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XCII. *On the Electrical Capacity of a long narrow Cylinder, and of a Disk of sensible Thickness.*

THE distribution of electricity in equilibrium on a straight line without breadth is a uniform one. We may expect, therefore, that the distribution on a cylinder will approximate to uniformity as the radius of the cylinder diminishes.

Let $2l$ be the length of the cylinder, and b its radius.

Let x be measured along the axis from the middle point of the axis, and let y be the distance of any point from the axis.

Let λ be the linear density on the curved surface of the cylinder; that is, let λdx be the charge on the annular element dx.

Let σ be the surface-density on the flat ends.

Then, at a point on the axis for which $\xi = x$, the potential (ψ) is

$$\psi = \int_{-l}^{+l} \lambda \left[(\xi - x)^2 + b^2\right]^{-\frac{1}{2}} dx + 2\pi \int_0^b \sigma\eta \left[(l-\xi)^2 + y^2\right]^{-\frac{1}{2}} dy + 2\pi \int_0^b \sigma\eta \left[(l+\xi)^2 + y^2\right]^{-\frac{1}{2}} dy,$$

the first integral being extended over the curved surface, and the other two over the positive and negative flat ends respectively.

If the electricity is in equilibrium in the cylinder, ψ must be constant for all points within the cylinder, and therefore for all points on the axis.

Also, by Art. 144 of *Electricity and Magnetism,* if ψ is constant for all points on the axis, it is constant for all points within the surface of the cylinder.

If we suppose λ and σ constant,

$$\psi = \lambda \log \frac{(f_1 + l - \xi)(f_2 + l + \xi)}{b^2} + 2\pi\sigma (f_1 + f_2 - 2l),$$

where f_1 and f_2 are the distances of the point (ξ) on the axis from the positive and negative edges of the curved surface.

At the middle of the axis,

$$\xi = 0, \quad \text{and} \quad f_1 = f_2 = \sqrt{l^2 + b^2},$$

$$\psi_{(0)} = 2\lambda \log \frac{f+l}{b} + 4\pi\sigma (f - l).$$

At either end of the axis, $\xi = l$, $f_1 = b$, $f_2 = 2l$, nearly,

$$\psi_{(l)} = \lambda \log \frac{4l}{b} + 2\pi\sigma b.$$

Just within the cylinder, when ξ is just less than l,

$$\frac{d\psi}{dx_{(l)}} = -\lambda\left(\frac{1}{b} - \frac{1}{f}\right) + 2\pi\sigma = 0 \quad \text{by hypothesis.}$$

Hence $$2\pi b\sigma = \lambda,$$

or the density on the end must be equal to that on the curved surface.

The whole charge is therefore $E = 2\pi b\sigma (2l + b)$.

The greatest potential is $$\psi_{(0)} = 2\pi b\sigma \left(2 \log \frac{2l}{b} + \frac{b}{l}\right).$$

The smallest potential is that at the curved edge, and is approximately

$$\psi_{(e)} = 2\pi b\sigma \left(\log \frac{4l}{b} + \frac{2}{\pi}\right);$$

and the capacity must lie between

$$\frac{E}{\psi_{(0)}} = \frac{2l + b}{2 \log \frac{2l}{b} + \frac{b}{l}} \quad \text{and} \quad \frac{E}{\psi_{(e)}} = \frac{2l + b}{\log \frac{4l}{b} + \frac{2}{\pi}}.$$

These are the limits between which Cavendish shews that the capacity must lie. When the cylinder is very narrow, the upper limit is nearly double the lower, so that we cannot obtain in this way any approximation to the true value.

To obtain an approximation, we may make use of the following method, in which we neglect the effect of the flat ends, and consider the cylinder as a hollow tube:—

Let Q be the potential energy of any arbitrary distribution of electricity on the cylinder,

$$Q = \tfrac{1}{2}\int_{-l}^{+l} \lambda\psi d\xi.$$

The charge is

$$E = \int_{-l}^{+l} \lambda d\xi.$$

Let us now suppose this charge to distribute itself so as to pass into the state of equilibrium; then the potential will become uniform, and equal, say, to ψ_0, and

$$Q_0 = \tfrac{1}{2}\psi_0 E.$$

If K is the capacity of the conductor,

$$E = K\psi_0, \quad \text{and} \quad K = \frac{1}{2}\frac{E^2}{Q_0}.$$

Since Q, the potential energy due to any arbitrary distribution of the charge, may be greater, but cannot be less, than Q_0, the energy of the same charge in equilibrium, the capacity may be greater, but cannot be less, than

$$\frac{1}{2}\frac{E^2}{Q} \quad \text{or} \quad \frac{\left[\int_{-l}^{+l}\lambda d\xi\right]^2}{\int_{-l}^{+l}\lambda\psi d\xi}.$$

This inferior limit of the capacity is greater than that derived from the maximum value of the potential, and, as we shall see, often gives a very close approximation to the truth. Thus, if we suppose, in the case of the cylinder, λ to be uniform,

$$E = 2\lambda l, \qquad Q = 2\lambda^2 l\left(\log\frac{f+2l}{b} - \frac{f-b}{2l}\right),$$

where $f^2 = 4l^2 + b^2$. For a long narrow cylinder,

$$K_0 > \frac{l}{\log\frac{4l}{b} - 1}.$$

To obtain a closer approximation, let us suppose the distribution to be of any form, and to be expressed in the form of a series of harmonics.

