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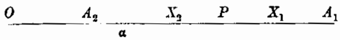
LXXXV. *On a Paradox in the Theory of Attraction.*

LET A_1A_2 be a straight line, P a point in the same, X_1, X_2 corresponding points in the segments PA_1, PA_2.

Let the distances of these points from the origin O measured in the positive



direction be a_1, a_2, p, x_1, x_2, respectively, and let the equation of correspondence between x_1 and x_2 be

$$\frac{1}{x_1-p}-\frac{1}{a_1-p}=\frac{1}{p-x_2}-\frac{1}{p-a_2} \quad \text{.....................(1).}$$

If x_1 and x_2 vary simultaneously,

$$\frac{dx_1}{(x_1-p)^2}=-\frac{dx_2}{(p-x_2)^2} \quad \text{..............................(2).}$$

Hence x_1 and x_2 move in opposite directions, and the lengths of the corresponding elements dx_1 and dx_2 (considered both positive) are as the squares of their respective distances from the point P.

If therefore AB is a uniform rod of matter attracting inversely as the square of the distance, the attractions of the corresponding elements on a particle at the point P will be equal and opposite.

Now by giving values to x_1, varying continuously from a_1 to p, we may obtain a corresponding series of values of x_2, varying from a_2 to p, and since every corresponding pair of elements dx_1 and dx_2 exert equal and opposite attractions on a particle at P, we might conclude that the attraction of the whole segment A_1P on a particle at P is equal and opposite to that of the segment A_2P on the same particle.

But it is still more evident that if A_2P is the greater of the two segments, and if we cut off $Pa = PA_1$ the attractions of Pa and PA_1 on the particle at P will be equal and opposite. But the attraction of PA_2 exceeds that of Pa by the attraction of the part aA_2, therefore the attraction of PA_2 exceeds that of PA_1 by a finite quantity, contrary to our first conclusion.

Hence our first conclusion is wrong, and for this reason. The attractions of any two corresponding segments A_1X_1 and A_2X_2 are exactly equal, but however near the corresponding points X_1 and X_2 approach to P, the attraction of each of the parts X_1P and X_2P on P is infinite, but that of X_2P exceeds that of X_1P by a constant quantity, equal to the attraction of A_2a on P.

This method of corresponding elements leads to a very simple investigation of the distribution on straight lines, circular and elliptic disks and solid spheres and ellipsoids of fluids repelling according to any power of the distance.

The problem has been already solved by Green* in a far more general manner, but at the same time by a far more intricate method.

We have, as before, for corresponding values of x_1 and x_2

$$\frac{1}{x_1-p} - \frac{1}{a_1-p} = \frac{1}{p-x_2} - \frac{1}{p-a_2} \quad\text{.......................(1).}$$

Transposing

$$\frac{1}{x_1-p} + \frac{1}{p-a_2} = \frac{1}{p-x_2} + \frac{1}{a_1-p} \quad\text{.......................(2).}$$

Multiplying

$$\frac{(x_1-a_1)(x_1-a_2)}{(x_1-p)^2(a_1-p)(p-a_2)} = \frac{(a_2-x_2)(a_1-x_2)}{(p-x_2)^2(a_1-p)(p-a_2)} \quad\text{...............(3).}$$

If we write

$$(a_1-x_1)(x_1-a_2) = y_1^2 \quad\text{.............................(4),}$$

$$(a_1-x_2)(x_2-a_2) = y_2^2 \quad\text{.............................(5),}$$

we find from equation (3)

$$\frac{x_1-p}{y_1} = \frac{p-x_2}{y_2} \quad\text{..................................(6).}$$

Let ρ_1, ρ_2 be the densities and s_1, s_2 the sections of the rod at the corresponding points X_1 and X_2, and let the repulsion of the matter of the

* George Green. "Mathematical Investigations concerning the laws of the Equilibrium of Fluids analogous to the electric fluid, with other similar researches." *Transactions of the Cambridge Philosophical Society*, 1833. (Read Nov. 12, 1832.) Ferrers' Edition of Green's Papers, p. 119.

