The Two-Capacitor Problem Reconsidered

Raymond P. Mayer, John R. Jeffries, and George F. Paulik

Abstract— **The two-capacitor problem involves connecting a charged capacitor to an uncharged capacitor and accounting for the difference in energy between the initial and equilibrium states. Heat due to electrical resistance in the connecting wires is usually cited for the energy loss. In this paper, the wires are assumed to be perfectly conducting and without electrical resistance. The circuit then behaves as a loop antenna and radiates energy in the form of EMR. All loss of energy in the system can be accounted for through EMR considerations. Examples illustrate the rate of decay of the current in the circuit.**

I. Introduction

A circuit analysis problem involving two capacitors is common to a variety of electrical engineering and physics texts (e.g., [1]–[4]). A capacitor, C_1, is given a charge, Q_0, and then connected to an uncharged capacitor, C_2, as shown in Fig. 1. After the switch is closed, the two-capacitor circuit is assumed to approach an equilibrium in which both capacitors are at the same potential. A short calculation reveals that the initial energy, $Q_0^2/2C_1$, is greater than the equilibrium energy, $(C_1/(C_1+C_2))Q_0^2/2C_1$, and one is asked to account for the missing energy. The generally accepted explanation states that energy is lost through heat due to electrical resistance in the connecting wires and that there is also the possibility of an energy loss through electromagnetic radiation (EMR) [4] and [5].

This points out an ambiguity in the two-capacitor problem. For example, no resistance is indicated in Fig. 1 yet the explanation assumes an electrical resistance in the connecting wires. It is clear that Fig. 1 is not to be regarded as a valid circuit diagram but instead as a schema of physical components. It should be analyzed only after accounting for the effects present in the circuit when the switch is closed. Such an analysis is made in [5] after introducing inductance and electrical resistance elements to Fig. 1.

This paper presents an analysis of the two-capacitor circuit without adding an electrical resistance, i.e., the wires are perfectly conducting. Zero resistance wires are not unrealistic considering the phenomenon of superconductivity. An inductance element is introduced to take into account the self-inductance, L, of the circuit as in [5]. The resulting LC circuit would behave as a simple harmonic oscillator. However, this is not physically correct as the circuit would never approach an equilibrium and would never lose energy. This contradicts the fact that such an alternating current would induce an oscillating magnetic dipole and emit energy via EMR [6]–[9].

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R. P. Mayer and J. R. Jeffries are with the Science and Mathematics Department, West Shore Community College, 3000 N. Stiles, Scottville, MI 49454.

G. F. Paulik is with the Mathematics Department, University of Iowa, Iowa City, IA 52242.

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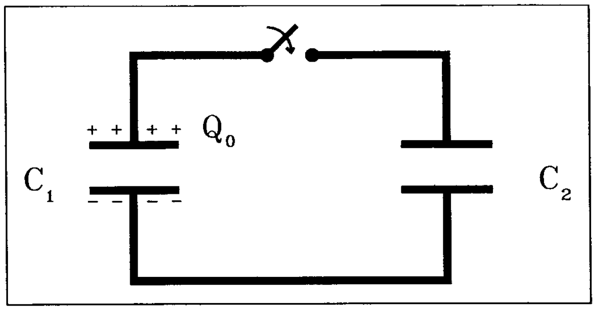

Fig. 1. The two-capacitor problem as typically presented. The first capacitor is charged and then is connected to the second by means of a switch.

In an attempt to resolve this apparent contradiction, the circuit is modeled as a loop antenna with an alternating current which decays as EMR is emitted. To represent the effect of energy loss via EMR, a radiation resistance element, R_{rad}, is added to the circuit diagram. The resulting LCR_{rad} circuit allows one to account for the difference in energy between the initial and equilibrium states solely through EMR considerations. A short calculation using "standard" values [5] for L and C illustrates the radiated power and decay of the current.

II. Analysis

As is generally done in elementary circuit analysis problems, it is assumed that:

A1) Switches close instantaneously without arcing and there is no arcing between the capacitor plates.

A2) Connecting wires and circuit elements do not add electrical resistance to that otherwise specified.

A3) Capacitors are ideal and without dielectric.

