Motion of extended charges in classical electrodynamics

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(Received 28 April 1976; revised 19 July 1976)

The Lorentz–Dirac theory of radiation reaction on the motion of point charges is beset by the well known problems of runaway solutions and preacceleration. We examine the classical theory of extended charged particles and obtain a differential-difference type equation of motion. Analysis of this equation reveals that the theory is internally consistent (i.e., no runaways or acausality) whenever the size of the particle exceeds the classical radius (defined as the radius for which the electrostatic self-energy equals the mass of the particle). A specific example is presented which explicitly shows the different character of the extended and point charge solutions.

I. INTRODUCTION AND REVIEW OF THE LORENTZ–DIRAC THEORY

The treatments of radiation reaction in advanced textbooks on classical electrodynamics are based upon the Lorentz–Dirac theory. For the case of point charges, this theory leads to the well known result[1]

$$m_0\ddot{\mathbf{R}}(t) = \mathbf{F}(t) - \delta m\ddot{\mathbf{R}}(t) + m\tau\dddot{\mathbf{R}}(t), \tag{1}$$

where m_0 is the mechanical mass, $\mathbf{F}(t)$ is the external force, δm is the electrostatic self-energy ($m_0 + \delta m \equiv m$, the observed mass), and $\tau = 2e^2/3mc^3 \simeq 10^{-23}$ sec represents the time required for light to traverse the classical electron radius. This theory contains the well known difficulties that the self-energy δm is infinite, that runaway solutions occur (i.e., accelerations grow exponentially even in the absence of external forces), and that the theory is acausal (the charged particle responds to forces before they are "turned on"). The first problem can be handled by "renormalization": that is, it is possible to avoid reference to δm by working solely with the physical mass $m = m_0 + \delta m$, whose value is taken from experiment. The problem of runaway solutions is not handled quite so easily. In the Lorentz–Dirac–Rohrlich[2] treatment, arguments are advanced that Newtonian initial conditions (position and velocity) should suffice to specify the particle's motion, and, since Eq. (1) is a third-order linear differential equation, one additional condition must be imposed. Now, the general solution to Eq. (1) is

$$m\ddot{\mathbf{R}}(t) = e^{t/\tau}\left[m\ddot{R}(0) - \frac{1}{\tau}\int_0^t dt' e^{-t'/\tau}\mathbf{F}(t')\right] \tag{2}$$

and clearly, if we choose the initial condition

$$m\ddot{\mathbf{R}}(0) = \frac{1}{\tau}\int_0^\infty dt' e^{-t'/\tau}\mathbf{F}(t'), \tag{3}$$

runaway behavior is suppressed.[3] Nevertheless, the theory is still plagued by preacceleration. Specifically if we combine Eqs. (2) and (3), we have the final result[1]

$$m\ddot{\mathbf{R}}(t) = \int_0^\infty ds e^{-s}\mathbf{F}(t + \tau s), \tag{4}$$

an equation which demonstrates that the particle samples the force a time τ into the future. Equation (4) represents the standard nonrelativistic theory of radiation damping and solutions of this equation (along with its relativistic generalization) have been studied in detail[4] for various forms of the external force $\mathbf{F}(t)$. Within experimental limits, and in the relevant domain of validity, the Lorentz–Dirac theory successfully describes the motion of charged particles.

There remains the question of internal consistency. That is, the solutions represented by Eq. (4) exhibit acausal behavior and presumably this should be considered unacceptable in an internally consistent classical theory of charged particle motion. The standard response to this criticism is that the time scale over which the preacceleration occurs is $\sim 10^{-23}$ seconds for an electron and, since this represents a time in the quantum domain, no *observable* acausality is predicted at the classical level.[5]

These properties of the Lorentz–Dirac theory of point charges are all explained clearly in modern textbooks on classical electrodynamics.[1,2] What is not stressed is the fact that the classical theory of *extended* charged particles is *free* of the difficulties of runaway solutions and preacceleration whenever the size of the particle exceeds its classical radius (defined as the radius for which the electrostatic self-energy equals the mass of the particle). This result has been available in the research literature for many years.[6] Since, for simple charge distributions, the calculations required to demonstrate these conclusions are not difficult, we would like to present them to a broader readership.