The potential due to any such distribution at a given point may be expressed in terms of spherical harmonics of the second kind. See Ferrers' *Spherical Harmonics*, Chap. v.

If we write $$\alpha = \frac{1}{2}\frac{r_2 + r_1}{l}, \quad \beta = \frac{1}{2}\frac{r_2 - r_1}{l},$$

where r_1 and r_2 are the distances of a given point from the ends of the line, and if the linear density is expressed by

$$\lambda = \Sigma A_i P_i\left(\frac{\xi}{l}\right),$$

where P_i is the zonal harmonic of degree i, then the potential at the given point (α, β) is $$\psi = \Sigma A_i Q_i(\alpha)\, P_i(\beta),$$

where Q_i is the zonal harmonic of the second kind, and is of the form

$$Q_i(\alpha) = P_i(\alpha)\log\frac{\alpha+1}{\alpha-1} + R_i(\alpha),$$

where $R_i(\alpha)$ is a rational function of α of $(i-1)$ degrees, and is such that $Q_i(\alpha)$ vanishes when α is infinite, thus:

$$Q_0(\alpha) = \log\frac{\alpha+1}{\alpha-1},$$

$$Q_1(\alpha) = \alpha\log\frac{\alpha+1}{\alpha-1} - 2,$$

$$Q_2(\alpha) = \left(\frac{3}{2}\alpha^2 - \frac{1}{2}\right)\log\frac{\alpha+1}{\alpha-1} - 3\alpha,$$

$$Q_3(\alpha) = \left(\frac{5}{2}\alpha^3 - \frac{3}{2}\alpha\right)\log\frac{\alpha+1}{\alpha-1} - 5\alpha^2 + \frac{4}{3},$$

$$Q_4(\alpha) = \left(\frac{5.7}{8}\alpha^4 - \frac{3.5}{4}\alpha^2 + \frac{3}{8}\right)\log\frac{\alpha+1}{\alpha-1} - \frac{5.7}{4}\alpha^3 + \frac{5.11}{12}\alpha.$$

At a point at a very small distance b from the line, if we write

$$L = \log\frac{r_1 + l - \xi}{b} + \log\frac{r_2 + l + \xi}{b},$$

the potential due to the distribution whose linear density is

$$\lambda_i = A_i P_i\left(\frac{x}{l}\right)$$

is approximately

$$\psi_i = A_i P_i\left(\frac{\xi}{l}\right)\left[L - 2\left(i - \frac{i(i-1)}{2.2} + \frac{i(i-1)(i-2)}{2.3.3} - \frac{i(i-1)(i-2)(i-3)}{2.3.4.4} + \&c.\right)\right];$$

thus, if

$$\lambda_0 = A_0,$$

$$\lambda_1 = A_1 \frac{x}{l},$$

$$\lambda_2 = A_2 \left(\frac{3}{2} \frac{x^2}{l^2} - \frac{1}{2} \right),$$

$$\lambda_3 = A_3 \left(\frac{5}{2} \frac{x^3}{l^3} - \frac{3}{2} \frac{x}{l} \right),$$

$$\lambda_4 = A_4 \left(\frac{5 \, . \, 7}{8} \frac{x^4}{l^4} - \frac{3 \, . \, 5}{4} \frac{x^2}{l^2} + \frac{3}{8} \right);$$

then approximately

$$\psi_0 = A_0 L,$$

$$\psi_1 = A_1 \frac{\xi}{l} (L - 2),$$

$$\psi_2 = A_2 \left(\frac{3}{2} \frac{\xi^2}{l^2} - \frac{1}{2} \right) (L - 3),$$

$$\psi_3 = A_3 \left(\frac{5}{2} \frac{\xi^3}{l^3} - \frac{3}{2} \frac{\xi}{l} \right) \left(L - \frac{11}{3} \right),$$

$$\psi_4 = A_4 \left(\frac{5 \, . \, 7}{8} \frac{\xi^4}{l^4} - \frac{3 \, . \, 5}{4} \frac{\xi^2}{l^2} + \frac{3}{8} \right) \left(L - \frac{25}{6} \right).$$

