to assume that when terms of the second order are included, this effect on the wave-length is not subject to correction. As the circumstances of the reflexion are thus not known with sufficient exactness, it is necessary to fall back on general principles. Now Professor LODGE has emphasized the fact that, when a beam of light traverses a complete circuit in a medium containing moving bodies but devoid of magnetic intensity, the change of phase produced by their motion is null to the first order of small quantities. If it were exactly null, or null to the second order, the result of MICHELSON would follow; and it would seem also that MICHELSON'S result

favours somewhat the exact validity of this principle. The exactness of this circuital principle seems to be required also by the argument (§ 79) from the equilibrium of exchange in an enclosure. For if when a system of rays pass from a point to its image-point their relative differences of phases were not the same to a small fraction of a wave-length whether the bodies are at rest or in motion, it would follow that the distribution of the energy in the diffraction pattern which forms the physical image would depend on the movement of the bodies. Thus concentration of the radiation might be produced by movements of the transparent bodies, which are subject to control.

The present discussion supposes the motion of the transparent bodies to be practically uniform; the condition $\mu^2 (u\,dx + v\,dy + w\,dz)$ an exact differential would be violated inside a transparent body in rapid rotation, but then (§ 98) the formula of FRESNEL would require correction owing to the space-rate of variation of the velocity of the material medium.

Experiments by Professor OLIVER LODGE.

81. Since this account of the theory was written, Professor LODGE has kindly made some experiments on the effect produced by a magnetic field on the velocity of light, which considerably affect its aspect. By surrounding the path of the beam of light in his interference apparatus* by coils carrying currents, he realized what was equivalent to a circuit of 50 feet of air magnetized to $\pm$ 1400 c.g.s.; and he would have been able to detect a shift in the fringes, between beams of light traversing this circuit in opposite directions, of $\frac{1}{50}$ of a band, or say with absolute certainty $\frac{1}{20}$ of a band, either way. Four coils were employed, each 18 inches long and with 7000 turns of wire; and they were excited by a current of 28 ampères at 230 volts, involving nearly 9 horse-power. The result was wholly negative; and in consequence the velocity of light cannot be altered by as much as 2 millimetres per second for each c.g.s. unit of magnetic intensity. The cyclic æthereal flow in a magnetic field must therefore be very slow; but the radiation traversing it is of course very fast.

To bring this result into line with the present theory we are compelled to assume that the density of the æther is at least of the same order of magnitude as the densities of solid and liquid matter, at any rate if we must adhere to the view that the motion of the æther carries the light with it. This hypothesis is of a somewhat startling character; the density under consideration belongs however to an intangible medium and is not apparently amenable in any way to direct perception; it is on a different plane altogether from the density of ordinary matter, and is in fact most properly considered simply as a coefficient of inertia in the analytical expression for the energy.

* O. J. LODGE, "Aberration Problems," 'Phil. Trans.,' A, 1893. [There are also some earlier experiments by CORNU.]

82. The maximum electric force which air can sustain at ordinary temperatures and pressures is about 130 c.g.s.; and on POUILLET'S data the maximum electric force involved in the solar radiation, near the Sun's surface, is about 30 c.g.s., a value which would be much increased on more recent estimates. One result of taking a high value for the æthereal density would be that in the most intense existing field of radiation we are certain of being still far from the limits of perfect elasticity of the comparatively free æther.

The kinetic energy in the free æther is the square of the magnetic intensity divided by 8π; and this must be $\frac{1}{2}\rho v^2$, where ρ is its density and v its velocity. Now from Professor LODGE'S result the velocity corresponding to the c.g.s. unit of magnetic force is less than ·2 centimetre per second; hence the inertia of the æther must exceed twice that of water. The elasticity must of course be taken large in proportion to the density, in order to preserve the proper velocity of radiation. In view of the very great intensity of the chemical and electrical forces acting between the atoms in the molecule, values even much greater than these would not appear excessive. But on the other hand such a value of the density requires us to make the æther absolutely stationary except in a magnetic field, in order to avoid hydrodynamical forcives between moving bodies. The residual forcive between bodies at rest in a field of æthereal motion, due to very slight defect of permeability, has already been shown, after Lord KELVIN'S illustration, to simulate diamagnetism; and the fact that there exist no powerfully diamagnetic substances is so far a confirmation of the present hypothesis. The view that the magnetic field of a current involves only slight circulation of the fluid æther is also in keeping with the account which has been given (§ 46) of the genesis of such a field.

On Magneto-Optic Rotation.

83. The rotation of the plane of polarization of light in a uniform magnetic field depends on the interaction of the uniform velocity of the æther, which constitutes that field, with the vibrational velocity which belongs to the light-disturbance. The uniform flow in the medium we may consider to be connected with a partial orientation of the vortex-molecules; the chemical or hydrodynamic vibrations, in other words vibrations of the magnetism, can now be propagated in waves, and it is natural to expect that the propagation of the light will be somewhat affected by this regularity. Now for the light-waves the motion that is elastically effective is the rotation $d/dt\,(f, g, h)$; and the varying part of the velocity of an element of volume containing the rotational motion of the magnetic vortices which is to some extent interlinked with the motion of the light-waves, is proportional to

$$\frac{d}{d\theta}(\xi, \eta, \zeta), \quad \text{where } \frac{d}{d\theta} = \alpha_0\frac{d}{dx} + \beta_0\frac{d}{dy} + \gamma_0\frac{d}{dz},$$

$(\alpha_0, \beta_0, \gamma_0)$ being the imposed magnetic field. This variation is caused by alteration of the vibrational velocity of a particle owing to its change of position as it is carried along in the magnetic field, analogously to the origin of the corresponding term in the acceleration of an element of the medium, in the equations of hydrodynamics. There may exist a term in the energy, resulting from this interaction, of the form

$$C'\left(\frac{d\xi}{d\theta}\frac{df}{dt}+\frac{d\eta}{d\theta}\frac{dg}{dt}+\frac{d\zeta}{d\theta}\frac{dh}{dt}\right);$$

and I have elsewhere* tried to show that, on a *consensus* of various reasons, this term, originally given by MAXWELL, must be taken as the correct representation of the actual magneto-optic effect. The term is extremely small, and is distinct from the direct effect of the motion of the æther (§ 79), which is irrotational; it leads to an acceleration of one kind of circularly polarized light, and a retardation of the other kind, which are of equal amounts.

It was this phenomenon of magneto-optic rotation that gave the clue to MAXWELL's theory of the electric field. As has recently been remarked by various authors,† the deduction from it, that magnetic force must be a rotation of the luminiferous medium, is too narrow an interpretation of the facts; the identification of magnetic force with rotation has however hitherto been retained as an essential part of most theories of the æther.

84. It is to be observed that the magneto-optic terms in the energy of the medium do not depend essentially on any averaging of the effect of molecular discreteness, in the same way as dispersive terms or structural rotatory terms. The problem of reflexion is, in the magnetic field, perfectly definite; and the boundary conditions at the interface can all be satisfied, provided we recognize a play of electromotive pressure at the interface, which assists in making the stress continuous,‡ and which

* "On Theories of Magnetic Action on Light . . . ." 'Report of the British Association,' 1893. Any other energy-term containing the same differential operators would however equally satisfy these conditions; for example $d\xi/d\theta\ df/dt$ might be replaced by $d\xi/dt\ df/d\theta$ or even by $f d^2\xi/d\theta dt$, so far as the equations of bodily propagations are concerned. Such forms would be discriminated by the theory of reflexion. As the term in the energy is related to the motion of the medium, in must involve $d/d\theta$; and this circumstance, combined either with the character of the optical rotation produced, or with the present hypothesis which requires that the term involves (f, g, h), suffices to limit it to one of these types; *cf. loc. cit.*, § 3.

† *E.g.*, H. LAMB, "On Reciprocal Theorems in Dynamics," 'Proc. Lond. Math. Soc.,' vol. 19, 1888, where the remark is actually made that a distribution of vortices with their axes along the direction of the field might account for the magnetic rotation of the light.

‡ J. LARMOR, 'Report of the British Association,' 1893; G. F. FITZGERALD, 'Phil. Trans.,' 1880. Professor FITZGERALD informs me that he has for some time doubted the view that the magnetic force can be solely a rotation in the medium, on the ground that the magnetic tubes of a current-system are circuital and have no open ends, making it difficult to imagine how alteration of the rotation inside

is required on account of this interaction of the linear motion of the medium with the rotational motion of the waves. The chief obstacle in the way of a complete account of the magnetic phenomena of reflexion appears to be the uncertainty with respect to the proper mathematical representation of ordinary metallic reflexion.

On Radiation.

85. In accordance with this theory, radiation would consist of rotational waves sent out into the æther from the vibrations somehow set up in the atomic charges. It has been observed (§ 65) that the characters and periods of these electric vibrations, and of the radiations they emit, depend only on the relative positions and motions of the vortex-atoms in the molecule, and are quite unaffected, except indirectly, by irrotational motion (magnetic intensity) in the æther which they traverse. The mode of propagation of electric vibrations in free æther cannot be interfered with by the bodily motion of the medium, however intense, except in so far as the motion of the medium carries the electrical waves along with it; a result justifying the DOPPLER principle which is applied to the spectroscopic determination of stellar motions. It also follows that radiation will not be set up by motions of the surrounding free æther, except in so far as the molecules are dissociated or their component atoms violently displaced with respect to each other. To allow the radiation to go on, such displacement must result on the whole in the performance of work against electric attractions, at the expense of the heat energy and chemical energy of the system, which must thus be transformed into electrical energy before it is radiated away. The radiation of an incandescent solid or liquid body is maintained by the transfer of its motion of agitation into electrical energy in the molecules, and thence into radiation. This action goes on until a balance is attained, so that as much incident radiation is absorbed by an element of volume as it gives out in turn; when this state is established throughout the field of radiation the bodies must be at the same temperature.

Conversely, the absorption of incident radiation by a body results finally in a diffusion of its energy into irregular material motions or heat, directed motion always implying magnetic force.

86. There appear to be experimental grounds for the view that a gas cannot be made to radiate [at any rate with the definite periods peculiar to it] by merely heating it to a high temperature, so that radiation in a gas must involve chemical action or, what is the same thing, electric discharge. This would be in agreement with the conclusion that motion of a molecule through the æther, however the latter is disturbed, will not appreciably set up electric vibrations, unless it comes well within

them could be produced; also that a flow along these tubes need not produce any disturbance in the other properties of the electric field [; also that the magnetic rotation being a purely material phenomenon, whose direction is not subject to any definite law, it must be of a secondary character].

range of the chemical forces of another molecule; and it implies that the encounters of the molecules that are contemplated in the kinetic theory of gases are not of so intimate a character* as the encounters in a solid or liquid mass; in the latter case there is perhaps not sufficient space for free repulsion, and the molecules become so to speak jammed together. In the theory of exchanges of radiation, a gas would thus act simply as a medium for the transfer of radiations from one surface to another without itself adding to or subtracting from them.

It follows from the second law of Thermodynamics that the heat-equivalent of the radiation of a given substance rises with the temperature, and this may be extended to each separate period in the radiation; this is however a theorem of averages not directly applicable to single molecules.

It seems a noteworthy consequence of the foregoing that the kinetic theory of gases is valid without taking any account of radiation. Without some tangible mode of presentation such as the mechanism of radiation here put forward, there would be a strong temptation to assume that the interchange of energy in that theory must take place not only between the different free types of vibration of the molecule (*i.e.* hydrodynamical vibrations of the vortices), but that also there is even in the steady state continual interchange with the æther. According to the present views such interchange would involve dissociation in the molecules; and there exist in fact observations relative to the action of ultra-violet radiations in producing discharge of electricity across a gas and consequent luminosity in it, a phenomenon which very probably depends on dissociation. Whether the ideas here indicated turn out to be tenable or not, they at all events may serve to somewhat widen our range of conceptions.

87. The result that the electric vibrations of a molecule depend on its configuration and the relative motion of its parts, not directly on its motion of translation through the æther, seems also to be of importance in connexion with the fundamental fact that the periods of the radiations corresponding to the spectral lines of any substance are precisely the same whatever be its temperature. The lines may broaden out owing to frequency of collisions due to increase of density or rise of temperature of the substance, but their mean period does not change. If we consider a system of ordinary hydrodynamical isolated vortex-atoms, a rise of temperature is represented by increase of the energy, and that involves an expansion of each ring and a diminution of its velocity of translation; such an expansion of the ring would in turn alter the periods of its electric vibrations. The question arises, how far the action of the atomic charge will modify or get rid of these two fundamental objections to a vortex-atom theory of gases. Independently of this, it seems quite reasonable to hold that in the case of atoms paired together into molecules by their electrical and chemical forces, the size and configuration of the rings will be

* [The difficulty of chemical combination of dry gases confirms this conclusion; as also for example the fact that molecular impacts do not explode a mixture like hydrogen and chlorine.]

determined solely by these forces, which are far more intense than any forces due to mere translation through the medium; and then, when radiation occurs as the result of some violent disturbance, or of dissociation of the molecule, it will have subsided before any sensible change of size due to slowly-acting hydrodynamical causes could have occurred. As was pointed out by MAXWELL, the definiteness of the spectral lines requires that at least some hundreds of vibrations of a molecule must be thrown off before they are sensibly damped; and on this view there is ample margin for such a number.

On these ideas the velocity of translation of a molecule in a gas would not be connected with the natural hydrodynamical velocity of a simple vortex-atom, but would rather be determined by the circumstances of collisions, as in the ordinary kinetic theory of gases. The configuration of a molecule, which determines its electric periods, would also be independent of the movements of translation and rotation, which constitute heat and are the concern of the kinetic theory of gases.

Introduction of the Dissipation Function.