Our paper is organized as follows. Section II starts with a sketch of the derivation of the Abraham–Lorentz equation for the motion of a particle with an extended charge distribution [Eq. (1) represents the point charge limit of this equation]. We rewrite this equation as a differential-difference equation and examine its solutions, finding that the classical theory is consistent (no runaways or acausal behavior) whenever the characteristic size of the charged particle exceeds its classical radius, $r_0 \sim e^2/mc^2$. Finally, in Sec. III we examine numerically the solutions to the differential-difference equation of motion for a simple case and compare this to the Lorentz–Dirac result. Section IV contains a brief summary and discussion of the results.

II. MOTION OF AN EXTENDED CHARGE

The starting point of this work is the standard Abraham–Lorentz expression for the radiation reaction force

$$m_0\ddot{\mathbf{R}}(t) = \mathbf{F}(t) - \frac{2e^2}{3c^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\gamma_n\,\mathbf{R}^{(n+2)}(t), \quad (5a)$$

where

$$\gamma_n = \int d\mathbf{x}d\mathbf{x}'\rho(x')|\mathbf{x}-\mathbf{x}'|^{n-1}\rho(x). \quad (5b)$$

This equation is derived in detail in, for example, the textbook of Jackson.[1] Basically, one starts with the Lorentz force equation expressing the interaction of a charged particle with its electromagnetic self-fields,

$$m_0\ddot{\mathbf{R}}(t) = \mathbf{F}_{\text{ext}}(t) + \mathbf{F}_{\text{self}}(t), \quad (6a)$$

where

$$\mathbf{F}_{\text{self}} = e\int d\mathbf{x}\left[\rho\mathbf{E}_{\text{self}} + \frac{1}{c}\mathbf{J}\times\mathbf{B}_{\text{self}}\right] \quad (6b)$$

and $(\rho,\mathbf{J})$ is the (charge, current) distribution of the particle $[\int d^3x\rho(x) \equiv 1]$. Next, the fields are eliminated in favor of the self-potentials, which are in turn replaced by the retarded solutions of the Maxwell field equations. The result is an equation which contains only the external force and the coordinates of the particle. A restriction to small velocities results in Eq. (5), where the appearance of all time derivatives of the velocity is due to expanding the retarded fields about the time t.

Since $\gamma_n \propto L^{n-1}$, where L is the characteristic dimension of the particle, Eq. (5) clearly reduces to Eq. (1) in the point limit ($L \to 0$). Instead of following this path, we choose to analyze Eq. (5) for a finite charge distribution. For simplicity, we shall present in the text the particular case of a rigid spherical shell of charge[6]

$$\rho(x) = (1/4\pi L^2)\delta(|\mathbf{x}| - L). \quad (7)$$

Straightforward evaluation of Eq. (5b) then yields the coefficients

$$\gamma_n = [2/(n+1)](2L)^{n-1} \quad (8)$$

which, when inserted into Eq. (5a), allows us to sum the series to obtain

$$\ddot{\mathbf{R}}(t) = \frac{\mathbf{F}(t)}{m(1-c\tau/L)} + \xi\left[\dot{\mathbf{R}}\left(t - \frac{2L}{c}\right) - \dot{\mathbf{R}}(t)\right], \quad (9)$$

where

$$\xi \equiv \frac{(c/2L)(c\tau/L)}{(1-c\tau/L)} \quad (10)$$

and

$$m = m_0 + \delta m = m_0 + 2e^2/3Lc^2. \quad (11)$$

Taylor expansion of the time-shifted velocity in Eq. (9) obviously reproduces the original series. This linear differential-difference equation determines the motion of a charged shell, and we now investigate its solutions. In the Appendix, we show that the equation of motion is of the differential-difference type for an arbitrary spherically symmetric charge distribution.[7]

We start with an analysis of Eq. (9) in the absence of external forces. Assuming a solution of the form $\mathbf{R}(t) = \mathbf{R}(0)e^{\beta t/\tau}$, we have (eliminating the root $\beta = 0$)

$$\beta/\tau = \xi[e^{-2L\beta/c\tau} - 1], \quad (12)$$

and we must find the complex roots of this equation. The appearance of any root β with positive real part corresponds to a runaway solution. Introducing the dimensionless variables $g = 1/[(L/c\tau) - 1]$ and $\eta = 2L\beta/c\tau \equiv \mu + i\nu$, Eq. (12) becomes

$$\mu = g[e^{-\mu}\cos\nu - 1], \quad (13a)$$

$$\nu = -ge^{-\mu}\sin\nu. \quad (13b)$$

Clearly, Eq. (13a) does not admit positive solutions for μ when $g > 0$, meaning that there are no runaway solutions for $L > c\tau$. On the other hand, for $g < 0$, there is always a solution with $\nu = 0$ and μ positive. Furthermore, there are an infinite number of complex conjugate solutions with μ negative, corresponding to damped oscillatory motion. Figure 1 shows the trajectories of several of the roots, as well as the runaway root, plotted as a function of $\alpha \equiv L/c\tau$.