If we write $\mathfrak{L}$ for $\log \frac{(4l^2 + b^2)^{\frac{1}{2}} + 2l}{b}$, or approximately $\mathfrak{L} = \log \frac{4l}{b}$, we find, to the same degree of approximation,

$$\int \lambda_0 \psi_0 d\xi = 4 A_0^2 l (\mathfrak{L} - 1),$$

$$\int \lambda_0 \psi_2 d\xi = \int \lambda_2 \psi_0 d\xi = -\tfrac{2}{3} A_0 A_2 l,$$

$$\int \lambda_2 \psi_2 d\xi = 2 A_2^2 l \left(\tfrac{2}{5} \mathfrak{L} - \frac{101}{75} \right),$$

$$\int \lambda_0 \psi_4 d\xi = -\tfrac{1}{5} A_0 A_4 l,$$

$$\int \lambda_2 \psi_4 d\xi = -\tfrac{2}{7} A_2 A_4 l,$$

$$\int \lambda_4 \psi_4 d\xi = 2 A_4^2 l \left(\tfrac{2}{9} \mathfrak{L} - \frac{6989}{5670} \right).$$

Determining A_2 so as to make $\int (\lambda_0+\lambda_2)(\psi_0+\psi_2)\,d\xi$ a minimum, and remembering that $E=2lA_0$, we find

$$A_2 = A_0 \frac{5}{6} \cdot \frac{1}{\mathfrak{L} - \frac{101}{30}},$$

and we obtain as a second approximation

$$K_2 > \frac{l}{\mathfrak{L} - 1 - \frac{5}{36}\,\frac{1}{\mathfrak{L} - \frac{101}{30}}}.$$

Unless the length of the cylinder considerably exceeds 7·245 times its diameter, this approximation is of little use, for this ratio makes A_2 infinite. It shews, however, that when the ratio of the length to the diameter increases without limit, the electric density becomes more nearly uniform, and the expression for K_0 approximates to the true capacity.

We may go on to a third approximation by determining A_2 and A_4 so that $\int (\lambda_0+\lambda_2+\lambda_4)(\psi_0+\psi_2+\psi_4)\,d\xi$ shall be a minimum; whence

$$A_2 = A_0 \frac{5}{6} \frac{\mathfrak{L} - \frac{3373}{630}}{\left(\mathfrak{L} - \frac{101}{30}\right)\left(\mathfrak{L} - \frac{6989}{1260}\right) - \frac{45}{196}},$$

$$A_4 = A_0 \frac{9}{20} \frac{\mathfrak{L} - \frac{457}{210}}{\left(\mathfrak{L} - \frac{101}{30}\right)\left(\mathfrak{L} - \frac{6989}{1260}\right) - \frac{45}{196}},$$

and

$$K_4 > \frac{l}{\mathfrak{L} - 1 - \frac{5}{36}\,\frac{1}{\mathfrak{L} - \frac{101}{30}} - \frac{9}{400}\,\frac{\left(\mathfrak{L} - \frac{457}{210}\right)^2}{\left(\mathfrak{L} - \frac{101}{30}\right)\left[\left(\mathfrak{L} - \frac{101}{30}\right)\left(\mathfrak{L} - \frac{6989}{1260}\right) - \frac{45}{196}\right]}}.$$

When $\mathfrak{L}$ is very great, the distribution of electricity is expressed by the equation

$$\lambda = A_0 \left[1 + \frac{1}{\mathfrak{L}} \frac{7}{32} \left\{ 9 \frac{x^4}{l^4} - 2 \frac{x^2}{l^2} - \frac{17}{15} \right\} \right],$$

which shews that, as the ratio of the length to the diameter increases, the density becomes more nearly uniform, and the deviation from uniformity becomes more confined to the parts near the ends of the cylinder.

To indicate the character of the approximation, I have calculated $\mathfrak{L}$ and the three terms of the denominator of K_4 for different values of the ratio of l to b. When this ratio is less than 100, the third term is unavailable.

$\frac{l}{b}$	$\mathfrak{L}$	1st term.	2nd term.	3rd term.
10	3·68888 .	2·68888	− 0·43151	
20	4·38203 .	3·38203	− 0·13680	
30	4·78749 .	3·78749	− 0·09775	
50	5·29832 .	4·29832	− 0·07191	
100	5·99146 .	4·99146	− 0·05291	− 0·13566
1000	8·29405 .	7·29405	− 0·02818	− 0·00892.

Examples of the application of the method to the calculation of the capacities of a cylinder in presence of a plane conducting surface, and in presence of another equal cylinder, will be given in the notes to the forthcoming edition of Cavendish's *Electrical Researches*, as illustrations of measurements made by Cavendish in 1771.