88. The original structure of Analytical Dynamics, as completed by the work of LAGRANGE, POISSON, HAMILTON, and JACOBI, was unable to take a general view of frictional forces; one of the most important extensions which it has since received, from a general physical standpoint, has been the introduction of the Dissipation Function by Lord RAYLEIGH. He has shown* that in all cases in which the frictional stress between any two particles of the medium is proportional to their relative velocity, when the motion is restricted to be such as maintains geometrical similarity in the system—*i.e.* in all cases in which, $(x_1y_1z_1)$ and $(x_2y_2z_2)$ being the two particles, the components of the frictional stress between them are

$$\mu_x(\dot{x}_1 - \dot{x}_2),\ \mu_y(\dot{y}_1 - \dot{y}_2),\ \mu_z(\dot{z}_1 - \dot{z}_2),$$

where μ_x, μ_y, μ_z are any functions of the co-ordinates—the virtual work of the frictional forces in any geometrically possible displacement may be derived from the variation of a single function $\mathfrak{F}$. The virtual work for the two particles just specified is in fact

$$\mu_x(\dot{x}_1 - \dot{x}_2)\,\delta(x_1 - x_2) + \mu_y(\dot{y}_1 - \dot{y}_2)\,\delta(y_1 - y_2) + \mu_z(\dot{z}_1 - \dot{z}_2)\,\delta(z_1 - z_2);$$

and for the whole system it will be found by addition of such expressions as this. Now if we form the variation, with respect to the velocities alone, of the expression

$$\mathfrak{F} = \tfrac{1}{2}\Sigma\{\mu_x(\dot{x}_1 - \dot{x}_2)^2 + \mu_y(\dot{y}_1 - \dot{y}_2)^2 + \mu_z(\dot{z}_1 - \dot{z}_2)^2\},$$

* 'Proc. Lond. Math. Soc.,' 1873; 'Theory of Sound,' I., 1877, § 81. [An analytical function of this kind occurs however incidentally in the 'Mécanique Analytique,' Section viii., § 2.]

and in it replace the variations of the velocities by the variations of the corresponding co-ordinates, we shall have just obtained this virtual work. This function $\mathfrak{F}$ may now be expressed in terms of any generalized co-ordinates that may be most convenient to represent the configuration of the system for the purpose in hand, and the virtual work of the viscous forces for any virtual displacement specified by variations of these co-ordinates will still be derived by this rule. "But although in an important class of cases the effects of viscosity are represented by the function $\mathfrak{F}$, the question remains open whether such a method of representation is applicable in all cases. I think it probable that it is so; but it is evident that we cannot expect to prove any general property of viscous forces in the absence of a strict definition which will enable us to determine with certainty what forces are viscous and what are not."*

89. The general variational equation of motion of the viscous system will in fact be

$$\int (\delta \mathrm{T} - \delta \mathrm{W} - \delta' \mathfrak{F})\, dt = 0,$$

wherein δ represents variation with respect to the co-ordinates and velocities of the system, while δ' represents variations with respect to the velocities only, the differentials of the velocities being in the result of the latter variation replaced by differentials of the corresponding co-ordinates.†

90. The importance of this analysis in respect to problems in the theory of radiation is fundamental. If a radiation maintains its period of vibration unaltered in passing through a viscous medium, it follows necessarily that the viscous forces of the medium are of the type above specified. If the elastic forces were not linear functions of the displacements and the viscous forces linear functions of the velocities, the period of a vibration would be a function of its amplitude; and thus a strong beam of homogeneous light, after passing through a film of metal or other absorbing medium, would come out as a mixture of lights of different colours. So long as we leave on one side the phenomena of fluorescence, we can therefore assert that the laws of absorption must be such as are derivable from a single dissipation function, of the second degree in the velocities, which is appropriate to the medium.

* Lord RAYLEIGH, 'Theory of Sound,' § 81. [An extension of the range of the function is easy after the method of LAGRANGE, *loc. cit.* It is worthy of notice that we can also formulate a function of mutual dissipation between two interacting media.]

† It may be observed that the use of this variational equation would form the most elegant method of deriving the ordinary equations of motion of material dissipative systems in which the value of $\mathfrak{F}$ is known. For example the equations of motion of a viscous fluid in cylindrical, polar, or any other type of general co-ordinates, may be derived at once from the expressions for the fundamental functions in these co-ordinates, without the necessity of recourse to the complicated transformations sometimes employed. *Cf.* "Applications of Generalized Space Co-ordinates to Potentials and Isotropic Elasticity," 'Trans. Camb. Phil. Soc.,' XIV., 1885.

Recapitulation of the Vibrational Qualities of the Æther.

91. On the present extension of MACCULLAGH's scheme, the properties of the æther in a ponderable medium, as regards those averaged undulations which constitute radiation, are to be derived from the following functions;
its kinetic energy

$$T = \tfrac{1}{2}\rho\int\left(\frac{d\xi^2}{dt^2} + \frac{d\eta^2}{dt^2} + \frac{d\zeta^2}{dt^2}\right)d\tau,$$

its potential energy

$$W = \tfrac{1}{2}\int(a^2f^2 + b^2g^2 + c^2h^2)\,d\tau, \text{ where } (f, g, h) = \text{curl }(\xi, \eta, \zeta),$$

its dissipation function, representing decay of the regularity of the motion,

$$\mathfrak{F} = \tfrac{1}{2}\int\left(a'^2\frac{df^2}{dt^2} + b'^2\frac{dg^2}{dt^2} + c'^2\frac{dh^2}{dt^2}\right)d\tau.$$

We may add as subsidiary terms the magneto-optic energy

$$T' = \int\left(\alpha^2\frac{d\xi}{d\theta}\frac{df}{dt} + \beta^2\frac{d\eta}{d\theta}\frac{dg}{dt} + \gamma^2\frac{d\zeta}{d\theta}\frac{dh}{dt}\right)d\tau,$$

where

$$\frac{d}{d\theta} = \alpha_0\frac{d}{dx} + \beta_0\frac{d}{dy} + \gamma_0\frac{d}{dz},$$

$(\alpha_0, \beta_0, \gamma_0)$ being the intensity of the imposed magnetic field;
and the optical rotational energy

$$W' = \int(\alpha'^2 f\nabla^2\xi + \beta'^2 g\nabla^2\eta + \gamma'^2 h\nabla^2\zeta)\,d\tau.$$

And there are also to be included the terms in W of higher orders, that produce regular (*i.e.* sensibly non-selective) dispersion of various kinds, of which the chief is

$$W_1 = \int\Phi\,\{(f, g, h), \nabla^2(f, g, h)\}\,d\tau,$$

where the symbol Φ in the integral denotes a lineo-linear function.

Throughout these equations, the elastic properties of the æther retain their purely rotational character; its internal elastic energy, its dissipation, and its connexions with other interlinked motions, depend on the rotation of its elements and not on their distortion or compression. A partial exception occurs in the magneto-optic terms, which represent interaction with a motion of partly irrotational character; and this exception is evidenced by the necessity which then arises of taking explicit account of incompressibility in order to avoid change from rotational to longitudinal undulation in a heterogeneous medium.

92. The question occurs, how far the form of these functions may be susceptible of alteration, so as thereby to amend those points in which the account given by the electric theory of light is at variance with observation, for example, in the problem of metallic reflexion. The form of the function $\mathfrak{F}$ is derived from the phenomena of electrical dissipation when the currents are steady or changing with comparative slowness; as in other cognate cases, it may be subject to modification when the rate of alternation is extremely rapid. But as the elastic quality of the medium is assumed to be determined by the components of its rotation, and not at all by distortion or compression, it seems natural to infer that the viscous resistance to change of the strain is determined in terms of the same quantities and therefore by a quadratic function of $d/dt\,(f, g, h)$. This argument, if granted, will carry with it the assertion of OHM's law of linear conduction in its general form, though probably with co-efficients depending on the period, for disturbances of all periods however small.

In the expressions for $\mathfrak{F}$ and W, as given above, the principal axes of the æolotropic conductivity are taken to coincide with the principal axes of the æolotropic electric displacement, a simplification which need not generally exist.

The fact that the electric dissipation-function does not involve the velocities of the material system shows that the forces derived from it are solely electromotive.

93. It seems clear that viscous terms alone could not possibly in any actual medium be so potent as to reduce the real part of the complex index of refraction suitable to metallic media to be a negative quantity. Such a state of matters arising from purely internal action involves instability; while on the contrary the general influence of viscosity is to improve rather than to diminish the dynamical stability of a system. This phenomenon, if indeed it is here properly described, must therefore be due to the support and control of some other vibrating system; an explanation which has been proposed is to adopt the views of YOUNG and SELLMEIER, and ascribe its origin to a near approach between the periods of hydrodynamical vibrations of the atoms in the molecule and the simultaneous rotational vibrations of the æther produced by the light waves. A theory like this is however usually held as part of the larger view which represents ordinary refraction as the result of synchronism of periods and consequent absorption in the invisible part of the spectrum; while, in the above, the main part of the refraction is ascribed to defect of elasticity due to mobile atomic charges. It seems natural therefore to look for some other explanation of the discrepancies between theory and observation in ordinary metallic reflexion; and the idea suggests itself that if the opacity near the surface were so great as to

cause sensible absorption in a very small fraction of a wave-length, the analytical formulæ might be entirely altered.

Sir GEORGE STOKES* has however supported the view that besides the effects due to simple absorption, metals probably also show reflexion phenomena involving change of phase, such as were originally discovered by AIRY for the diamond, and were afterwards found in other highly refractive substances. These effects, which were extended by JAMIN to ordinary media, have been eliminated by Lord RAYLEIGH for the case of water by cleansing of the surface, by which means the sharpness of the optical transition would be improved. The phenomena for the case of diamond were long ago classed by GREEN† as a result of gradual transition; and this might be expected to be more marked between hard substances whose optical properties are very different. On this view we may not be driven to try the hypothesis of extreme absorption in the interficial layer, which is unsatisfactory for the same reasons as apply to KIRCHHOFF's doctrine of extraneous forces; the quality above mentioned, for which Sir GEORGE STOKES proposes the name of the adamantine property, being sufficient.

Reflexion by Partially Opaque Media.

94. The ordinary formulæ for reflexion at the surface of an absorbing medium may now be derived from the analytical functions which express the averaged dynamical constitution of the æther for the case of its vibrations in ponderable bodies. If the general argument is correct, it is to be expected that these formulæ would be verified for reflexion at the surfaces of such media as are not too highly absorbent in comparison with the length of the wave. There are in fact two extreme cases; first the reflexion of electromagnetic waves of sensible length from metallic surfaces, where the reflexion is complete and there is no absorption at all; and second the reflexion of waves from perfectly transparent media, where the reflexion is incomplete because part of the energy goes on in the transmitted wave. The reflexion of light from metals may conceivably be more nearly akin to the first of these limiting cases than to the second; but for media more transparent than metals we should expect closer agreement with the ordinary theory, now to be developed.

95. The general variational equation of the motion is

$$\int (\delta T - \delta W - \delta' \mathfrak{F})\, d\tau = 0,$$

leading to

* In a note appended to a paper by Sir J. CONROY, "Some experiments on Metallic Reflexion," 'Roy. Soc. Proc.,' Feb., 1893.

† G. GREEN, "Supplement to a Memoir on the Reflexion and Refraction of Light," 'Trans. Camb. Phil. Soc.,' May, 1839.

$$\int dt \Big[\rho \int \left\{ \frac{d\xi}{dt}\frac{d\delta\xi}{dt} + \frac{d\eta}{dt}\frac{d\delta\eta}{dt} + \frac{d\zeta}{dt}\frac{d\delta\zeta}{dt} \right\} d\tau$$
$$- \int \left\{ a^2 f \left(\frac{d\delta\zeta}{dy} - \frac{d\delta\eta}{dz} \right) + b^2 g \left(\frac{d\delta\xi}{dz} - \frac{d\delta\zeta}{dx} \right) + c^2 h \left(\frac{d\delta\eta}{dx} - \frac{d\delta\xi}{dy} \right) \right\} d\tau$$
$$- \int \left\{ a'^2 \frac{df}{dt} \left(\frac{d\delta\zeta}{dy} - \frac{d\delta\eta}{dz} \right) + b'^2 \frac{dg}{dt} \left(\frac{d\delta\xi}{dz} - \frac{d\delta\zeta}{dx} \right) + c'^2 \frac{dh}{dt} \left(\frac{d\delta\eta}{dx} - \frac{d\delta\xi}{dy} \right) \right\} d\tau = 0.$$

On integrating by parts so as to eliminate the differential coefficients of the variation $\delta(\xi, \eta, \zeta)$, and neglecting the terms relating to the limits of the time, this gives the integral with respect to time of the expression

$$- \int \rho \left\{ \frac{d^2\xi}{dt^2}\delta\xi + \frac{d^2\eta}{dt^2}\delta\eta + \frac{d^2\zeta}{dt^2}\delta\zeta \right\} d\tau$$
$$- \int \left\{ \left(\frac{dc^2h}{dy} - \frac{db^2g}{dz} \right)\delta\xi + \left(\frac{da^2f}{dz} - \frac{dc^2h}{dx} \right)\delta\eta + \left(\frac{db^2g}{dx} - \frac{da^2f}{dy} \right)\delta\zeta \right\} d\tau$$
$$+ \int \{ (mc^2h - nb^2g)\,\delta\xi + (na^2f - lc^2h)\,\delta\eta + (lb^2g - ma^2f)\,\delta\zeta \}\, dS$$
$$- \int \left\{ \frac{d}{dt}\left(\frac{dc'^2h}{dy} - \frac{db'^2g}{dz} \right)\delta\xi + \frac{d}{dt}\left(\frac{da'^2f}{dz} - \frac{dc'^2h}{dx} \right)\delta\eta + \frac{d}{dt}\left(\frac{db'^2g}{dx} - \frac{da'^2f}{dy} \right)\delta\zeta \right\} d\tau$$
$$+ \int \left\{ \frac{d}{dt}(mc'^2h - nb'^2g)\,\delta\xi + \frac{d}{dt}(na'^2f - lc'^2h)\,\delta\eta + \frac{d}{dt}(lb'^2g - ma'^2f)\,\delta\zeta \right\} dS.$$

Hence the equations of propagation of vibrations are of the type

$$\rho\frac{d^2\xi}{dt^2} + \frac{dc^2\zeta}{dy} - \frac{db^2\eta}{dz} + \frac{d}{dt}\left(\frac{dc'^2\zeta}{dy} - \frac{db'^2\eta}{dz} \right) = 0,$$

that is

$$\rho\frac{d^2\xi}{dt^2} + \frac{dc_1^2\zeta}{dy} - \frac{db_1^2\eta}{dz} = 0,$$

where

$$(a_1^2, b_1^2, c_1^2) = \left(a^2 + a'^2\frac{d}{dt},\ b^2 + b'^2\frac{d}{dt},\ c^2 + c'^2\frac{d}{dt} \right).$$

Thus on the assumption that the principal axes of the dissipation function are the same as those of the optical elasticity, the equations of propagation in absorptive crystalline media differ from those of transparent media only by the principal indices assuming complex values.