Two points must be stressed. First, the runaway behavior appears only when $L < c\tau$, corresponding to a negative mechanical mass, $m_0 < 0$. However, the classical theory is well defined only when $m_0 > 0$ and then there are no runaways. Note that the runaway behavior is not peculiar to the point limit but appears whenever the radius is less than the classical radius. A second question involves the interpretation of the roots of the homogeneous equation. The appearance of an infinite number of damped oscillatory solutions corresponds directly to the fact that the differential-difference equation of motion requires specification of *all* time derivatives of the velocity at one time. The as-

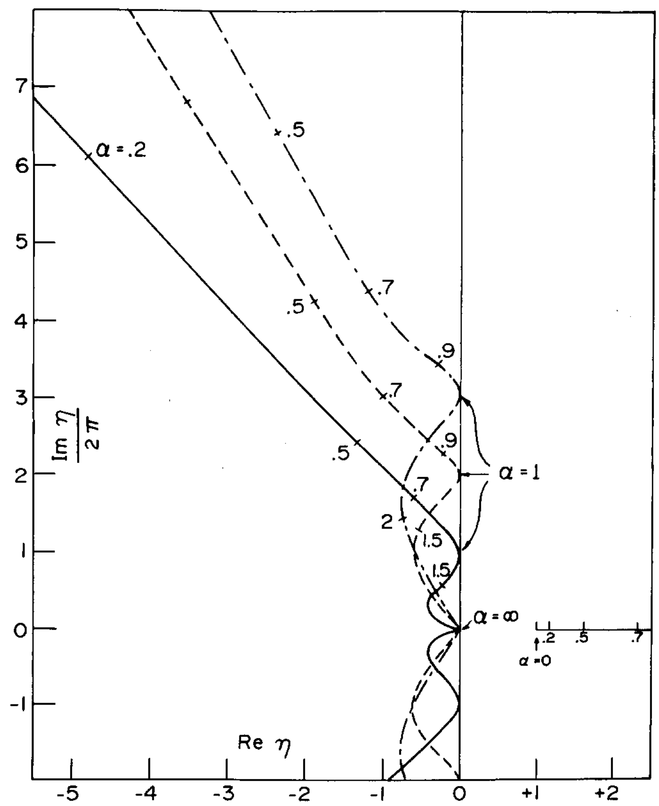

Fig. 1. Complex trajectories (as a function of L) of the solutions to Eq. (12). For $\alpha > 1$, there are damped ($\text{Re}\,\eta < 0$) oscillatory solutions. When $\alpha < 1$, the runaway root "appears" on the positive real axis. As $\alpha \to 0$, all trajectories go to $-\infty$ except for the runaway mode, which approaches $\eta = +1$.

sumption of Newtonian initial conditions, as made in the Lorentz–Rohrlich[2] theory, is neither necessary nor justified, since an infinite number of field degrees of freedom have been eliminated from the coupled equations governing the interacting system of particle plus fields. Physically this means that, when the external force ceases to act on the particle, a finite time is required for the particle to "settle down" to a constant velocity.

Finally, we discuss preacceleration. Using a Fourier transform, we can rewrite Eq. (9) as

$$m\ddot{\mathbf{R}}(t) = \int_{-\infty}^{+\infty} dt' G(t-t')\mathbf{F}(t'), \tag{14}$$

where the response function is

$$G(t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(e^{i\omega(t-t')} \left\{ 1 + \frac{i}{\omega}\left(\frac{c}{2L}\right)\left(\frac{c\tau}{L}\right) \times \left[\exp\left(-i\frac{2L\omega}{c}\right) + i\frac{2L\omega}{c} - 1 \right] \right\}^{-1} \right). \tag{15}$$

Causality demands that $G[(t-t') < 0] = 0$, which is equivalent to the denominator in Eq. (15) having no zeroes in the lower half of the complex ω plane. However, this condition corresponds to Eq. (12) having no roots for $Re\beta > 0$ [i.e., if we let $\omega \to -i\beta/\tau$, the zeroes of the denominator are seen to be the roots of Eq. (12)]. Therefore, our conclusion is that, if $L > c\tau$ (or, $m_0 > 0$), the classical theory is consistent, there being neither runaway solutions nor acausal response.

