On the Force Exerted on a Magnetic Particle by a Varying Electric Field.

By J. G. LEATHEM, M.A., D.Sc., Fellow of St. John's College, Cambridge.

(Communicated by Sir J. Larmor, F.R.S. Received May 21,—
Read June 26, 1913.)

With a view to explaining magnetism as a purely electrical phenomenon it is customary in the modern theory of electromagnetism to define magnetic force as a solenoidal vector whose curl is 4π times the electric current, and the magnetisation of a material element as half the angular moment of the motion of electricity in the element.* But it is clear that the definitions remain unjustified until it has been shown that the relations of these quantities to one another and to the other quantities of the theoretical formulation are the same as the relations which experience indicates as subsisting between the corresponding physical quantities.

It is, therefore, an essential part of the test of the theory to ascertain the theoretical value of the force exerted by a varying external field upon a magnetic particle.

The particle is supposed to contain electric charge, whether continuously or discretely distributed it is not necessary at the moment to specify. This charge is supposed to be in motion relative to the particle, and the force exerted on the particle by the electromagnetic field is simply the resultant of the forces which the field exerts on the electric charge.

Let w denote the velocity of an origin O situated in the particle and moving with it, and let an element of electric charge de belonging to the particle have co-ordinates x, y, z referred to non-rotating axes with O as origin. Then the velocity of de has components $w_x+\dot{x}$, $w_y+\dot{y}$, $w_z+\dot{z}$, and may be denoted by $w+u$.

The external field is specified by the magnetic intensity H, (α, β, γ), and the æthereal displacement D, (f, g, h). If these letters without suffix denote the values at O, the value at (x, y, z) is derived by means of the operator of Taylor's expansion, namely

$$\exp\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right).$$

In terms of the electromagnetic system of units the force on an element of charge is

$$\{4\pi c^2 \mathrm{D}+[w+u,\ \mathrm{H}]\}\ de,$$

* *Cf.* H. A. Lorentz, 'Encyk. der Math. Wiss.,' vol. v, 2, p. 181; and Larmor, 'Æther and Matter,' § 64.

where c is the velocity of light, and the square bracket denotes the vector-product. Consequently the resultant force on the particle is

$$\mathrm{F} = \int \left\{ \exp\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \right) \right\} \{4\pi c^2 \mathrm{D} + [w+u, \mathrm{H}]\}\, de,$$

the integral being extended to all the charges in the particle. Here the exponential operator applies only to the components of D and H; it will, of course, be expanded, and all but the earlier terms will be considered negligible.

The following notation is convenient:—

$$\int de = e, \qquad \int (x, y, z)\, de = (p_x, p_y, p_z),$$

$$\int (x^2, y^2, z^2, yz, zx, xy)\, de = (q_{11}, q_{22}, q_{33}, q_{23}, q_{31}, q_{12}).$$

Here e is the algebraic total charge of the particle, p_x, p_y, p_z are the components of its electric polarisation, while the q's are analogous to moments and products of inertia and may be called the 'second electric moments' of the particle.

If we assume that the second moments are so small as to be negligible we get

$$\mathrm{F}_x = \left(e + p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z} \right)(4\pi c^2 f + w_y\gamma - w_z\beta)$$
$$+ \int \left(1 + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \right)(\dot{y}\gamma - \dot{z}\beta)\, de.$$

Now the components of magnetisation m_x, m_y, m_z, are defined by the relations

$$\int (y\dot{z} - z\dot{y}, \quad z\dot{x} - x\dot{z}, \quad x\dot{y} - y\dot{x})\, de = 2\,(m_x, m_y, m_z);$$

and we note further that

$$2\int (x\dot{x}, y\dot{y}, z\dot{z})\, de = \frac{d}{dt}(q_{11}, q_{22}, q_{33}),$$

$$\int (y\dot{z} + z\dot{y}, \quad z\dot{x} + x\dot{z}, \quad x\dot{y} + y\dot{x})\, de = \frac{d}{dt}(q_{23}, q_{31}, q_{12});$$

hence $\int y\dot{z}\, de = \tfrac{1}{2}\dot{q}_{23} + m_x, \qquad \int z\dot{y} = \tfrac{1}{2}\dot{q}_{23} - m_x,$

with other similar equalities. Accordingly

$$\mathrm{F}_x = \left(e + p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z} \right)(4\pi c^2 f + w_y\gamma - w_z\beta)$$
$$+ m_y\frac{\partial\beta}{\partial x} + m_z\frac{\partial\gamma}{\partial x} - m_x\left(\frac{\partial\beta}{\partial y} + \frac{\partial\gamma}{\partial z} \right)$$

+ terms involving time-fluxes of the q's.

This formula is general. With a view to considering a purely magnetic particle we may suppose that the total charge e is zero, and that there is no electric polarisation, so that the first term of F_x is zero.

Let us further suppose that there is such permanence in the configuration or average configuration of electric charge in the particle that there is a set of axes (possibly rotating, provided the rotation be not extremely rapid), with O as origin, with respect to which the second electric moments are constant, or have constant average values. When this holds good the time-fluxes of the q's are either zero or (for rotating axes) small of the same order of smallness as the q's themselves, and so may be neglected.