96. To determine how the absorption affects the interfacial conditions on which the solution of the problem of reflexion depends, let us transform the axes of co-ordinates so that the interface becomes the plane of yz, and $(l, m, n) = (1, 0, 0)$. The potential energy function and the dissipation function will now be quadratic functions of the rotation and its velocity respectively, U and U′ say, as in § 14; and we can now

incidentally extend our view to the case in which these functions have not the same principal axes. The variational equation of motion is represented by the vanishing of the time-integral of the expression

$$
\begin{aligned}
&-\int \rho \left\{ \frac{d^2\xi}{dt^2}\delta\xi + \frac{d^2\eta}{dt^2}\delta\eta + \frac{d^2\zeta}{dt^2}\delta\zeta \right\} d\tau \\
&-\int \left\{ \left(\frac{d}{dy}\frac{dU}{dh} - \frac{d}{dz}\frac{dU}{dg} \right)\delta\xi + \left(\frac{d}{dz}\frac{dU}{df} - \frac{d}{dx}\frac{dU}{dh} \right)\delta\eta + \left(\frac{d}{dx}\frac{dU}{dg} - \frac{d}{dy}\frac{dU}{df} \right)\delta\zeta \right\} d\tau \\
&+\int \left\{ \left(m\frac{dU}{dh} - n\frac{dU}{dg} \right)\delta\xi + \left(n\frac{dU}{df} - l\frac{dU}{dh} \right)\delta\eta + \left(l\frac{dU}{dg} - m\frac{dU}{df} \right)\delta\zeta \right\} dS \\
&-\int \left\{ \frac{d}{dt}\left(\frac{d}{dy}\frac{dU'}{dh} - \frac{d}{dz}\frac{dU'}{dg} \right)\delta\xi + \frac{d}{dt}\left(\frac{d}{dz}\frac{dU'}{df} - \frac{d}{dx}\frac{dU'}{dh} \right)\delta\eta + \frac{d}{dt}\left(\frac{d}{dx}\frac{dU'}{dg} - \frac{d}{dy}\frac{dU'}{df} \right)\delta\zeta \right\} d\tau \\
&+\int \left\{ \frac{d}{dt}\left(m\frac{dU'}{dh} - n\frac{dU'}{dg} \right)\delta\xi + \frac{d}{dt}\left(n\frac{dU'}{df} - l\frac{dU'}{dh} \right)\delta\eta + \frac{d}{dt}\left(l\frac{dU'}{dg} - m\frac{dU'}{df} \right)\delta\zeta \right\} dS.
\end{aligned}
$$

The equations of propagation are therefore of type

$$\rho\frac{d^2\xi}{dt^2} + \frac{d}{dy}\frac{dU_1}{dh} - \frac{d}{dz}\frac{dU_1}{dg} = 0$$

where

$$U_1 = U + \frac{d}{dt}U'.$$

The boundary condition demands in general the continuity of the expression

$$\int \left\{ \left(m\frac{dU_1}{dh} - n\frac{dU_1}{dg} \right)\delta\xi + \left(n\frac{dU_1}{df} - l\frac{dU_1}{dh} \right)\delta\eta + \left(l\frac{dU_1}{dg} - m\frac{dU_1}{df} \right)\delta\zeta \right\} dS$$

in crossing the interface; for the special case of $(l, m, n) = (1, 0, 0)$, this involves continuity in η, ζ, dU_1/dg and dU_1/dh.

Thus, under the most general circumstances, the inclusion of opacity is made analytically by changing the potential energy-function from U to U_1, where U_1 is still a quadratic function, but with complex coefficients. If U and U′ have their principal axes in the same directions, a change of the principal indices of refraction of the medium from real to complex values suffices to deduce the circumstances both of propagation and of reflexion of light in partially opaque substances from the ones that obtain for perfectly transparent media. In all cases however the function U_1 has three principal axes of its own, whose position depends on the period of the light.

Dynamical Equations of the Primordial Medium.

97. The medium by means of which we have been attempting to co-ordinate inanimate phenomena is of uniform density, if there be excepted the small volumes

occupied by possibly vacuous cores of the vortex atoms. Its motion is partly hydrodynamical and irrotational, and is partly of rotational elastic quality. Its equations of motion are, for the averaged displacements which represent the general circumstances of crystalline quality,

$$\rho \frac{D^2\xi}{dt^2} + \frac{dc^2h}{dy} - \frac{db^2g}{dz} + \frac{dp}{dx} = 0$$

$$\rho \frac{D^2\eta}{dt^2} + \frac{da^2f}{dz} - \frac{dc^2h}{dx} + \frac{dp}{dy} = 0$$

$$\rho \frac{D^2\zeta}{dt^2} + \frac{db^2g}{dx} - \frac{da^2f}{dy} + \frac{dp}{dz} = 0,$$

where (ξ, η, ζ) is the linear displacement, (f, g, h) is its vorticity or curl, and p is a hydrostatic pressure in the medium, the symbol D^2/dt^2 denoting the acceleration of a moving particle as contrasted with the rate of change of velocity at a fixed point.

98. These equations represent the general circumstances of the propagation of radiation through the medium; and in them the velocity of translation of the medium due to vortices in it has been averaged. But if we desire to investigate in detail the motion and vibrations of a single vortex-ring or a vortex-system in a rotationally elastic fluid medium, it is of course not legitimate to average the motion of translation near the ring. The determination of the circumstances of the influence of a moving medium on the radiation also requires a closer approximation. Considering therefore the free æther, which is devoid of crystalline quality, and substituting

$$\frac{d}{dt}(\xi, \eta, \zeta) = (u + u_1, \quad v + v_1, \quad w + w_1),$$

so as to divide the velocity into two parts one of which represents the translation of the medium and the other its vibration, we have

$$\frac{D}{dt} = \frac{d}{dt} + (u + u_1)\frac{d}{dx} + (v + v_1)\frac{d}{dy} + (w + w_1)\frac{d}{dz},$$

so that

$$\frac{D}{dt}(u + u_1) = \frac{\delta u}{dt} + \frac{\delta u_1}{dt} + u_1\frac{du}{dx} + v_1\frac{du}{dy} + w_1\frac{du}{dz}$$

very approximately where

$$\frac{\delta}{dt} \text{ represents } \frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}.$$

Hence separating the hydrodynamical part in the form

$$\rho\frac{\delta}{dt}(u, v, w) = -\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)p_0,$$

which represents irrotational motion except in the vortices, there remain vibrational equations of the type

$$\rho\left(\frac{\delta u_1}{dt} + u_1\frac{du}{dx} + v_1\frac{du}{dy} + w_1\frac{du}{dz}\right) + \alpha^2\left(\frac{dh_1}{dy} - \frac{dg_1}{dz}\right) + \frac{dp_1}{dx} = 0.$$

In a region in which the velocity of translation (u, v, w) is uniform, the radiation is thus simply carried on by the motion of the medium.

99. The vibrational motion which is propagated from an atom is interlinked with the motion of translation of the medium, only through the hydrostatic pressures which must be made continuous across an interface; the form of the free surface has in fact to be determined so as to adjust these pressures at each instant. To fix our ideas, let us consider for a moment the problem of the vibrations of a single ring with vacuous core, moving by itself through the medium, in the direction of its axis, with a given atomic electric charge on it. To obtain a solution we assume that the radius vector of the cross section of the core varies with the time according to the harmonic function suitable to its types of simple vibration; and we determine the irrotational motion in the medium that is produced by this motion of the surface of the core, and calculate the pressure p_0 at the free surface. Next we determine the vibrational rotation (f, g, h) that is conditioned by the same vibratory movement of the surface of the core, while it is independent of the inertia of the hydrodynamical motion in the medium; this has also to satisfy the condition that the tangential components of the rotation are null all over the surface, so that there may be no electromotive tangential traction on it. In order to satisfy all these surface conditions it will usually be necessary to introduce an electromotive pressure p_1 into the equations of vibration, although this was not required in the problem of reflexion at a fixed interface; in other words the pressure in that problem was quite unaffected and therefore left out of account. The magnitude of this pressure is then to be calculated from the solution; and the condition that it is equal and opposite at the free surface, to the pressure p_0 of hydrodynamical origin, gives an equation for the period of the vibrations of the type assumed. If on the other hand the core is taken to consist of spinning fluid devoid of rotational elasticity, instead of vacuum, the conditions at its surface will be modified.

100. If the form of the ring is such that the period of its hydrodynamic vibration is large compared with that of the corresponding electric vibration, an approximate solution is much easier; it is now only necessary to suppose that on each successive configuration of the core there is a distribution of static electricity in equilibrium, and to allow for the effect of this distribution on the total pressure which must vanish at a free surface.

In this case the electric vibrations will continue for a comparatively long time, until all the energy of the disturbance in the molecule is radiated away, but they will be of very small intensity. The vibrations of an electric charge over a conducting atom which is not a vortex ring are practically dead-beat, and could not give rise to continued radiation of definite periods: but the case is different here, and the vibrations will go on until the energy of the disturbance of the steady motion of the vortex-ring atom has all been changed into electrical waves.

Now the periods of the principal hydrodynamical vibrations of a single ring may be regarded as the times that would be required for disturbances of the different permanent types to move round its core with velocities of the same order of magnitude as the actual velocity of translation of the ring through the medium; while the periods of the electrical vibrations are the times that would be required for electric disturbances to move round the core with velocities of the same order as the velocity of radiation. The first of these periods is for an isolated ring very much the greater, so much so that electric vibrations could hardly be excited at all by vibrations of the atom comparatively so slow. But in the case of a molecule there would also be much smaller hydrodynamical periods, due to the interaction between neighbouring parts of the paired rings, which may be expected to maintain electrical vibrations in the manner above described; and in the case of an isolated ring the periods which involve crimping of the cross section may produce a similar effect, though they cannot involve a sensible amount of energy.

When the core is of the same density as the surrounding fluid, and there is no slip at its surface, the hydrodynamical pressure across the interface will be continuous in the steady motion of the ring; therefore the above electric pressure must be uniform all over the interface; that is, the electric force must be constant over it, as well as the electric potential. These conditions determine the form of the interface in the steady motion; and the rotational motion of the core is then determined, through its stream function, so as to have given total amount and to be continuous with the circulatory irrotational motion just outside it.

On Gravitation and Mass.

101. The hypothesis of finite though very small compressibility of the æther has occasionally been kept in view in the foregoing analysis, in the hope that it may lead to results having some affinity to gravitation. There does not appear however to be any correspondence of this kind. A tentative theory has already been proposed and examined by W. M. HICKS, which makes gravitation a secondary effect of those vibrations of vortices in an incompressible fluid which consist in pulsations of volume of their vacuous cores. But the periods of such vibrations are not very different from the periods of their other types; and the theory cannot be said to be successful, the objections to it being in fact fully stated by its author.*

* W. M. HICKS, 'Proc. Camb. Phil. Soc.,' 1879; 'Roy. Soc. Proc.,' 1883; also 'Phil. Trans.,' 1883, p. 162.

Let us now consider the effect of a compressional term in the potential energy of the medium, of the form

$$\tfrac{1}{2} \mathrm{A} \int \left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} \right)^2 d\tau, \quad \text{say } \tfrac{1}{2} \mathrm{A} \int \varpi^2 \, d\tau,$$

where $-\varpi$ is the compression in the medium. The variation of this term will be

$$\mathrm{A} \int \varpi \, (l \, \delta x + m \, \delta y + n \, \delta z) \, d\mathrm{S} - \mathrm{A} \int \left(\frac{d\varpi}{dx} \, \delta x + \frac{d\varpi}{dy} \, \delta y + \frac{d\varpi}{dz} \, \delta z \right) d\tau.$$

Thus there will be added to the right-hand side of the equations of vibration new terms, giving in all

$$\rho \frac{d^2\xi}{dt^2} + \frac{dc^2h}{dy} - \frac{db^2g}{dz} - \mathrm{A} \frac{d\varpi}{dx} = 0$$

$$\rho \frac{d^2\eta}{dt^2} + \frac{da^2f}{dz} - \frac{dc^2h}{dx} - \mathrm{A} \frac{d\varpi}{dy} = 0$$

$$\rho \frac{d^2\zeta}{dt^2} + \frac{db^2g}{dx} - \frac{da^2f}{dy} - \mathrm{A} \frac{d\varpi}{dz} = 0.$$

It follows that ϖ satisfies the equation

$$\rho \frac{d^2\varpi}{dt^2} = \mathrm{A} \nabla^2 \varpi;$$

so that the compressional wave is propagated independently of the rotational one, of which the circumstances are given by equations of the type

$$\rho \frac{d^2 f}{dt^2} = - \frac{d}{dx} \left(\frac{da^2f}{dx} + \frac{db^2g}{dy} + \frac{dc^2h}{dz} \right) + \nabla^2 a^2 f.$$

In the discussion of the reflexion of light it has been shown that the same absolute separation of compression and rotation is manifested in the passage across an interface into a new medium; so that however heterogeneous the medium be rendered by the presence of vortex-atoms, these two types of disturbance are still quite independent of each other.

The alteration in the electrostatic equations which would be produced by this compressional quality has already been given; if the value of the modulus A is extremely great, this alteration will be quite unnoticeable. In that case, waves of compression will be propagated with extremely great velocity, so that as regards compression the medium will assume almost instantly an equilibrium condition, for which therefore $\nabla^2 \varpi = 0$.

It follows that the value of the integral $\int d\varpi/dn \, . \, d\mathrm{S}$ is the same for all boundaries which contain inside them the same atoms. If we want to make this integral constant throughout time, we may imagine that the medium was originally in equilibrium without compression, and was then strained by altering the volume of each electrically charged atom by a definite amount. The state of strain thus represented *in the æther* has a pressure at each point equal to A multiplied into the gravitation potential of a mass equal to this constant, supposed placed at the atom. Its energy is however

$$\tfrac{1}{2}\,\mathrm{A}\int\varpi^2\,d\tau, \quad \text{instead of} \quad -\tfrac{1}{2}\,\mathrm{A}\int\left(\frac{d\varpi^2}{dx^2}+\frac{d\varpi^2}{dy^2}+\frac{d\varpi^2}{dz^2}\right)d\tau,$$

which it ought to be* if it were gravitational energy; so that there is no means of explaining gravitation here.

102. If we could imagine for a moment that the electric charges of the two ions in a molecule do not exactly compensate each other, but that there is a slight excess always of the same sign, we should have a *repulsive* force of gravitational type, transmitted by a stress in a rotational æther. A term of this form in the energy, if it were kinetic instead of potential, would account for gravity. The question thus suggested is, whether the kinetic energy of the primordial medium has been sufficiently expressed, in view of the inherent rotational quality in its elements. It was proved by LAPLACE that the velocity of gravitation must be enormously great compared with that of light; so that the gravitational energy, whatever its origin, must preserve a purely statical aspect with respect to all the other phenomena that have been here under discussion.

The objection has been raised, by CLERK MAXWELL and others, to the vortex-atom theory of matter, that it can give no account of mass for the case of sensible bodies. But it may be urged that mass is a dynamical conception, which in complicated cases it would be hard to define exactly or give an account of. The clearest view of dynamics would appear to be the one maintained by various writers, notably by L. N. M. CARNOT and by KIRCHHOFF, that the function of that science is to correlate, or give a general formula for, the sequence of physical phenomena. The ultimate formula which is, it is hoped, to embrace the physical universe is the law of Least Action; and the ultimate definition of mass is to make it a coefficient in the kinetic part of the energy-function of the matter in that formula. As the theories here discussed are referred to the single basis of this law of Least Action, the objection that they do not take account of mass can hardly be prohibitive; though they may not be able to explain how the idea of mass is originated by aggregation of terms in that equation.

103. It is conceivable that the rotational elasticity of the fundamental medium is really due to a rotatory motional distribution in it, which resists disturbance from

* *Cf.* MAXWELL, "A dynamical theory of the Electromagnetic Field," § 82, 'Phil. Trans.,' 1864.

its steady equilibrium state with excessively great effective elasticity, while the tractions necessary to equilibrate a free boundary are non-existent. Such a hypothesis looks like explaining one æther by means of a new one, but it is perhaps not really more complicated than the facts; on our present principle of interpretation, the change of gravitation in the field due to a disturbance at any point must have been propagated somehow, while in the machinery that transmits electric and luminiferous disturbances no elasticity has yet been recognized anywhere near intense enough to take part in such a propagation.