III. AN EXAMPLE

In this section, we present a comparison between the solutions of the Lorentz–Dirac equation [Eq. (4)] and the differential-difference equation [Eq. (9)] for a simple driving force. First, if we consider an infinite step-force

$$\mathbf{F}(t) = m\mathbf{k}\theta(t), \tag{16}$$

Eq. (9) can be solved iteratively in time intervals $2L/c$. For the nth time interval $(n-1) < ct/2L < n$, we have

$$\ddot{\mathbf{R}}_n(t) = \frac{\mathbf{k}}{[1-(c\tau/L)]} \sum_{j=0}^{n-1} \frac{\{\xi[t-j(2L/c)]\}^j}{j!} \times \exp\{-\xi[t-j(2L/c)]\}. \tag{17}$$

A finite step-force can always be written as the difference of two infinite steps, namely (assuming $t_0 > 0$),

$$\mathbf{F}(t) = m\mathbf{k}\theta(t)\theta(t_0 - t) = m\mathbf{k}\theta(t) - m\mathbf{k}\theta(t-t_0). \tag{18}$$

Consequently, Eq. (17) also leads to the result for the finite step, and this is shown in Fig. 2 for a particular case ($L/c\tau = 1.5$ and $t_0 = 6\tau$). The solution of the standard Lorentz–Dirac equation for the finite step force is shown on the same figure. This simple example clearly demonstrates the different character of the solutions: the Lorentz–Dirac solution displays preacceleration, while the solution of the differential-difference equation is causal and exhibits damped oscillatory motion after the force is turned off.

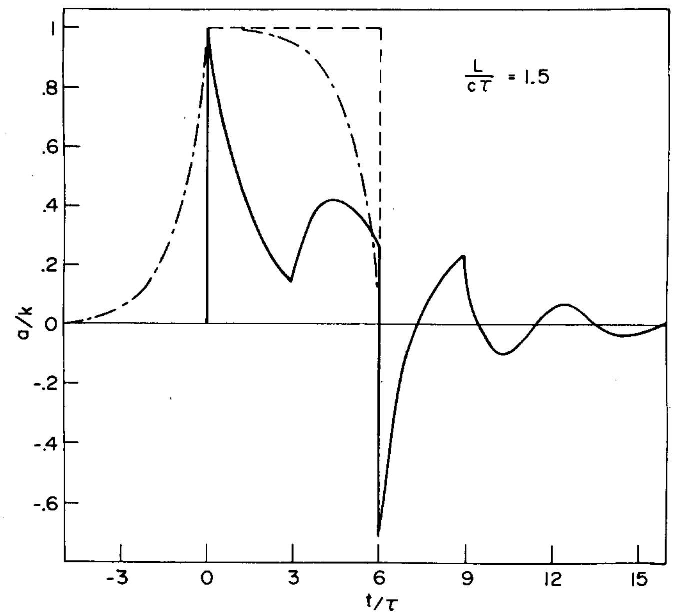

Fig. 2. Acceleration $a = \ddot{R}$ as a function of time for the step force of Eq. (18), with $t_0 = 6\tau$ and $L = 1.5c\tau$. Dashed curve is the solution without radiation reaction, while the dot-dash and solid curves are the solutions to Eqs. (4) and (9), respectively.

IV. SUMMARY AND CONCLUSIONS

We have examined the effects of radiation reaction on the motion of a nonrelativistic extended charged particle.[8] In standard textbooks on electrodynamics,[1] the Lorentz–Dirac–Rohrlich theory of point charges is presented, together with the attendant problems of runaway solutions and preacceleration. However, we have shown directly from the equation of motion that neither of these problems arises for the case of extended charges whenever the mechanical mass of the particle is positive. In effect, classical electrodynamics is a consistent theory only in describing the motion of charges with a characteristic charge radius greater than the classical radius.

The example of the charged shell, discussed in detail, is of particular pedagogic value since the calculations are easily carried through analytically. The resulting differential-difference equation is interesting in several respects. The time difference $2L/c$ appearing in Eq. (9) is the time required for light to cross the particle diameter and reflects the retarded nature of the reactive force. The initial conditions required to specify the solution of the equation (namely, specification of the velocity over a finite time interval or of all derivatives of the velocity at one time) appropriately reflect the fact that an infinite number of field degrees of freedom have been eliminated from the coupled equations describing the interacting system.