Electric Capacity of a Disk of sensible Thickness.

We may apply the same method to determine the capacity of a disk of radius a and thickness b, b being very small compared with a.

We may begin by supposing that the density on the flat surfaces is the same as when the disk is infinitely thin.

Let α and β be the elliptic co-ordinates of a given point with respect to the lower disk, or in other words let the greatest and least distances of the point from the edge of the disk be $a(\alpha+\beta)$ and $a(\alpha-\beta)$.

The distance of the given point from the axis is

$$r = a\alpha\beta \quad \text{.....................................(1)},$$

and its distance from the plane of the lower disk is

$$z = a\,(\alpha^2 - 1)^{\frac{1}{2}}\,(1 - \beta^2)^{\frac{1}{2}} \quad \text{.............................(2)}.$$

If we write

$$a^2 p^2 = a^2 - r^2 \quad \text{..................................(3)},$$

then, if A_2 is the charge of the upper disk, distributed as when undisturbed by the lower disk, the density at any point is

$$\sigma = \frac{A_2}{2\pi a^2 p} \quad \text{..................................(4)}.$$

If A_1 is the charge of the lower disk, also undisturbed, the potential at the given point due to it is

$$\psi = A_1 a^{-1} \operatorname{cosec}^{-1} \alpha \quad \text{.............................(5)},$$

or, if we write

$$\alpha^2 = \gamma^2 + 1 \quad \text{..................................(6)},$$

$$\psi = A_1 a^{-1}\left(\frac{\pi}{2} - \tan^{-1}\gamma\right) \quad \text{.........................(7)}.$$

We have next to find the relation between p and γ when the given point is in the upper disk, and therefore $z = b$.

Equation (2) becomes

$$b^2 = a^2\gamma^2\,(1 - \beta^2) \quad \text{...............................(8)},$$

and

$$p^2 = 1 - \alpha^2\beta^2 \quad \text{......................................(9)},$$

or

$$p^2 = \frac{b^2}{a^2\gamma^2} - \gamma^2 + \frac{b^2}{a^2} \quad \text{...............................(10)}.$$

Since the given point is on the upper disk, and since b is small, p must be between 1 and 0, and γ between $\frac{b}{a}$ and $\left(\frac{b}{a}\right)^{\frac{1}{2}}$; and between those limits we may write, as a sufficient approximation for our purpose,

$$\tan^{-1}\gamma = \gamma = \frac{b}{a}\left[1 + \left(\frac{b}{a}\right)^{\frac{1}{2}}\right]\left[p + \left(\frac{b}{a}\right)^{\frac{1}{2}}\right]^{-1} \quad \text{.................(11)}.$$

We have now to find the value of the surface integral of the product of the density into the potential taken over the upper disk, or

$$\int 2\pi r\,dr\,\sigma\psi = A_1 A_2 a^{-1}\left(\frac{\pi}{2} - \int_0^1 \tan^{-1}\gamma\,dp\right) \quad \text{.................(12)}.$$

Substituting the value of $\tan^{-1}\gamma$ from (11), the integral in (12) becomes

$$A_1A_2a^{-1}\left\{\frac{\pi}{2}-\frac{b}{a}\left[1+\left(\frac{b}{a}\right)^{\frac{1}{2}}\right]\log\left[\left(\frac{a}{b}\right)^{\frac{1}{2}}+1\right]\right\} \quad\ldots\ldots\ldots\ldots\ldots(13).$$

The corresponding quantity for the action of the upper disk on itself is got by putting $A_1=A_2$ and $b=0$, and is

$$A_2^2a^{-1}\frac{\pi}{2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

In the actual case, $A_1=A_2=\frac{1}{2}E$, when E is the whole charge; and the lower limit of the capacity is therefore

$$K>\frac{2a}{\pi-\frac{b}{a}\left[1+\left(\frac{b}{a}\right)^{\frac{1}{2}}\right]\log\left[\left(\frac{a}{b}\right)^{\frac{1}{2}}+1\right]} \quad\ldots\ldots\ldots\ldots\ldots\ldots(15);$$

but since we have assumed that b is very small compared with a, we may express our result with sufficient accuracy in the form

$$K=\frac{2}{\pi}\left(a+\frac{1}{2\pi}b\log\frac{a}{b}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

or, the capacity of two equal disks is equal to that of a single disk whose circumference exceeds that of either disk by $b\log\frac{a}{b}$.

If the space between the disks is filled up, so as to form a single disk of sensible thickness, there will be a certain charge on the cylindric surface; but, at the same time, the charge on the inner sides of the disks will vanish, and that on the outer sides of the disks will be diminished, so that the capacity of a disk of sensible thickness is very little greater than that given by (16).