We may not surmount the difficulty by the assumption that, in addition to the finite resistance to rotation which is the cause of the propagation of the radiation, the medium also possesses an enormously greater static resistance to rotations of some more fine-grained structure, and that the surface integral of the rotation over any surface enclosing a vortex-atom is a positive constant, of course definite and unchangeable in value for each atom; for this would lead to gravitational repulsion instead of attraction. The term must be in the kinetic energy, not in the potential energy of the medium.

104. In a representation of a magnetic or other medium,* imagined to be composed of gyrostatic elements spinning indifferently in all directions, and linked into a system by an arrangement like idle-wheels between them, in fact by an ideal system of universal ball-bearings, the kinetic energy function would have a rotatory part

$$T = \tfrac{1}{2} C \int \left(\frac{df^2}{dt^2} + \frac{dg^2}{dt^2} + \frac{dh^2}{dt^2} \right) d\tau,$$

where (f, g, h) is the absolute rotation of an element, which is supposed from the connecting mechanism to be a continuous function of position in the system.

We would have therefore

$$\delta T = C \int \left\{ \frac{df}{dt} \frac{d}{dt} \left(\frac{d\delta\zeta}{dy} - \frac{d\delta\eta}{dz} \right) + \frac{dg}{dt} \frac{d}{dt} \left(\frac{d\delta\xi}{dz} - \frac{d\delta\zeta}{dx} \right) + \frac{dh}{dt} \frac{d}{dt} \left(\frac{d\delta\eta}{dx} - \frac{d\delta\xi}{dy} \right) \right\} d\tau$$

$$= \int \{\ldots\}\, dS - C \int \left\{ \frac{d^2}{dt^2} \left(\frac{dh}{dy} - \frac{dg}{dz} \right) \delta\xi + \frac{d^2}{dt^2} \left(\frac{df}{dz} - \frac{dh}{dx} \right) \delta\eta + \frac{d^2}{dt^2} \left(\frac{dg}{dx} - \frac{df}{dy} \right) \delta\zeta \right\} d\tau.$$

Thus the kinetic forcive which is the equivalent of the actual applied forcive in the medium per unit volume, arising from its potential energy and such extraneous forces as act on it, is

$$C \frac{d^2}{dt^2} \operatorname{curl} (f, g, h), \quad \text{or} \quad - C \frac{d^2}{dt^2} \nabla^2 (\xi, \eta, \zeta).$$

If we suppose the displacement (ξ, η, ζ) to be originally derived from a potential

* *Cf.* MAXWELL'S "Hypothesis of Molecular Vortices," 'Treatise,' §§ 822–7.

function ϕ, this kinetic forcive exists only where there is some portion of the ideal mass-system of which ϕ is the potential; the spin in the medium thus produces no forcive anywhere except in the spinning parts.

We may imagine this medium to be a hydrodynamical one such as could sustain vortex-motion; then this kinetic forcive is confined to the vortices. Throughout a small volume containing a vortex, the aggregate of this forcive is

$$-\mathrm{C}\int\frac{d^2}{dt^2}\nabla^2(\xi,\eta,\zeta)\,d\tau\,;$$

of which the part outside the core of the vortex is

$$-\mathrm{C}\int\frac{d^2}{dt^2}\left(\frac{d}{dx},\ \frac{d}{dy},\ \frac{d}{dz}\right)\nabla^2\phi\,d\tau,$$

and is therefore null, so that this quantity $\int\nabla^2(\xi,\eta,\zeta)\,d\tau$ may be taken as an intrinsic constant for any particular isolated vortex throughout all time. Again, its value is the same for the regions bounded by all surfaces which include the same vortices; thus there is a kinetic reaction proportional to the second differential coefficient with respect to time of the amount of this particular constant thing that is carried by the vortices contained in the element of volume. If we attach in thought this forcive to a moving element of volume containing the vortices, instead of to the fixed element of volume, it will vary jointly as the amount of this thing that belongs to the vortex-group, and the acceleration of the element of volume in space; and its aggregate amount will not be affected by interaction between the vortices of the group. This appears to introduce the dynamical notion of mass and acceleration of matter; and this illustration has been furnished by a function representing energy of spin in the medium, which exists only where that spin is going on, *i.e.* in the vortices. The remaining part of the kinetic energy of the medium, which is the whole of the kinetic energy of that part of the medium not occupied by vortices, is translational as above and equal to

$$\tfrac{1}{2}\rho\int\left(\frac{d\xi^2}{dt^2}+\frac{d\eta^2}{dt^2}+\frac{d\zeta^2}{dt^2}\right)d\tau.$$

105. To make a working scheme we must suppose a layer of the medium, possessing actual spin, to cover the surface of each coreless vortex-atom; we might imagine a rotationless internal core which allowed no slipping at the surface, and this spin would be like that of a layer of idle-wheels which maintained continuity between this core and the irrotational circulatory motion of the fluid outside. A gyrostatic term in the kinetic energy thus appears to introduce and be represented by the kinetic idea of mass of the matter; it enters as an æolotropic coefficient of inertia for each vortex, but when averaged over an isotropic aggregate of vortices, it leads to a scalar coefficient for a finite element of volume.

If the core of the vortex-atom is not vacuous but consists as in ordinary vortices of spinning fluid, here devoid of rotational elasticity, the rotational kinetic energy of the vortex as distinguished from translational energy will be a possible source of the phenomena of mass; but to possess such energy the medium must have some ultimate structure, for in an infinitely small homogeneous element of volume the ratio of the rotational to translational part of the kinetic energy would be infinitely small. Such a structure, confined to the cores of the vortices, need not be in contradiction with MAXWELL'S principle that the constitution of a perfect fluid cannot be molecular.

[*Added June* 14, 1894.]

On Natural Magnets.

106. Lord KELVIN* has pointed out that the forcive between a pair of rigid cores in a fluid, with circulatory irrotational motion through their apertures, is equal but opposite to the forcive between the corresponding steady electric currents as expressed by the electrodynamic formulæ. The reason of this difference lies in the circumstance that the connexions and continuity of the fluid system prevent the circulation round any core from varying, so long as that core is unbroken; while the constraints must be less complete in the electrodynamic problem, because the currents change their values by induction. These constant circulations are of the nature of the constant momenta belonging to cyclic motions of dynamical systems; and it is known that when such constant momenta are introduced into the expression for the energy in place of the corresponding velocities, the type of the general dynamical equation is thereby altered.† The modification which the equation of Least Action must undergo under these circumstances has been investigated on a previous occasion.‡ In the case of fluid circulation, when the cores are so thin as to interpose no sensible obstacle to the flow, the sign of that part of the kinetic energy which involves the cyclic constant of the motion has merely to be changed; in other words this energy is for the purpose of the modified dynamical equations to be treated as potential instead of kinetic. In all cases in which co-ordinates of a dynamical system can be ignored by elimination in this manner the energy function consists of two parts, one a quadratic function of the velocities of the bodies, the other a quadratic function of the constant momenta: in the case just mentioned the former part is negligible, so that the part whose sign is to be changed is practically the total energy.

* Lord KELVIN (Sir W. THOMSON), "Hydrokinetic Analogy," 'Proc. Roy. Soc., Edin.,' 1870; 'Papers on Electrostatics and Magnetism,' p. 572. Also KIRCHHOFF, 'Crelle,' 1869.

† ROUTH, "Stability of Motion," 1877, ch. 4, §§ 20 seq.; THOMSON and TAIT, "Natural Philosophy," ed. 2, 1879, §§ 319, 320; VON HELMHOLTZ, "On Polycyclic Systems," 'Crelle,' 1884–1887.

‡ "Least Action," 'Proc. Lond. Math. Soc.,' XV., March 1884. (On p. 182 the electrodynamic energy is quoted with the wrong sign.)

The validity of the application of the Lagrangian equations in the unmodified form to electric currents, as in the discussion in this paper, thus requires that there is no intrinsic cyclosis in the motions which exist in the electrodynamic field. The conductors must therefore all form practically incomplete circuits, in which the flow may be maintained and altered by means of what are effectively breaches in the continuity of the medium; and as a further consequence, arising from such breaches of continuity, the mechanical forcives between the conductors will not now be wholly due to ordinary fluid pressure.

In an ordinary electric circuit, the circulation of the medium is thus maintained around the conducting part of the circuit by electric convection or displacement across the open or electrolytic part, by means of a process in which the rotational elasticity of the medium is operative. We may imagine this electric convection to be performed mechanically, and to be the source of the energy of the current: the force-component corresponding to the dynamical velocity which represents the current will then be the electric force which does work in the convection of charged ions. If this convection ceased, the circulatory motion which constitutes the magnetic field of the current (*i.e.* its momentum) would be stopped by the elasticity of the medium; and by altering the velocity of this convection, we have the means of adding to or subtracting from the circulatory motion, the change of kinetic energy so produced being derived from the electric force which resists convective displacement. This mode of mechanical representation suffices to include all the phenomena of ordinary electric currents. On the other hand, in a molecular circuit there is no electric convection, but only a permanent fluid circulation through it, such as would be self-subsisting, by aid of fluid pressure only, when the core is fixed, and could not in any case be permanently altered, on account of the rotational elasticity.

In the establishment of an ordinary current in an open circuit, the rotational elasticity of the medium acts very nearly as a constraint, on account of the great velocity of electric propagation; and there is therefore at each instant only an insignificant amount of energy involved in it. But notwithstanding, if there are other open conducting circuits in the neighbourhood the action of this elasticity in establishing the current will be partly directed by them and relieved by circulation round them. The final result for maintained currents is however irrotational motion through the circuits; the kinetic energy is sufficiently represented, for slow changes, by the ordinary electrodynamic formula for linear currents; and it is directly amenable to the Lagrangian analysis. If the currents move in each others' fields, with external agencies to prevent their strengths from altering, these agencies must supply twice as much energy as is changed into mechanical work in the movement, in accordance with a theorem of Lord KELVIN'S.

Conversely, assuming that the electromagnetic energy is kinetic, it would seem that we are required by LENZ'S law to take the currents in ordinary electric circuits to be of the nature of velocities, in the dynamical theory; though in the essentially different configuration of an Amperean magnetic molecule, the circulation which corresponds most closely to the current is more allied to a generalized momentum.

The energies of magnetic vortex atoms would have to be introduced with changed sign into the modified equation of Least Action, and this will involve the presence in the modified function of terms containing the electric generalized velocities in the first degree. Unless the cross sections of the rings are very small compared with their diameters, there will also occur terms involving products of the strengths of the vortices and the velocities of the movements of the rings. For two stationary thin rigid cores of very narrow section, the mutual forcive due to fluid pressure will thus be equal but opposite to the forcive between the corresponding electric currents; the general features of this result are in fact easily verified by consideration of the distribution of velocity, and therefore of pressure, in the steady fluid motion of the medium.

107. The serious difficulty presents itself that the mutual attractions of natural magnets are actually in the same direction as those of the equivalent electric currents, and not, as would appear from this theorem, in the opposite direction. In the first place however, the theorem is proved only for rigid cores, held in the circulating fluid medium, and the forcive in question is simply the resultant of fluid pressures over the surfaces of the cores. In the case of vortex atoms with vacuous cores, such a pressure would not exist at all. And when we consider individual molecules, the question is also mixed up with the unsolved problem of the nature of the inertia of a vortex molecule.

It may be of use to examine separately the distribution of kinetic energy which the presence of two vortex aggregates implies in the medium surrounding them and between them, as distinguished from the kinetic energy inside them which is in direct relation with intermolecular forces. Let us take Lord KELVIN's illustration, a set of open rigid tubes in a frictionless fluid, through each of which there is circulatory motion. "When any change is allowed in the relative positions of two tubes by which work is done, a *diminution* of kinetic energy of the fluid is produced within the tubes, and at the same time an *augmentation* of its kinetic energy in the external space. The former is equal to double the work done; the latter is equal to the work done; and so the loss of kinetic energy from the whole liquid is equal to the work done."* The distribution of energy in the medium, outside two vortex aggregates, thus varies in the same way and with the same sign as the energy of the field of the corresponding magnets, as of course it ought to do. And the question is suggested, are we allowed to turn the difficulty as to the nature of the inertia of the vortex atoms by considering the magnetic forcive between two permanent aggregates as derived from the transformation of the kinetic energy in the medium between them?

The motion of the medium between them may be set up by the proper impulsive pressures over the surfaces of the aggregates, just as the magnetic field is determined

* Lord KELVIN, "Electrostatics and Magnetism," 1872, § 737.

by the distribution of magnetic intensity over the outer boundary of the magnets. And the principle of energy by itself shows that if we bound the two aggregates by moving surfaces which always pass through the same particles of the medium, the increment of the kinetic energy outside is equal to the work done *in the actual motion* by the pressures transmitted across the surfaces of the two aggregates; though we are unable to extend this result to arbitrary virtual displacements of the surfaces. Nor is the method of § 58 now applicable to complete the proof, because it is impossible to have an equipotential surface surrounding a magnetic system.

108. In all theories which ascribe the induction of electric currents to elastic action across the intervening medium, a discrepancy arises when the induction is produced by movement through a steady magnetic field: for in such cases there is no apparent play of electric force across the field. This difficulty may perhaps disappear, on the present view, when we regard such a field, not as an absolutely steady motion like fluid circulation round fixed cores, but as the statistically steady residue of elementary elastic disturbances sent out through the medium by the molecular discharges which maintain the inducting currents, or by changes of orientation and other disturbances of the molecules of the permanent magnets, such as are involved in any kinetic theory of matter. These elastic disturbances do not spread out indefinitely as waves, but come to an end when the medium has attained a new steady state which they have been instrumental in forming. The progress and decay of each small disturbance generates a current on the secondary system, whose integrated amount would be null if that system were at rest: but in the actual circumstances of movement during the progress of the induction there will be a residual value. The aggregate of such differences between elementary direct and reverse induced currents would constitute the observed total current. Thus as regards induction, change of the magnetic field of a permanent magnet would act in the same way as that of an ordinary current, notwithstanding that if each molecule of the magnet were held fixed there might (§ 106) be no induction.

On these grounds, the field of a permanent magnet would be regarded, not as a steady circulation of the æther, absolutely devoid of elastic reaction, but as the statistically steady resultant of the changing fields of the incessantly moving molecules which make up the magnet. The steady field of motion associated with a fixed magnetic molecule would be maintained by fluid pressure alone: but when the molecule is rotated, some agency is required to prevent slip during the establishment of the new steady motion; and in this way the elasticity may come into play. In ordinary hydrodynamics, the process of the establishment of a fluid motion is kept out of sight: it is simply assumed that the motion can be set up without slip, and that it is set up practically instantaneously throughout the field. In the present problem on the other hand, something formally equivalent to slip does occur across the dielectric gaps in each electric circuit; and this circumstance modifies the process of establishment of the motion.

This explanation if valid, would carry with it, by virtue of the principle of energy, the observed law of attraction of a permanent magnet on an ordinary electric current; and also, provided we could assume the law of action and reaction to be applicable, that of a magnetic field on the aggregate constituting a permanent magnet. And as in the case of currents maintained steady, when two permanent magnets move each other the energy in the medium surrounding them is increased by the mechanical work done, but the energy in their interiors is diminished by twice that amount.