Finally, we remark that, while the motion of point charges cannot be described in a *classically* consistent fashion, it has been shown[9,10] that the *quantum* theory of nonrelativistic point charges is free of runaways and of acausal behavior and that the correspondence limit of the quantum solutions agrees with those given above in the classical domain of validity.[11] It is only in this sense that one should discuss point charges in classical electrodynamics.

APPENDIX

Here, we show how the classical equation of motion of an extended charge, Eq. (5), can be written in the form of a differential-difference equation.[7] Basically, we shall rewrite the last term in Eq. (5a) as

$$\frac{2e^2}{3c^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!c^n} \gamma_n \mathbf{v}^{(n+1)}(t)$$

$$= m \int_0^1 dy \mathbf{v}\left(t - \frac{2L}{c} y\right) f(y), \quad \text{(A1)}$$

where L is the radius beyond which $\rho(x) = 0$ and where $f(y)$ is an as yet unspecified function. Assuming a spherically symmetric charge distribution, the structure dependent coefficients γ_n become [see Eq. (5b)]

$$\gamma_n = \frac{2\pi^2(2L)^{n+5}}{n+1} \int_0^1 dy y^{n+1} \int_0^{1-y} dy' \times (y^2 - y'^2)\rho[(y + y')L]\rho(|y - y'|L). \quad \text{(A2)}$$

Inserting this into Eq. (A1) and expanding the velocity on the right-hand side of Eq. (A1) about the time t, we can identify

$$f(y) = \frac{2c}{L}\left(\frac{c\tau}{L}\right)(4\pi L^3)^2 \int_0^{1-y} dy' \times (y^2 - y'^2)\rho[(y + y')L]\rho(|y - y'|L), \quad \text{(A3)}$$

$$m_0\dot{\mathbf{v}}(t) = \mathbf{F}_{\text{ext}}(t) + m \int_0^1 dy f(y) \mathbf{v}\left(t - \frac{2L}{c} y\right). \quad \text{(A4)}$$

The differential-difference equation of motion for the charged shell, Eq. (9), is clearly a particular example of Eq. (A4) and our earlier discussion about the initial conditions necessary to specify the solution applies here as well. Note that

$$\int_0^1 dy f(y) = 0, \quad \text{(A5)}$$

which guarantees that the uniform velocity solution exists when $\mathbf{F}_{\text{ext}} = 0$.

*Work supported in part through funds provided by the Energy Research and Development Administration under Contract AT(11-1)-3069.

†Work supported by the Energy Research and Development Administration under Contract W-7405-ENG-36.

[1]See, for example, J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).

[2]F. Rohrlich, *Classical Charged Particles—Foundations of Their Theory* (Addison-Wesley, Reading, 1965).

[3]Another justification of this prescription is the principle of "undetectability of small charges" (see Ref. 2). This is the requirement that the solutions of Eq. (4) have the correct behavior in the limit that the charge goes to zero.

[4]G. N. Plass, Rev. Mod. Phys. **33**, 37 (1961).

[5]Note, however, that this response really addresses itself to the domain of validity of the theory and not to its internal consistency; these are two distinct questions. For example, Newtonian mechanics is an internally consistent theory even outside its domain of applicability, the latter being determined by relativistic mechanics.

[6]For a review of theories of an extended electron, see T. Erber, Fortschr. Phys. **9**, 343 (1961). A fairly extensive bibliography on the subject is also given in Ref. (9).

[7]Other approaches in the case of an arbitrary spherically symmetric charge distribution can be found in D. Bohm and M. Weinstein, Phys. Rev. **74**, 1789 (1948) and in D. J. Kaup, Phys. Rev. **152**, 1130 (1966).

[8]As far as we know, a fully consistent relativistic treatment of the radiation damping effects on the motion of an extended charged particle has not been carried out. One of the difficulties is the proper inclusion of elastic deformation, since a rigid charge distribution is not compatible with special relativity. Treatment of the relativistic point charge can be found in Ref. 4, where it is shown that the same problem of runaways and acausality appear.

[9]E. J. Moniz and D. H. Sharp, Phys. Rev. D **10**, 1133 (1974).

[10]E. J. Moniz and D. H. Sharp, "Radiation Reaction in Nonrelativistic Quantum Electrodynamics" (unpublished).

[11]For further discussion of this point, see F. Rohrlich, Acta Phys. Austriaca **41**, 375 (1975).