rod vary inversely as the n^{th} power of the distance, then the condition of equilibrium of a particle at P under the action of the elements dx_1 and $-dx_2$ is

$$\rho_1 s_1 dx_1 (x_1 - p)^{-n} = -\rho_2 s_2 dx_2 (p - x_2)^{-n} \quad\text{.....................(7).}$$

Eliminating dx_1 and dx_2 by means of equation (2), we find

$$\rho_1 s_1 (x_1 - p)^{2-n} = \rho_2 s_2 (p - x_2)^{2-n} \quad\text{.......................(8),}$$

and from this by means of equation (6) we obtain

$$\rho_1 s_1 y_1^{2-n} = \rho_2 s_2 y_2^{2-n} \quad\text{...............................(9),}$$

as the condition of equilibrium between the elements.

The condition of equilibrium is therefore satisfied for every pair of elements by making

$$\rho s y^{2-n} = \text{constant} = C \quad\text{..........................(10).}$$

In a uniform rod s is constant, so that the distribution of density is given by the equation

$$\rho = C y^{n-2} \quad\text{..................................(11).}$$

If $n = 2$, as in the case of electricity, the density is uniform.

We have already shewn that when the density is uniform a particle not at the middle of the rod cannot be in equilibrium, but on the other hand any finite deviation from uniformity of density would be inconsistent with equilibrium. We may therefore assert that the distribution of the fluid when in equilibrium is not absolutely uniform, but is least at the middle of the rod, while at the same time the deviation from uniformity is less than any assignable quantity.

If the force is independent of the distance, $n = 0$ and

$$\rho = C y^{-2} \quad\text{..................................(12),}$$

or if r is the distance from the middle of the rod, $2l$ being the length of the rod,

$$\rho = \frac{C}{l^2 - r^2} \quad\text{.................................(13).}$$

If C were finite, the whole mass would be infinite. Hence if the mass of fluid in the rod is finite it must be concentrated into two equal masses and placed at the two ends of the rod.

Let us next consider a disk on which two chords are drawn intersecting at the point P at a small angle θ, and let corresponding elements be taken of the two sectors so formed.

In this case the section of either sector is proportional to the distance of the element from the point of intersection, and therefore the two sections are proportional to the values of y at the two elements. Hence if ρy^{3-n} is constant, the particle at the point of intersection will be in equilibrium.

If the edge of the disk is the ellipse whose equation is

$$1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}=0 \quad\text{.............................(14)},$$

and if at any point within it

$$1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}=p^2 \quad\text{.............................(15)},$$

and if the length of a diameter parallel to the given chord is $2d$, then the value of y for any point of the chord is

$$y=pd \quad\text{.............................(16)}.$$

Hence if

$$\rho=Cp^{n-3} \quad\text{.............................(17)},$$

a particle placed at any point of the disk will be in equilibrium under the action of any pair of sectors formed by chords intersecting at that point, and therefore it will be absolutely in equilibrium.

When as in the case of electricity, $n=2$,

$$\rho=Cp^{-1} \quad\text{.............................(18)},$$

the known law of distribution of density.

If the repulsion were inversely as the distance, the fluid would be accumulated in the circumference of the disk, leaving the rest entirely empty.

If the force were inversely as the cube of the distance, the density would be uniform over the surface of the disk.

Lastly, let us consider a solid ellipsoid, the equation of the surface being

$$1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}=0,$$

and at any point within it let

$$1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}=p^2.$$

At any point of a chord drawn parallel to a diameter whose length is $2d$ the value of y is pd.

If we consider a double cone of small angular aperture whose vertex is at a given point, and whose axis is this chord, the sections at two corresponding elements are in the ratio of the squares of the distances of the elements from the given point, and therefore in the ratio of the values of p^2 at these elements. Hence the condition to be satisfied is

$$\rho p^{4-n} = C, \text{ a constant.}$$

If this condition be fulfilled the fluid will be in equilibrium at every point of the ellipsoid.

If $n = 2$, $$\rho = Cp^{-2}$$

is the condition of equilibrium. But if C is finite the whole mass of the fluid in the ellipsoid if distributed according to this law of density would be infinite. Hence if the whole quantity of fluid is finite it must be accumulated entirely on the surface, and the interior will be entirely empty, as we know already.

If the force is inversely as the fourth power of the distance the density within the ellipsoid will be uniform.