Whatever be the value of these remarks, it would seem that the difficulty with respect to permanent magnets can hardly be insuperable, as it must attach in some form to any theory which makes magnetic energy kinetic. For, on that hypothesis, this energy must be wholly cyclic when there are only permanent magnets on the field; and its sign would therefore have to be changed, just as above, in forming dynamical equations which take separate account of each magnetic molecule. If on the other hand the statistical view above adopted is allowed, the complication introduced by intermolecular actions will be avoided, and only the averaged action between the two systems will remain.

On the Electrodynamic Equations.

109. The kinetic energy of the electric medium is

$$\mathrm{T} = \tfrac{1}{2}\int\left(\frac{d\xi^2}{dt^2} + \frac{d\eta^2}{dt^2} + \frac{d\zeta^2}{dt^2}\right) d\tau.$$

Let us transform this expression to new variables (f, g, h) which represent the components of the absolute rotation at each point; and let us suppose that there is nowhere any discontinuity or defect of circuital character in these quantities. We must therefore assign to them very large but not infinite values in an indefinitely thin superficial layer of the conductors, which shall be continuous with their actual values outside and their null value inside that surface.* The object of doing this is to abolish all surface-integral terms which would otherwise enter, on integration by parts, at each interface of discontinuity; the surface-integral terms that belong to the infinitely distant boundary need not concern us, except in cases where radiation plays a sensible part.

We may show as in § 52 that under these circumstances

$$\mathrm{T} = \frac{1}{8\pi}\int\left(\frac{df}{dt}\mathrm{F} + \frac{dg}{dt}\mathrm{G} + \frac{dh}{dt}\mathrm{H}\right) d\tau,$$

where

$$(\mathrm{F}, \mathrm{G}, \mathrm{H}) = \int\frac{d\tau'}{r'}\,\frac{d}{dt}\,(f', g', h'),$$

r' being the distance of the element $d\tau'$ from the element $d\tau$.

* The procedure of this section leaves out dissipation, and so confines the currents to the surfaces of the conductors.

It is of necessity postulated throughout that (f, g, h) is circuital, for it is the curl of (ξ, η, ζ); that is, the proper current sheet must always be taken to exist at the surface of the conductor in order to complete the electric displacement in the medium. It follows as in § 57, but only under this proviso, that the magnetic force is the curl of MAXWELL's vector potential (F, G, H) of the current-system.

The transformation of the kinetic energy T to the directly elastic co-ordinates (f, g, h) is thus established; and the dynamical equation of the medium is

$$\delta\int(\mathrm{T} - \mathrm{W})\,dt = 0$$

in which the time is to remain unvaried. In order however to obtain equations wide enough to allow of the restriction of (f, g, h) to circuital character, which is now no longer explicitly involved, we must incorporate this restriction in the variational equation after the manner of EULER and LAGRANGE, and so make

$$4\pi\delta\int dt\,(\mathrm{T} - \mathrm{W}) + \delta\int dt\int d\tau\psi\left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right) = 0,$$

and restrict the function of position ψ subsequently so as to satisfy the circuital relation. Thus

$$\delta\int dt\int\left\{\tfrac{1}{2}\left(\mathrm{F}\frac{df}{dt} + \mathrm{G}\frac{dg}{dt} + \mathrm{H}\frac{dh}{dt}\right) - \mathrm{W} + \psi\left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right)\right\} d\tau = 0.$$

Now in all cases in which the kinetic energy of a dynamical system involves the velocities but not the co-ordinates, the result of its variation is the same as if the momenta, such as F, G, H, in the expression in terms of momenta and velocities, were unvaried, and the result so obtained were doubled. Thus we have here

$$\int dt\int\left\{\left(\mathrm{F}\frac{d\delta f}{dt} + \mathrm{G}\frac{d\delta g}{dt} + \mathrm{H}\frac{d\delta h}{dt}\right) + \psi\left(\frac{d\delta f}{dx} + \frac{d\delta g}{dy} + \frac{d\delta h}{dz}\right)\right\} d\tau = \int dt\,\delta\mathrm{W};$$

or, integrating by parts and omitting the boundary terms for the reasons above given,

$$\int dt\int\left\{\left(-\frac{d\mathrm{F}}{dt} - \frac{d\psi}{dx}\right)\delta f + \left(-\frac{d\mathrm{G}}{dt} - \frac{d\psi}{dy}\right)\delta g + \left(-\frac{d\mathrm{H}}{dt} - \frac{d\psi}{dz}\right)\delta h\right\} d\tau = \int dt\,\delta\mathrm{W}.$$

Therefore throughout the system the forcive corresponding to the displacement (f, g, h) is

$$(\mathrm{P}, \mathrm{Q}\ \mathrm{R}) = -\left(\frac{d\mathrm{F}}{dt} + \frac{d\psi}{dx},\ \frac{d\mathrm{G}}{dt} + \frac{d\psi}{dy},\ \frac{d\mathrm{H}}{dt} + \frac{d\psi}{dz}\right).$$

109*. When however we consider the case of conductors in motion, so that their current sheets, instead of being referred to fixed axes, are carried on along with them, we shall have to refer the medium and therefore also the above variational operation to a moving scheme of axes or more generally to a moving space; and this will be accomplished if we include in d/dt (F, G, H) not only ordinary partial differential coefficients with respect to the time, but also the rate of change due to alteration of position of the point considered owing to the movement of the space to which it is referred.

The result of this reference to moving space, for the case in which it moves like a body of invariable form, is worked out as in MAXWELL, 'Treatise,' § 600, and leads to his well-known equations of electric force. These equations are however expressed with equal generality by eliminating the adjustable quantity ψ, thus obtaining for any complete circuit, with this extended meaning of d/dt,

$$\int(\mathrm{P}\,dx + \mathrm{Q}\,dy + \mathrm{R}\,dz) = -\frac{d}{dt}\int(\mathrm{F}\,dx + \mathrm{G}\,dy + \mathrm{H}\,dz)$$
$$= -\frac{d}{dt}\iint(la + mb + nc)\,d\mathrm{S}.$$

As this relation retains the same form whether referred to fixed or to moving space, it expresses the FARADAY-MAXWELL law that under all circumstances the electromotive force referred to a circuit, fixed or moving, is equal to the rate of diminution of the magnetic induction through its aperture.

The expressions for the electric force thus determined are merely *formulæ* for the kinetic reaction of the disturbed medium, which must be at each instant balanced by the forces of the elastic strain which is the other aspect of the efficient cause of the phenomena. Thus they do not imply any conclusion that in all material dielectrics, whether gaseous or liquid or solid, the motion of the matter produces an electric effect which is objectively the same for all; the equations referred to moving space apply in fact quite as readily to the free æther itself as to a moving material medium, provided the currents as well as the electric force are referred to the moving space.

In any actual problem, the quantity ψ, which enters into the electric force, is made determinate by means of the circuital condition to be satisfied by the currents throughout the dielectric: as a matter of convention we may if we please take ψ to include the electric potential of charges on the conductors which are the terminal aspects of the elastic strain in the dielectric, but nothing essential is perhaps gained by such a course, unless in the case of slow movements.

110. If however we were to adopt, on the lines of HELMHOLTZ's theory of 1870, a different procedure and assume that the vector (F, G, H) is a physical entity as distinct from a mathematical expression, and so assign a definite physical formula for it, which must from our actual knowledge be of the type

$$\mathrm{F} = \int\frac{u}{r}\,d\tau + \int\left(\mathrm{B}\frac{d}{dz} - \mathrm{C}\frac{d}{dy}\right)\frac{1}{r}\,d\tau,$$

it would follow that the circumstances of the induced electric force are not determined merely by the distribution of magnetic induction in the field, but involve the actual distribution of electric current and of magnetism throughout all space. For there are very various distributions of electric current and magnetism in the more distant parts of space which lead to the same distribution of magnetic induction in the neighbourhood of the system in which the currents are induced: these would be equivalent as regards the magnitudes of induced currents, but not as regards the distribution of induced electric force.

This state of things would not be inconsistent with general principles. The electric influence arising from a disturbance of one system is propagated elastically to other systems across the intervening medium, the propagation being nearly instantaneous without showing any sensible trace of the disturbance during its transit through the medium, and this on account of the high elasticity and consequent great velocity of propagation. The magnetic field is a residual effect of this propagation; that field is sufficient to represent the aggregate features of the result in cases in which the current is mostly conducted, but it need not represent the features of the propagation in detail. There are in fact cases in which induction takes place across a space in which there is at no time any sensible electric or magnetic force at all: for example the starting of a current in a ring electro-magnet induces in this way a current in any outside circuit which is linked with the ring: the elastic propagation here leaves no trace in the form of motion of the æther or magnetic force.

111. When the velocity of electric propagation may be taken as indefinitely great compared with the velocities of the conductors in the field, the phenomena of induced currents will depend only on the *relative* motion of the inducing and induced systems; thus we may simplify the conditions by taking the induced system at rest subject to the electric influences sent out from an inducing system in motion and otherwise changing. Now in this simpler case the electric intensity consists of two parts, one of them required to keep the current going against the viscous resistance of the conductor and the elastic resistance of the dielectric, and the other a free disturbance which will be continually cancelled with the velocity of radiation as fast as it is produced. The latter part therefore practically does not exist in ordinary problems of induction, in which the movements are slow compared with the velocity of light. Thus the elastic displacement of the electric medium may be taken as in internal equilibrium by itself in all such cases; there can be no free electric force inside a conductor, and the electric charge, if any, will reside on its surface. The amount of this superficial charge will be the time-integral of the displacement current which is involved in the total current, and which is wholly in the outside dielectric. Now the determination of the complete current is a perfectly definite problem, on the principles of AMPÈRE and FARADAY: thus the electric force at any point and the static electrification on the conductors are also on the same principles definite and determinate, subject to this proviso of slow movement of the bodies concerned.

Conclusion.

112. The foundation of the present view is the conception of a medium which has the properties of a perfect incompressible fluid as regards irrotational motion, but is at the same time endowed with an elasticity which allows it to be the seat of energy of strain and to propagate undulations of transverse type ; and the question discussed is how far such a simple type of medium affords the means of co-ordination of physical phenomena. This idea of a medium with fluid properties at once disposes of the well-known difficulties which pressed on all theories that imposed on the æther the quality of solidity. If the objection is taken, which has been made against the ordinary vortex-atom theory of matter—that a perfect fluid is a mathematical abstraction which does not exist in nature, and the objective existence of which has not been shown to be possible,—the conclusive reply is at hand that the rotational elasticity with which the medium is here endowed effectually prevents any slip or breach such as would be the point of failure of a simple fluid medium without some special quality to ensure continuity of motion. On this head it will be sufficient to refer to some remarks of Sir G. G. STOKES* on a cognate topic. If therefore it is objected that we have no experience of a medium whose elasticity depends on rotation and not on distortion, the reply is that we can form no notion of the structure of a continuous frictionless fluid medium, unless we endow it with just some such elastic property in order to maintain its continuity.

The idea of representing magnetic force in the equations of electrodynamics by the velocity of the electric medium has been tried already, for example by HEAVISIDE and by SOMMERFELD, not to mention EULER. The objection however has been taken by BOLTZMANN and also by VON HELMHOLTZ that it would be impossible on such a theory for a body to acquire a charge of electricity. A cardinal feature in the electrical development of the present theory is on the other hand the conception of intrinsic rotational strain constituting electric charge, which can be associated with an atom or with an electric conductor, and which cannot be discharged without rupture of the continuity of the medium. The conception of an unchanging configuration which can exist in the present rotational æther is limited to a vortex ring with such associated intrinsic strain : this is accordingly our specification of an atom. The elastic effect of convection through the medium of an atom thus charged is equivalent to that of a twist round its line of movement : such a twist is thus a physical element of an electric current.

113. The chief result of the discussion is that a rotationally elastic fluid æther gives a complete account of the phenomena of optical transmission, reflexion, and refraction, in isotropic and crystalline media, coinciding in fact formally in its wider features with the electric theory of light ; and that it gives a complete account of

* Sir G. G. STOKES, "On the Constitution of the Luminiferous Æther," 'Phil. Mag.,' 1848, 'Collected Papers,' vol. 2, p. 11.

electromotive phenomena in electrostatics and electrodynamics. It assigns correctly the magnetic rotatory action on light to a subsidiary term of definite type in the energy function of a material medium; while to avoid a magnetic translatory action of such amount as would be detectable, it is compelled to assign a high value to the coefficient of inertia of the free æther. In unravelling the detailed relations of æther to matter it is not very successful, any more than other theories; but it suggests a simple and precise basis of connexion, in that form of the vortex-atom theory of matter to which it leads; and even should the present mode of representation of the phenomena become on further development in this direction definitely untenable, it may still be of use within its limited range as illustrating wider views of possibilities in that field. The theory also leads to the correct expressions for the ponderomotive class of electrostatic and electrodynamic phenomena, or rather it is not in disagreement with them; for here again knowledge of the details of the relation between the æther and the matter is defective, and thus for example the law of the attraction between permanent magnets is left unexplained. It supplies also a more definite view of the essentially elastic origin of all electrodynamic action than has perhaps hitherto been obtained, especially in cases of induction by motion across a steady magnetic field.

[Added August 13, 1894.]

Introduction of Free Electrons.

114. The conclusion to which we are led in § 107 is that a simple vortex-atom theory is not in a position to attempt to explain the law of the forcive between permanent magnets, if only for the reason that on such a theory no explanation of the inertia of matter has yet been developed. This difficulty is, however, not peculiar to the present special view of the electric field; any representation of a magnetic molecule, which assigns to it a purely cyclic motional constitution, is subject to an equal or greater difficulty in explaining why it is that the law of the forcive between magnets is the same as between currents, and not just the reverse.

What is required in order to obtain a decisive positive result is, that the assumption of a purely cyclic character for the motions associated with permanent magnets shall be avoided by giving the elasticity of the medium some kind of grip on them. The movements of rotation and vibration of the simple vortices which constitute a vortex-aggregate are not competent to secure this, however sudden they may be, for in the irrotational fluid motion the constraint of the rotational elasticity has only to reduce a labile condition of the medium into a stable one; thus there is no sensible play of elastic energy introduced, such as would be required to explain induction in a steady magnetic field.

One way of bringing about this desired interaction of magnetic with elastic energy, at the same time safeguarding the permanence of the atomic current, would be to make it a current of convection, *i.e.* to suppose the core of the vortex-ring to be made up of discrete electric nuclei or centres of radial twist in the medium. The circulation of these nuclei along the circuit of the core would constitute a vortex which can move about in the medium, without suffering any pressural reaction on the circulating nuclei such as might tend to break it up; the hydrodynamic stability of the vortex, in fact, suffices to hold it together. But its strength is now subject to variation owing to elastic action, so that the motion is no longer purely cyclic. A magnetic atom, constructed after this type, would behave like an ordinary electric current in a non-dissipative circuit. It would for instance be subject to alteration of strength by induction when under the influence of other changing currents, and to recovery when that influence is removed; in other words the Weberian explanation of diamagnetism would now hold good.