Thus, for a purely magnetic particle,

$$F_x = m_y \frac{\partial \beta}{\partial x} + m_z \frac{\partial \gamma}{\partial x} - m_x \left(\frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z} \right),$$

$$= m_x \frac{\partial \alpha}{\partial x} + m_y \frac{\partial \beta}{\partial x} + m_z \frac{\partial \gamma}{\partial x},$$

$$= \left(m_x \frac{\partial}{\partial x} + m_y \frac{\partial}{\partial y} + m_z \frac{z}{\partial z} \right) \alpha + m_y \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) - m_z \left(\frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x} \right);$$

or, vectorially,

$$F = \left(m_x \frac{\partial}{\partial x} + m_y \frac{\partial}{\partial y} + m_z \frac{\partial}{\partial z} \right) H + [m, \text{ curl } H],$$

$$= \left(m_x \frac{\partial}{\partial x} + m_y \frac{\partial}{\partial y} + m_z \frac{\partial}{\partial z} \right) H + 4\pi [m, \dot{D}].$$

The first term of F is the ordinary formula for the force exerted on a magnetic particle, regarded as a polarised combination of positive and negative magnetism, by a field of magnetic force. The second term is rather unexpected; it represents a mechanical force exerted on a magnet by a current of æthereal displacement, perpendicular to the current and to the magnetic moment, and proportional to the product of the two and the sine of the angle between them. If experimental evidence were definitely against the existence of such a force the theory would be at fault.

It might seem possible to test the matter by hanging a small magnet horizontally between the horizontal plates of a charged condenser and then effecting a non-oscillatory discharge of the condenser. If the upper plate were originally charged with positive electricity, the displacement current on discharge would be upwards, and an eastward impulse on the magnet might be looked for. But when it is remembered that the formula is in terms of electromagnetic units it will be seen that the charge on the condenser required to impart sensible motion to the magnet would probably be enormously great. Thus an experimental test may well be out of the question.*

With regard to the hypothesis of the exact or average constancy of the values of the second electric moments of a magnetic particle, it is to be remarked that exact permanence of configuration in a whirling distribution of electricity is to be looked for only when the rotation is entirely about one axis round which the distribution is circularly symmetrical. This does not seem to be a probable state of affairs in a magnetic particle. On the other hand an average permanence of configuration may be claimed to exist for quite a complicated system of orbital motions of separate electrons provided the geometrical configuration of the orbits be permanent. All that is required for permanence of the average electric configuration is that the time-average be taken for an interval of time which is great compared with all the periods that the various electrons take to describe their respective orbits, or, if one orbit be described by several electrons, the interval between successive recurrences of the same electric configuration in that orbit. If the velocities in the orbits are very great only a very minute interval of time need be taken in order to get constant time-averages. In the case of a magnetic particle possessing as a whole a rotatory motion of not very great rapidity the permanence of the time-averages of the second moments would be with respect to moving axes.

It is indeed conceivable that the duration of any obtainable displacement current might be too short to permit the substitution of time-averages for a more accurate tracing of the changes in the configuration of charge in a single particle; in the case, however, of a magnet made up of a large number of particles any resultant effect would correspond to an average for all the particles, in which average the fortuitous character of the instantaneous circumstances for a single particle would be obliterated by force of numbers and the probably quite irregular distribution of phase.

Sir Joseph Larmor, to whom the writer is indebted for suggestions and criticisms, suggests as interesting the following aspect of the supplementary term in the above hypothetical expression for the force on a magnetic particle :—

If a magnet were merely a whirling distribution of electricity then the forces acting on a region of it ought, like those on any other distribution of electricity, to be expressible as the result of a quasi-stress over the boundary and a quasi-momentum in the region. But the commonly assumed forces

* Another possible difficulty is that the displacement current might alter the state of magnetisation, so that m would be a function of D and D. In so far as this held good the result would be uncertain.

on a medium of magnetic quality are not so expressible;* consequently the force on a whirl of electricity is not completely expressed by the usual formula of magnetic type in terms of its equivalent magnetic moment. The addition of the above obtained subsidiary term of much smaller order is just what is needed to restore the possibility of a stress-momentum specification. The term being too small for any light to be thrown upon its existence by direct experiment, there is no reason for excluding it; if we postulate the universality of the stress-momentum representation we must retain it.

For the sake of completeness it may be mentioned that, to the degree of approximation above contemplated, the torque on a particle is

$$[p, \{4\pi c^2 D + [w, H]\}] + [m, H] + \text{terms involving time-fluxes of the second electric moments.}$$

The part of this applicable to a purely magnetic particle is the same as is got from ordinary magnetic theory.

It is to be noted that the expressions here discussed refer only to the action of an external field on a particle. They do not include the action upon an electron of its own field or of the field due to other electrons belonging to the same particle. Thus radiation and electromagnetic inertia do not enter into the discussion.

* Larmor, "Dynamical Theory, etc.," 'Phil. Trans.,' A, 1897, vol. 190, § 39.