The monad elements (§ 70) out of which a magnetic molecule of this kind is built up are electric centres or nuclei of radial rotational strain. From what is known of molecular magnitudes, in connexion with electrochemical data, it would appear that to produce an intensity of magnetization of 1700 c.g.s., which is about the limit attainable for iron, these monad charges—or *electrons*, as we may call them, after Dr. JOHNSTONE STONEY—must circulate very rapidly, in fact with velocities not many hundred times smaller than the velocity of radiation.* Even a single pair of electrons revolving round each other at such a rate as this would produce a practically perfect secular vortical circulation in the medium; so that a magnetic molecule may quite well be composed of a single positive or right-handed electron and a single negative or left-handed one revolving round each other in this manner. We may in fact rigorously apply to the present problem the principle used by GAUSS for the discussion of secular effects in Physical Astronomy. Instead of proceeding by addition of the elementary effects produced by a planet as it moves from point to point of its orbit, GAUSS pointed out that the secular results as distinguished from mere periodic alternations are the same as if the mass of the planet were supposed permanently distributed round its orbit so that the density at any point is inversely proportional to the velocity the planet would have when at that point. Just in the same way here, the steady flow of the medium, as distinguished from vibrational effects, is the same as if each electron were distributed round its circular orbit, thus forming effectively a vortex-ring, of which however the intensity is subject to variation owing to the action of other systems.†

* Let q be the ionic charge, v its velocity, A the area of the orbit and l its length, n the number of atoms in 1 cub. centim.; then $n.q/l.v.\mathrm{A} = 1700$. From electrochemical data we may take $nq = 10^3$, and from molecular dimensions $\mathrm{A}/l = \frac{1}{2}.10^{-8}$; whence $v = 3.10^8$, which is of the order of about one hundredth of the velocity of radiation. This would make the periodic time come out about 10 times the period of luminous radiations.

† It may be observed that for the case of a simple diad molecule, composed of two equal and opposite

This mode of representation would leave us with these electrons as the sole ultimate and unchanging singularities in the uniform all-pervading medium, and would build up the fluid circulations or vortices—now subject to temporary alterations of strength owing to induction—by means of them.

115. It may be objected that a rapidly revolving system of electrons is effectively a vibrator, and would be subject to intense radiation of its energy. That however does not seem to be the case. We may on the contrary propound the general principle that whenever the motion of any dynamical system is determined by imposed conditions at its boundaries or elsewhere, which are of a steady character, a steady motion of the system will usually correspond, after the preliminary oscillations, if any, have disappeared by radiation or viscosity. A system of electrons moving steadily across the medium, or rotating steadily round a centre, would thus carry a steady configuration of strain along with it; and no radiation will be propagated away except when this steady state of motion is disturbed.

It is in fact easy to investigate the characteristics of this strain-configuration when the electric system is moving with constant velocity, say in the direction of the axis of x with velocity c. By § 97, the dynamical equations of the surrounding medium are

$$\left(\frac{d^2}{dt^2} - a^2\nabla^2\right)(f, g, h) = 0,$$

referred to co-ordinates fixed in space. The equations determining the disturbance relative to the electric system are derived by changing the co-ordinate x to a new relative co-ordinate x', equal to $x - ct$; this leaves spacial differentiations unaltered, but changes d/dt into $d/dt - cd/dx'$, thus giving

$$\left\{(a^2 - c^2)\frac{d^2}{dx'^2} + a^2\frac{d^2}{dy^2} + a^2\frac{d^2}{z^2}\right\}(f, g, h) = \left(\frac{d^2}{dt^2} - 2c\frac{d^2}{dx'dt}\right)(f, g, h).$$

In a steady motion the right-hand side of this equation would vanish; and the conditions of steady motion are thus determined by the solution of the ordinary potential equation for a uniaxial medium. The constants involved in the values of f, g, h so determined are connected by the fact that at a boundary of the elastic medium the rotation (f, g, h) must be directed along the normal. It follows at once for example that for a spherical nucleus * the rotation is everywhere radial. As the

electrons rotating round each other in equal orbits, their secular effects just cancel each other, so that the molecule as a whole is non-magnetic. This exact cancelling will not however usually occur when there are more than two electrons in the molecule, or when a number of molecules are bound together in a group as in the case of an iron magnet. Similar considerations also apply as regards the average electric moment of a molecule, which is in fact the electric moment of the Gaussian secular equivalent above described.

* J. J. THOMSON, 'Recent Researches . . .,' 1893, pp. 16–22, where the existence of a superior limit (*infra*) to possible velocities was first pointed out: also HEAVISIDE, 'Phil. Mag.,' 1889, *cf.* 'Electrical Papers,' vol. 2, pp. 501 *seqq.* The problem of the dynamics of moving charges appears to have been first attacked on MAXWELL's theory by J. J. THOMSON, 'Phil. Mag.,' 1881.

velocity of the electric system is taken greater and greater the permeability, in the direction of its motion, of the uniaxial medium of the analogy becomes less and less, and the field therefore becomes more and more concentrated in the equatoreal plane. When the velocity is nearly equal to that of radiation, the electric displacement forms a mere sheet on this plane, and the charge of the nucleus is concentrated on the inner edge of this sheet. The electro-kinetic energy of a current-system of this limiting type is infinite (§ 52), and so is the electrostatic energy; thus electric inertia increases indefinitely as this state is approached, so that the velocity of radiation is a superior limit which cannot be attained by the motion through the æther of any material system.

Again, the steady electric field carried along with it by a system rotating about a fixed axis with angular velocity ω is to be obtained by changing d/dt in the elastic equations into $d/dt - \omega d/d\theta'$, where θ' denotes relative azimuth around the axis; they therefore assume the form

$$\left(\nabla^2 - \frac{\omega^2}{a^2}\frac{d^2}{d\theta'^2}\right)(f, g, h) = 0,$$

of which the solution would be difficult. And the equations of the relative steady field for the most general case of uniform combined translation and rotation of an electric system, supposed still of invariable shape, are expressed in like manner, by taking the central axis of the movement as the axis of x, in the form

$$\left(\nabla^2 - \frac{c^2}{a^2}\frac{d^2}{dx'^2} - \frac{\omega^2}{a^2}\frac{d^2}{d\theta'^2}\right)(f, g, h) = 0.$$

The circuital character of (f, g, h) will allow us to reduce these three variables in cases of symmetry to a single stream-function, of which the slope along the normal at the surface of the nucleus must be null.

Any deviation from this steady motion of a molecule, produced by disturbance, will result in radiation which will continue until the motion has again become steady. If we roughly illustrate by the phenomena of the Solar system, the mean circular orbits of the planets will represent the steady motion, while disturbances introduce planetary inequalities which would give rise to radiation of corresponding periods. An apparent obstacle to the application of this hypothesis to the theory of the spectrum is that such a steady motion is not unique, its periods depend on the energy of the system; but, from whatever cause, the chemical energy of a molecule (which is *electric*, therefore æthereal) has a definite value quite independent of the amount of *material* kinetic energy that may be involved in its temperature and capacity for heat. The periods of the vibrations would thus be fixed by the electric energy; while the prevailing character of the disturbances, which determines the relative intensities of the radiations, would depend on temperature. If there are lines in any spectrum which have this kind of origin, we should expect to find simple linear relations between the *reciprocals* of their periods or wave-lengths, as in the Planetary Theory.

On the other hand the sharpness of the spectral lines shows that the waves in the æther are absolutely simple harmonic, and this would point to atomic rather than molecular vibrations, were it not that the molecule is so small compared with a wave-length and also the periods far too great for such an origin.*

116. A difficulty has been felt as to how the centre of rotational strain which represents an electron is possible without a discrete structure of the medium; the following explanations may therefore be pertinent. In the first place, it is essential to any simple elastic theory of the æther that the charge of an ion shall be represented by some permanent state of strain of the æther, which is associated with the ion and carried along by it. Such a strain-configuration (in the light of what follows) can hardly be otherwise than symmetrical all round the ion; even if the nucleus be not itself symmetrical, this symmetry will be attained at a sufficient distance away from it. Now in an isotropic medium a steady configuration of strain of this kind must consist of a radial displacement such as we could imagine to be produced by an intrinsic pressure in the nucleus, or of a radial twist as above described, or it may combine the two. But for a great variety of reasons, electric and optical phenomena have no relation to any compression of the æther; therefore the notion of an intrinsic radial twist is the only representation that is available. An ideal process for the creation of such a twist-centre has already been described in § 51 for the case of the rotational æther. A filament of the æther ending at the nucleus is supposed to be removed, and the proper amount of circulatory motion is to be imparted to the walls of the channel so formed, at each point of its length, so as to produce throughout the medium the radial rotational strain that is to be associated with the electron; when this has been accomplished the channel is to be filled up again with æther which is to be made continuous with its walls. On now removing the constraint from the walls of the channel, the circulation imposed on them will tend to undo itself, until the reaction against rotation of the æther with which the channel has been filled up balances that tendency, and an equilibrium state thus supervenes with intrinsic rotational strain symmetrically surrounding the nucleus. If on the other hand the æther had the properties of an elastic solid, and resisted shear but not rotation, the equations of *bodily* elasticity would remain just the same (§ 19); but the surfaces of shear of such a nucleus would be conical, with the channel by which the shear is introduced as their common axis, and when the constraint is removed the rotation imposed on the surface of this channel will undo itself and the shear thus all come out again, because the medium with which the channel is now filled up opposes no resistance to being rotated. Thus an elastic solid æther does not admit of any configuration of intrinsic strain such as would be required to represent an electric charge; and this forms an additional ground for limitation of that medium to a rotationally elastic structure. For an isotropic medium must be either elastic like a solid or fluid,

* See G. JOHNSTONE STONEY, "On the Cause of Double Lines and of Equidistant Satellites in the Spectra of Gases," 'Trans. Roy. Dublin Soc.,' 1891.

or rotationally elastic, or it may combine these two properties; there is no* other alternative.

As to the intrinsic nature of the rotational elasticity of the free æther, although it is an important corroboration of our faith in the possibility of such a medium to have Lord KELVIN's gyrostatic scheme by which it might be theoretically built up out of ordinary matter, yet we ought not to infer that a rotational free æther is necessarily discrete or structural in its ultimate parts, instead of being a *continuum*. As a matter of history, the precisely similar argument has been applied to ordinary solids; the fact that deformation induces stress has been taken, apparently with equal force, as evidence of molecular structure in any medium which exhibits ordinary elasticity. It is necessary to put some limit to these successive refinements; there must be a final type of medium which we accept as fundamental without further analysis of its properties of elasticity or inertia: and there seems to be no adequate reason why we should prefer for this medium the constitution of an elastic solid rather than a constitution which distortion does not affect,—perhaps there is just the reverse.

117. The fluidity of the medium allows us to apply the methods of the dynamics of particles to the discussion of the motions through it of these electrons or strain-configurations, and their mutual influences. The potential energy of a system of moving electrons will be the energy of the strain in the medium; unless their velocities are appreciable compared with the velocity of radiation, this will be a function of their relative positions alone. The kinetic energy is that of the fluid circulation of the medium, which will under the same circumstances be a quadratic function of the velocity-components of the electrons, with coefficients which are functions of their relative positions. When however their velocities approach that of radiation the problem must be treated by the methods appropriate to a *continuum*, and cannot be formulated merely in terms of the positions of the electrons at the instant. It will suffice for the present to avoid the difficulties of the general case by supposing the velocities to be small, and the strain-configuration of each electron therefore carried on unaltered by it; as the correction required depends on $(c/a)^2$ it will possibly be negligible for any actual problem.

Let us then consider a single electron represented by a charge e moving along the direction of the axis of x with velocity v. The components of rotation in the medium due to its presence are at any instant $-e\,(d/dx,\ d/dy,\ d/dz)\,r^{-1}$, and those of the displacement current are derived from them by operating with the factor $-v\,d/dx$. This displacement current is the curl of the velocity of the medium, whence it may be easily verified that this velocity is $ev\,(0,\ -d/dz,\ d/dy)\,r^{-1}$, being a circulation round the line of motion of the electron.† The kinetic energy is thus $\frac{1}{2}(ev)^2\int(y^2+z^2)\,r^{-6}\,d\tau$;

* Professor FITZGERALD remarks that it might, conceivably, resist absolute linear displacement. A hypothesis of this sort, which is on a lower plane than those mentioned above, is in fact involved in the usual expositions of FRESNEL's dynamics of double refraction.

† It is to be observed that we cannot expect to obtain an expression for the displacement in the

which is equal to $4\pi/3a.(ev)^2$, if the nucleus which bounds internally the strained medium is spherical and of radius a. The potential energy of elastic strain in the medium is, on the same supposition, by the ordinary electrostatic formula, $\frac{1}{2}(e\text{V})^2/a$, where V is the velocity of electric propagation. We assume that the nucleus of the electron has no other intrinsic inertia of its own, and no other potential energy of its own; under these circumstances its potential and kinetic energies will be of the same order of magnitude only when its velocity is comparable with that of radiation. In that case the present formulæ are not applicable, except merely to indicate the orders of magnitude; but we can conclude that, in a steady molecular configuration of electrons, where there must be an increase of kinetic energy equal to the potential energy which has run down in their approach, the velocities of the constituent electrons must be comparable with that of radiation, just as the above estimate from magnetic data suggested.

Suppose there are two electric systems in the field producing velocities (u, v, w) and (u', v', w') respectively. The kinetic energy is now

$$\tfrac{1}{2}\int\{(u + u')^2 + (v + v')^2 + (w + w')^2\}\, d\tau,$$

of which the part that involves their mutual action is $\int(uu' + vv' + ww')\, d\tau$. If the velocity (u, v, w) belongs to an electron $(e, \mathbf{v})$ as above, the mutual part of the kinetic energy is $e\mathbf{v}\int(-\, v'\, d/dz + w'\, d/dy)\, r^{-1}\, d\tau$, or on integration by parts $-\, e\mathbf{v}\int(v'n - w'm)\, r^{-1}\, d\text{S} - e\mathbf{v}\int(dw'/dy - dv'/dz)\, r^{-1}\, d\tau$, of which the former part is null when the external boundary is very distant. Thus the mutual electro-kinetic energy is $-\, e\mathbf{v}\int r^{-1}\, df'/dt\, d\tau$, where f' is the component parallel to $\mathbf{v}$ of the electric displacement belonging to the other system.

If the other system is also an electron $(e', \mathbf{v}')$ the total electro-kinetic energy is

$$\text{T} = \tfrac{1}{2}\,\text{L}\,(e\mathbf{v})^2 + \tfrac{1}{2}\,\text{L}'\,(e'\mathbf{v})^2 + \text{M}\,.\,e\mathbf{v}\,.\,e'\mathbf{v}',$$

where L, L′ are as determined above, having the values $8\pi/3a$, $8\pi/3a'$ when the nuclei are spherical, while $\text{M} = r^{-1}\cos(ds\,.\,ds') + \frac{1}{2}\, d^2r/ds\, ds'$, in which ds, ds' are in the directions of $\mathbf{v}$, $\mathbf{v}'$, and r is the distance between the monads.* The potential energy is

$$\text{W} = \tfrac{1}{2}\,\text{A}\,(e\text{V})^2 + \tfrac{1}{2}\,\text{A}'\,(e'\text{V})^2 + \text{B}\,.\,e\text{V}\,.\,e'\text{V},$$

where A and A′ are as determined above, being the reciprocals of the radii when the nuclei are spherical, and $\text{B} = r^{-1}$. The equations of motion of the two electrons may

medium which is due to an electron; for the electron is part of the original constitution of the medium, and we cannot imagine it to be removed altogether. It may, however, be moved on into a new position, and we can then determine, as above, the displacement in the medium produced by this change of its locality.

* The calculation of M is given concisely by H. LAMB, 'Proc. Lond. Math. Soc.,' June 1883, p. 407; the result is given also by HEAVISIDE, 'Electrical Papers,' vol. 2, p. 501.

now be formed in the Lagrangian manner, and will hold good so long as the motions are fairly slow compared with radiation.

The question however arises whether we should not associate with the electric inertia of an ion of this kind a much greater inertia of matter to which the ion belongs. When we trace as above the consequences of refraining from doing so, we arrive at the result that these free electrons can be projected by their mutual actions, with velocities which are a considerable fraction of that of radiation. Bearing in mind the phenomena of the Solar corona and of comets' tails, and certain electric phenomena in vacuum tubes,* where some modification of the æther which affects light by reflexion and otherwise is projected with velocities of that order, there seems to be no reason for the summary exclusion of such an hypothesis as the present,† especially as an electrically neutral molecule could attain no such velocities, and would comport itself more like ordinary matter.

118. The circumstances of steady motion may be illustrated by a calculation for the case of two electrons; the same method would clearly also apply to a greater number. The kinetic energy of two electrons e_1 and e_2, whose co-ordinates are $(x_1\ y_1\ z_1)$ and $(x_2\ y_2\ z_2)$, moving under their mutual influence, is, by § 117,

$$T = \tfrac{1}{2}L_1 e_1^2 (\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \tfrac{1}{2}L_2 e_2^2 (\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) + \frac{e_1 e_2}{2r}(2\dot{x}_1\dot{x}_2 + \dot{y}_1\dot{y}_2 + \dot{z}_1\dot{z}_2),$$

the axis of x being parallel to their mutual distance r.

Let us take the case when they revolve steadily in the plane of xy with angular velocity ω round a common centre, at distances r_1, r_2 from it, where $r_1 + r_2 = r$. The kinetic reaction on e_1 resolved parallel to x is

$$\frac{d}{dt}\frac{dT}{d\dot{x}_1} - \frac{dT}{dx_1} = L_1 e_1^2 \ddot{x}_1 + e_1 e_2 \frac{d}{dt}\left(\frac{\dot{x}_2}{r}\right) - \frac{e_1 e_2}{2r^2}\cos\theta\,(2\dot{x}_1\dot{x}_2 + \dot{y}_1\dot{y}_2 + \dot{z}_1\dot{z}_2)$$

in which θ, the angle between r and x, is null; while $\ddot{x}_1 = \omega^2 r_1$, $\ddot{x}_2 = -\omega^2 r_2$, $\dot{y}_1 = \omega r_1$, $\dot{y}_2 = -\omega r_2$. On equating this to the electrostatic attraction, we have

$$\left(-L_1 e_1^2 r_1 - e_1 e_2 \frac{r_2}{r} + e_1 e_2 \frac{r_1 r_2}{2r^2}\right)\omega^2 = e_1 e_2 \frac{V^2}{r^2}.$$

Similarly

$$\left(-L_2 e_2^2 r_2 - e_1 e_2 \frac{r_1}{r} + e_1 e_2 \frac{r_1 r_2}{2r^2}\right)\omega^2 = e_1 e_2 \frac{V^2}{r^2}.$$

Hence

$$\left(L_1 e_1 - \frac{e_2}{r}\right) e_1 r_1 = \left(L_2 e_2 - \frac{e_1}{r}\right) e_2 r_2,$$

* Professor FITZGERALD suggests the addition to this list of auroras and magnetic storms.

† Professor J. J. THOMSON informs me that he finds the velocity of the negative rays in vacuum tubes to be about 2×10^7 c.g.s.

which determines the ratio of r_1 to r_2 in the steady motion; and then the value of ω gives the period of the rotation.

For example when the electrons are equal and opposite $e_1 = -e_2$, $L_1 = L_2$, and $r_1 = r_2$: thus the square of the velocity of either, $(\frac{1}{2}\omega r)^2$, is equal to $V^2/(2Lr - \frac{3}{2})$. For the case of a spherical nucleus of radius a, $L = 8\pi/3a$; thus the velocity of either must be considerably less than $\frac{1}{5}V$, which is small enough to allow this method to approximately represent the facts for that case.

It may be observed that in the general problem of the dynamics of a system of n electrons, the equations of conservation of momentum assume the forms

$$\frac{dT}{d\dot{x}_1} + \frac{dT}{d\dot{x}_2} + \ldots + \frac{dT}{d\dot{x}_n} = const.,$$

with similar equations in y and z. For the case of two electrons moving in the same line, the equations of energy and momentum determine the motion completely; their forms illustrate the complexity of the electric inertia which is involved.

119. In the general theory of electric phenomena it has not yet been necessary to pay prominent attention to the molecular actions which occur in the interiors of conductors carrying currents: it suffices to trace the energy in the surrounding medium, and deduce the forces acting on the conductors, considered as continuous bodies, from the manner in which this energy is transformed. The calculations just given suggest a more complete view, and ought to be consistent with it; instead of treating a conductor as a region effectively devoid of elasticity, we may conceive the ions of which it is composed as free to move independently, and thus able to ease off electric stress; the current will thus be produced by the convection of ionic charges. Now if all the atoms took part equally in this convection, their velocity would be exceedingly small; a current of i ampères per square centimetre would imply a velocity of about $10^{-4}\,i$ centimetres per second. The kinetic energy of an ion due to intrinsic electric inertia is, according to the formula above, $\frac{1}{2}\,8\pi/3a.\,(ev)^2$, where a is of order $< 10^{-8}$, e of order 10^{-21}; this would imply as above a centrifugal electric force of intensity $8\pi/3a.e.v^2/R$, which may be of order $10^{-19}\,i^2$, acting on this particular ion when it is going round a curve of radius R. Now even if the conductor were of copper, the slope of potential along it would be, with this current intensity, as much as $164i$. The effects of the intrinsic electric inertia are therefore so far quite beyond the limit of observation. We have however been taking the electric drift v to be the only velocity of the ions or electrons. If they possess a velocity of their own in fortuitous directions of order V, the average centrifugal electric force on an electron due to the current will possibly be as high as $8\pi/3a\,.\,e\,.\,vV/R$, because change of sign of V does not change the sign of the force. This would still hardly be detectable even if V were comparable with the velocity of radiation.

But an electric force of a cognate kind has in fact already been looked for and detected by E. H. HALL. When the current is moving in a field of magnetic force H at right angles to itself, there must be an electric force at right angles to both, acting on each particular ion, of which the intensity is vH.* For example if H were 10^3 c.g.s., this electric force would be $10^3 v$ c.g.s. or $10^{-5} v$ volts; in the rough estimates of the last paragraph it would be of order $10^{-1} i$, as compared with a slope of potential along the conductor of $164 i$; therefore it is quite amenable to observation, so that we must consider it more closely. As there are also an equal number of negative ions moving in the opposite direction, they must give rise to an opposite electric force acting on them; thus the total transverse electric force, as observed, will be reduced from the above value in the ratio $(v_2 - v_1) / (v_2 + v_1)$, where v_2 and v_1 are the velocities of drift of the positive and negative ions, which may be different just as KOHLRAUSCH found them to be in ordinary electrolysis. The absolute velocity V of an ion does not affect the result in this case. This view would therefore make the sign of the HALL effect depend on whether positive or negative ions conveyed most of the current.

120. The electromagnetic or mechanical forces acting on the conductors conveying the currents are on the other hand to be derived from the energy function, considered as potential after change of sign as in § 57, by the method of variations. For the reasons given above, the effect of the term $\Sigma \frac{1}{2} Le^2$, involving intrinsic electric inertia, is in the present problem inappreciable, except as giving a kind of internal gaseous pressure if the velocities of free electrons were comparable with that of radiation. The total electrokinetic energy is thus practically

$$\iint M i ds i' ds', \quad \text{where } M = r^{-1} \cos (ds, ds') + \tfrac{1}{2} d^2 r / ds\, ds';$$

and on the present hypothesis the energy may be considered to be correctly localized in this formula.

If the currents are uniform all along the linear conductors, the second term in M integrates to nothing when the circuits are complete, and we are thus left with the AMPÈRE-NEUMANN expression for the total energy of the complete currents, from which the Amperean law of force may be derived in the known manner by the method of variations. But it must be observed that, as the localization of the energy is in that process neglected, the legitimate result is that the forcive of AMPÈRE,

* It is assumed here that all forces of electric origin acting on the moving atomic charges are primarily electric forces; in accordance with the previous theory (§ 57) it is only the part of the energy-change which cannot be compensated by electromotive work, that reveals itself ultimately as a forcive working mechanically on the aggregates which constitute conducting bodies, or as heat in case it is too fortuitously constituted to admit of transformation into a regular mechanical working forcive. This ultimate destiny is independent of any question as to the origin of the inertia of the atoms. Thus the steady and unlimited fall of the electric resistance of metals with lowering of the temperature, found by DEWAR and FLEMING, shows that the frittering away of electric energy into heat in a metallic conductor depends upon the velocity of fortuitous agitation of the molecules, and would disappear when it ceased. The regular transfer of the electrons would thus involve no degradation of electric energy (§ 115), except so far as it is disturbed and mixed up by the thermal agitations of the molecules of the conductors. In electrolytes the dependence of the degree of ionisation on the temperature may mask the direct effect of the thermal agitations.

together with internal stress as yet undetermined between contiguous parts of the conductors, constitute the total electromagnetic forcive: it would not be justifiable to calculate the circumstances of internal mechanical equilibrium from the Amperean forcive alone, unless the circuits are rigid. For example, if we suppose that the circuits are perfectly flexible, we may calculate the tension in each, in the manner of LAGRANGE, by introducing into the equation of variation the condition of inextensibility. We arrive at a tension $i\int \mathrm{M}i'ds'$, where i is the current at the place considered; whereas the tension as calculated from AMPÈRE'S formula for the forcive would in fact be constant, the forcive on each element of the conductor being wholly at right angles to it.

The general case when the currents are not linear is also amenable to simple analysis. The energy associated with any linear element ids is $ids\int \mathrm{M}i'ds'$; which is equal to ids multiplied by the component of the vector-potential of the currents in the direction of ds, when the conduction and convection currents move round complete circuits. Thus, changing our notation, the energy associated with a current (u, v, w) in an element of volume $d\tau$ is $(\mathrm{F}u + \mathrm{G}v + \mathrm{H}w)\, d\tau$. In this expression (F, G, H) is the vector-potential of the currents; if there is also magnetism in the field, there will be a part of this vector-potential due to it, which may be calculated from the equivalent Amperean currents. Thus for a single Amperean circuit, $\mathrm{F} = i\int r^{-1}\, dx$, which by STOKES' theorem $= i\int(\mu d/dz - \nu d/dy)\, r^{-1}\, d\mathrm{S}$, where (λ, μ, ν) is the direction-vector of the element of area $d\mathrm{S}$; hence the magnetic part of the vector-potential is $(\mathrm{B}d/dz - \mathrm{C}d/dy,\ \mathrm{C}d/dx - \mathrm{A}d/dz,\ \mathrm{A}d/dy - \mathrm{B}d/dx)\, r^{-1}$, which agrees with the assumption in § 110. It will be observed that in the vector-potential of the field, as thus introduced, there is no indeterminateness; it is defined by the expression for the energy, as above.

We may complete this mode of expression of the energy by including the energy of the magnetism in the system due to the field in which it is situated. For a single Amperean atomic circuit it is $i\int(\mathrm{F}dx + \mathrm{G}dy + \mathrm{H}dz)$, which is by STOKES' theorem $i\int\{\lambda\, (d\mathrm{H}\,/\,dy - d\mathrm{G}\,/\,dz) + \ldots + \ldots\}\, d\mathrm{S}$; thus the energy of the magnets is $\int(\mathrm{A}\alpha + \mathrm{B}\beta + \mathrm{C}\gamma)\, d\tau$, where (α, β, γ) is the magnetic force due to the external field as usually defined; this follows from the formulæ for (F, G, H) already obtained. There is also the intrinsic energy of the magnets due to their own field; by the well-known argument derived from the work done in their gradual aggregation, the co-ordinated part of this is $\frac{1}{2}\int(\mathrm{A}\alpha_0 + \mathrm{B}\beta_0 + \mathrm{C}\gamma_0)\, d\tau$, where $(\alpha_0, \beta_0, \gamma_0)$ is the force of their own field. These terms will add on without modification to the other part of the electrokinetic energy for the purpose of forming dynamical equations, provided we assume as above that the magnetic motions are not of a purely cyclic character. This sketch will give an idea of how magnetism enters in a dynamical theory which starts from the single concept of electrons in movement.

The energy being thus definitely localized, and all the functions precisely defined, we derive in the Lagrangian manner the electric force

$$(\mathrm{P}, \mathrm{Q}, \mathrm{R}) = -\left(\frac{d\mathrm{F}}{dt} + \frac{d\Psi}{dx}, \ \frac{d\mathrm{G}}{dt} + \frac{d\Psi}{dy}, \ \frac{d\mathrm{H}}{dt} + \frac{d\Psi}{dz}\right)$$

when Ψ is some function of position as yet undetermined, whose value is to be adjusted to satisfy the restriction to circuital flow which the present analysis for conduction and convection currents involves. The electrodynamic forcive acting on the conductors carrying the currents is

$$(\mathrm{X}, \mathrm{Y}, \mathrm{Z}) = -\left(u\frac{d\mathrm{F}}{dx} + v\frac{d\mathrm{G}}{dx} + w\frac{d\mathrm{H}}{dx}, \ u\frac{d\mathrm{F}}{dy} + v\frac{d\mathrm{G}}{dy} + w\frac{d\mathrm{H}}{dy}, \ u\frac{d\mathrm{F}}{dz} + v\frac{d\mathrm{G}}{dz} + w\frac{d\mathrm{H}}{dz}\right);$$

but this involves, in addition to the usually recognized forcives of AMPÈRE's law and FARADAY's rule, a forcive in the direction ds of the resultant current Γ and equal to $-\Gamma\, d\mathrm{N}/ds$, where N is the component of the vector-potential in the direction of ds. This additional forcive may be represented as balanced by a tension $i\mathrm{N}$, in each filament or tube of flow carrying a current i, just as above. The existence of this tension seems to admit of easy test by a suitable modification of AMPÈRE's third crucial experiment.

It is now a simple matter to complete this theory, which at present applies to circuital convection and conduction currents, so as to include the effect of convection without this restriction. It will suffice to consider a uniform current i' flowing in an open path, thus accumulating electrification at one end and removing it from the other end. The second term in M when integrated with respect to ds' yields $ids \,.\, \tfrac{1}{2} i' \left|\frac{dr}{ds}\right|_1^2$; thus in the energy of the element of ids there is a term $ids \,.\, \tfrac{1}{2}\int \frac{d\rho}{dt}\cos\theta\, d\tau$, where θ is the angle between ds and the distance r of $d\tau$ from it, and $d\rho/dt$ is the rate of increase of the density of electrification at the element $d\tau$. Thus there is an additional electric force $-\tfrac{1}{2}\frac{d}{dt}\int\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)\frac{d\rho}{dt}\, d\tau$, and an additional electromagnetic force $\tfrac{1}{2}\int\left(\frac{y^2+z^2}{r^3}, \frac{z^2+x^2}{r^3}, \frac{x^2+y^2}{r^3}\right)\frac{d\rho}{dt}\, d\tau$, where (x, y, z) have reference to the element $d\tau$ as origin. These expressions are appropriate where, in place of following the convection of single electrons, we contemplate the change of electric density at a point in space; they suffer from an apparent want of convergency, which would be real were it not that $\int\rho\, d\tau$ is null.

121. It may be observed finally, that the question as to how far it is permissible to entertain the view that the non-electric properties of matter may also be deducible from a simple theory of free electrons in a rotationally fluid æther, has hardly here been touched upon. The original vortex-atom theory of matter has scarcely had a beginning made of its development, except in VON HELMHOLTZ's fundamental discovery of the permanence of vortices, and the subsequent mathematical discussions respecting their stability. How far a theory like the present can take the place of or supplement

the vortex theory, is therefore a very indefinite question. In the absence of any such clue, a guiding principle in this discussion has been to clearly separate off the material energy involving motions of matter and heat, from the electric energy involving radiation and chemical combination, which alone is in direct relation to the æther. The precise relation of tangible matter, with its inertia and its gravitation, to the æther is unknown, being a question of the structure of molecules; but that does not prevent us from precisely explaining or correlating the effects which the overflow of æthereal energy will produce on matter in bulk, where alone they are amenable to observation.

Optical Dispersion; and Moving Media.

122. The view of optical dispersion developed in the first part of this paper, on the basis of MacCullagh's analysis, has its foundation in the discreteness of the medium, the dispersion being assigned to residual terms superposed on the average refraction. The cause of the refraction itself is found in the influence of the contained molecules, which are constituted in part at least of mobile electrons and so diminish the effective elasticity of the medium. Now if these molecules formed a web permeating the medium, with connexions of its own, this web would act as an additional support, and the optical elasticity would, if affected at all, be increased. But it is different if the molecules are so to say parasitic, that is if they are configurations of strain in the æther itself, and their energy is thus derived directly from the æther and not from an independent source. To more clearly define the effective elasticity in that case, let us suppose a uniform strain of the type in question to be imparted to the medium by the aid of constraints; it follows from the linearity of the elastic relations that the stress involved in this superposed strain will be that corresponding to the elastic coefficient of the free æther, for there is by hypothesis no web involved with extraneous elasticity. Now suppose the constraints required to maintain this pure strain to be loosened; the molecules will readjust themselves into a new equilibrium position which involves less energy, and this diminution of the total energy of the strain implies a diminution of the corresponding effective elastic coefficient. This analysis has to do with the statical elasticity; in electrical terms it corresponds to the explanation of Faraday and Mossotti as to how it is that the ratio of electric force to electric induction is diminished by the presence of polarized molecules. If, however, in a problem of vibration, the displacement of the medium involved in the molecules thus settling down into a new conformation of equilibrium, after the constraints are removed, is comparable with that involved in the original strain, the kinetic energy of the medium will be affected by the molecules as well as the strain energy, and the circumstances of propagation will depend on the period of the waves. As the present theory involves altered effective elasticity but unaltered effective inertia, this dependence can be but slight; in other words the orientation of the molecules does not involve any considerable additional kinetic energy of displacement of the medium

in comparison with the work done by electric forces; just as was to be anticipated from § 117, where it has been shown that to produce a comparable motional effect very great velocity of translation or rotation of the molecule is requisite, not the comparatively small velocity of movement of the elements of the medium caused by a wave passing over it.

This amounts in fact to asserting that it is only the electric inertia of the molecules that affects the electric waves. Their material inertia is quite a different and secondary thing from the inertia of the æther;* on an electric theory it can have no direct influence on the radiation.

It seems clear also that if the molecules, in their relations to the æther, behave as systems of grouped electrons, their presence cannot disturb the fluidity of that medium, so that the foundation given above (§ 28) for MacCullagh's dispersion theory remains valid.

123. Let us contrast the merits of this view of dispersion with those of the type of theory in which it is ascribed to imbedded ponderable molecules. It has been shown,† that for an elastic-solid theory (or any theory treated by the method of rays, § 22) to give an account of the observed laws of reflexion at the surfaces of transparent media, the inertia may be supposed to vary from one medium to another, or else the rigidity, but not both. Thus, setting aside the latter alternative for other reasons, the molecules must act simply as a load upon the vibrating æther; this requires that their free periods must be very long compared with the period of the waves, which is a very reasonable hypothesis. But if the optical rigidity is absolutely the same for all media, we are bound to explain not only the dispersion, but the whole refraction, by the influence of the inertia of the load of molecules; thus to explain dispersion we have to take refuge in Cauchy's doctrine of simple discreteness of the medium.

Now let us formulate the problem of wave-propagation in a discrete medium of this kind. It will be a great simplification to consider stationary vibrations instead of progressive undulations; let us therefore combine two equal wave-trains travelling in opposite directions, and so obtain nodes and antinodes. We may imagine the continuity of the medium severed at two consecutive antinodes; thus the problem before us is to find the gravest free period of a block of the medium, forming half a wave-length, with its imbedded molecules. To represent in a simple manner the general features of this question, let us take Lagrange's problem of the vibrations of a stretched cord with n equidistant beads fixed on it. This will be a sufficient model of the case now in point, where the molecules act simply as a load; but if we are to consider possible influence of their free periods, so as to include anomalous dispersion as well as ordinary dispersion, we must also endow the beads of the model

* *Cf.* Lord Kelvin, 'Baltimore Lectures on Molecular Dynamics,' 1884, Lecture xx.

† Lord Rayleigh, 'Phil. Mag.,' Aug., 1871.

with free periods, which may be done by imposing an elastic restoring force on each.* In this latter case however the difficulty of representing the nature and origin of the restoring force detracts very seriously from the efficiency and validity of this mode of representation. Fortunately the simpler and more definite case is all we now require; when the mass is all concentrated in the beads, LAGRANGE finds that the velocity of propagation of a wave whose length contains n beads is $V_0 \sin \pi/2n \div /\pi/2n$.† For the case of an ordinary light-wave there are about 10^3 molecules in a wave-length, so that the dispersion for an octave should by this formula be about $\frac{1}{6}(\pi/2000)^2$ of the velocity, which is enormously smaller than the corresponding dispersion, usually about one per cent., of actual optical media.

Thus we must conclude that, while the present form of MACCULLAGH's theory ascribes refraction to the defect of elastic reaction of the molecules, and dispersion to the influence of their free periods, so also the elastic-solid theory must ascribe refraction to loading by the mass of the molecules, and dispersion to the influence of their free periods. In these respects the two theories run parallel, and there is not much to choose between them; a model constructed on either basis would fairly represent the phenomena of dispersion. The latter ascribes the influence of the matter to nodules of mass, in the æthereal, not by any means the material or gravitative sense, supposed distributed through the medium; the former finds its cause in the properties of the nuclei of intrinsic strain, or electrons. On either view, FRESNEL's laws of reflexion are a first approximation obtained by neglecting dispersion, and are as we know departed from by a medium which produces anomalous dispersion of the light, even for wave-lengths which suffer no sensible absorption.‡

* *Cf.* Lord KELVIN, 'Baltimore Lectures,' 1884.

† LAGRANGE, 'Méc. Anal.' ii., 6, § 30; RAYLEIGH, 'Sound,' § 120; ROUTH, 'Dynamics,' vol. 2, § 402.

‡ The most definite form which the YOUNG-SELLMEIER type of theory has yet assumed is that of Lord KELVIN ('Baltimore Lectures,' 1884). The author begins with an illustrative molecule, consisting of a core of very high inertia joined by elastic connexions to a chain of outlying satellites of which the last only is in connexion with the æther. The core being thus practically unmoved, the whole system is so to speak anchored to it, and the mass of the core does not come into account. Such an illustration gives very vivid representations of absorption and fluorescence. After working out the formula for the index of refraction in the manner of LAGRANGE's dynamics of linear systems, a transformation is suggested by consideration of the zeros and infinities of the function representing the index, which gives *à priori* a result whose validity is far wider than any special illustration, in the form

$$\mu^2 = 1 + \frac{c_1\tau^2}{\rho}\left\{-1 + \frac{q_1\tau^2}{\tau^2 - \kappa_1^2} + \frac{q_2\tau^2}{\tau^2 - \kappa_2^2} + \ldots\right\},$$

where τ is the period of the waves, $\kappa_1, \kappa_2 \ldots$ are the free periods of the molecule, and the coefficients $q_1, q_2 \ldots$ depend on the distribution of the energy of the steadily vibrating molecule amongst these periods. On this theory the æther is *not* simply loaded by the molecule, but the coefficient c_1 depends on the manner in which the molecule is anchored in space; the theory is accordingly in difficulties with regard to double refraction and reflexion (*loc. cit.*, Lecture xvi.), of which the former is not a dispersional phenomenon.

The analogous electric theory explained above appears to be free from these difficulties. The

124. The analogy just mentioned suggests a fresh search for a purely dynamical explanation of FRESNEL'S formula for the influence of motion of the medium on the velocity of light, of which we had previously to be content with an indirect demonstration on the basis of the law of entropy. In the first place, we shall consider the usually received proof,* on the theory of a loaded mechanical æther. Let ρ be the density of the æther and ρ' that of the load, and let ϑ be the displacement of the medium; the equation of propagation for the medium at rest is $(\rho + \rho')\, d^2\vartheta/dt^2 = \kappa\, d^2\vartheta/dx^2$; the equation for a medium in which the load ρ' is moving on with velocity v in the direction of propagation is

$$\rho \frac{d^2\vartheta}{dt^2} + \rho' \left(\frac{d}{dt} + v\frac{d}{dx}\right)^2 \vartheta = \kappa \frac{d^2\vartheta}{dx^2}.$$

We have clearly $\kappa/\rho = V^2$, $\kappa/(\rho + \rho') = V^2/\mu^2$, where V is the velocity of propagation in free æther; and on substituting $\vartheta = A \exp 2\pi/\lambda \,.\, \iota\,(x - V_1 t)$, we find for V_1 the velocity of propagation in the moving medium the value $V\mu^{-1} + v\,(1 - \mu^{-2})$, which is FRESNEL'S expression. This explanation precisely fits in with our previous conclusion, that on a mechanical theory the matter must affect the inertia but not at all the elasticity of the medium, except as regards the dispersion; and conversely, it may be used as independent evidence for that assumption.

The treatment of the same problem on the theory of a rotational æther follows a rather different course. By the hypothesis, the electric displacement or strain ϑ_2 due to orientation of the molecules may be treated as derived, by an equilibrium theory, from the inducing displacement ϑ_1 which belongs to the waves and provides the stress by which they are propagated. That part ϑ_2 of the electric displacement is in internal equilibrium at each instant with the displaced position of the molecules, and so furnishes no stress for the wave-propagation. The relation between ϑ_1 and the total displacement $\vartheta_1 + \vartheta_2$ is that of electrostatics, $\vartheta_1 + \vartheta_2 = K\vartheta_1$, where K is the effective specific inductive capacity of the medium. The equation of propagation when the medium is at rest is $\rho\, d^2(\vartheta_1 + \vartheta_2)/dt^2 = \kappa\, d^2\vartheta_1/dx^2$, showing that the

relation of the average disturbance of the molecule to the disturbance of the æther is there introduced simply by means of an experimental number, the specific inductive capacity of the medium. The correlative mechanical hypothesis would require us, not to anchor a massive core of the molecule in space, but to introduce a coefficient to express the ratio of the displacement of the molecule to the displacement of the medium on some appropriate kind of equilibrium theory,—thus in fact to directly load the æther, and refer only the variable part of dispersion to the free periods of the molecule; but such an idea would introduce all kinds of difficulties with respect to the kinetic theory of gases and material motions in general. In the electric theory these difficulties are evaded by the principle that the inertia of matter is different in kind from the inertia of æther; the one is subject to electromagnetic forcive, the other to electromotive forcive.

The recent discovery of an upper limit beyond which radiations that can travel in a vacuum do not travel across air, has an important bearing on the present subject.

* *Cf.* GLAZEBROOK, "On Optical Theories," 'Brit. Assoc. Report,' 1882.

velocity of the waves is $(\kappa/K\rho)^{\frac{1}{2}}$, so that $K = \mu^2$. The equation of propagation when the molecules are moving through the stationary æther with velocity v in the direction of the wave-motion, is

$$\rho \frac{d^2\vartheta_1}{dt^2} + \rho \left(\frac{d}{dt} + v\frac{d}{dx}\right)^2 \vartheta_2 = \kappa \frac{d^2\vartheta_1}{dx^2},$$

where $\vartheta_2 = (\mu^2 - 1)\vartheta_1$ as above. Thus, V_1 being the velocity of the wave, and V the velocity of propagation in free æther, we have just as before

$$V_1^2 + (V_1 - v)^2(\mu^2 - 1) = V^2,$$

giving very approximately $V_1 = V\mu^{-1} + v(1 - \mu^{-2})$, which is FRESNEL'S law.

The exact expression for V_1 merely modifies the first term of FRESNEL'S approximation by a correction involving $v^2(1 - \mu^{-2})$, which does not change sign with v; thus in the application to MICHELSON'S second-order experiment there is no essential modification, and his negative result remains outside the scope of this analysis.

125. An important corollary to the present theory is suggested and confirmed by the experiments of RÖNTGEN on the convection of excited dielectrics, mentioned above (§ 60). When a material dielectric is moved across an electric field, each ion of the group which constitutes one of its molecules produces its own convection current, composed partly of change of electric displacement in the surrounding free æther, but completed and made circuital by the actual convection of the ionic charge itself When, as in RÖNTGEN'S experiment, the configuration in space does not change by the motion, so that there is no displacement-current in the surrounding æther, it is easy to see that the total electromagnetic effect is the same as if the dielectric were magnetized to an intensity which is at each point the vector product of its velocity of movement and its electric moment per unit volume, the latter being $(K - 1)/4\pi$ times the electric force at the place. We have just seen (§ 124) that this is in accord with the optical aspect of convection of transparent matter.

I have much pleasure in expressing my deep obligation to Professor G. F. FITZGERALD for a very detailed and instructive criticism of this paper with which he has favoured me. I have been much guided by his comments in revising the paper, and would have made still more use of them but for the length to which it had already run. I need hardly state however that he is not to be held responsible for any of the views herein expressed.

My best thanks are also due to Mr. A. E. H. LOVE for a criticism at an earlier stage, from which I derived much advantage.