A4) The current is uniform throughout the length and cross section of the connecting wire.

At the instant the switch in Fig. 1 is closed, charge will begin to pass from one capacitor to the other and a current, I, will be present in the wires. In passing from a state of zero current to a state of nonzero current, the changing current induces a changing magnetic flux linkage, which in turn induces a back EMF given by $(-L)dI/dt$, where L is the geometric self-inductance of the circuit. In antenna circuit analysis, radiated energy is often regarded as equivalent to a fictitious I^2R heat dissipation. Accordingly, the circuit is assumed to contain a radiation resistance, R_{rad}, such that I^2R_{rad} is equal to the power of the radiated energy [6]. With regards to Kirchhoff's rules, an R_{rad} term is treated the same as an electrical resistance R term. The current in the resulting LCR_{rad} circuit satisfies the following well-known initial value problem:

$$L\frac{d^2I}{dt^2} + R_{\text{rad}}\frac{dI}{dt} + C^{-1}I = 0,$$
$$I(0) = 0,$$
$$\frac{dI}{dt}(0) = \frac{Q_0}{LC_1}, \tag{1}$$

where $C = C_1C_2/(C_1 + C_2)$.

Let c denote the speed of light. A loop antenna with diameter, d, driven harmonically at an angular frequency, ω, emits EMR of wavelength $\lambda = 2\pi c/\omega$ and has a radiation resistance given by [6]–[9]

$$R_{\text{rad}} = 1.25\pi^2(d\omega/c)^4. \tag{2}$$

Formula (2) is valid provided that:

A5) d is much greater than the cross section of the wire.

A6) d is less than $\lambda/4\pi$.

In order to use (2) in analyzing (1), one assumes:

A7) The circuit in Fig. 1 is in the shape of a circular loop.

A8) Formula (2) is valid for "undriven" underdamped circuits. Assumption A8) seems reasonable since (2) is independent of the magnitude of the current, and the current in an underdamped circuit oscillates as it decays.

The only difficulty encountered in the use of (2) in studying (1) is the fact that ω is unknown. Experience indicates that in many circuits radiation resistances are very small, which would suggest that $\omega \approx \omega_0 = (LC)^{-1/2}$. The following argument shows that this is indeed the case in the presence of certain easily verifiable assumptions.

The underdamped solution of (1) is

$$I(t) = (Q_0/LC_1\omega)e^{(-R_{\text{rad}}/2L)t}\sin(\omega t), \qquad t \geq 0, \tag{3}$$

where

$$\omega = ((LC)^{-1} - (R_{\text{rad}}/2L)^2)^{1/2}. \tag{4}$$

Note that ω appears in the R_{rad} term via (2). Thus, squaring both sides one sees that ω^2 is a (positive) root of the polynomial

$$p(x) = (25\pi^4d^8C/64Lc^8)\omega_0^2x^4 + x - \omega_0^2. \tag{5}$$

The derivative, p', is a cubic with one negative real root and p' is positive at values greater than this root. Since $p(0) < 0$, it follows that p has precisely one positive root. At $x = \omega_0^2$,

$$p(\omega_0^2) = (25\pi^4d^8/64L^5C^3c^8)\omega_0^2 = \omega_0^2/4Q^2. \tag{6}$$

Here $Q = (1/R_{\text{rad}})(L/C)^{1/2}$ is the quality factor of a circuit in which R_{rad} is given by (2) with $\omega = \omega_0$. Since $p(\omega_0^2)$ is positive it follows that

$$0 < \omega^2 < \omega_0^2. \tag{7}$$

Observing that $p'(x) > 1$ for $x > 0$, it follows from the mean value theorem that

$$0 < \omega_0^2 - \omega^2 < p(\omega_0^2) - p(\omega^2) = \omega_0^2/4Q^2. \tag{8}$$

TABLE I
CURRENT HALF-LIFE, $T_{1/2}$, DAYS

Loop Diameter d, meters	Capacitance C_1, μF		
	100	10	1
0.1	7900	79	0.79
1.0	2500	25	0.25
10.0	570	5.7	0.057

$Q \gg 1$ in (8) implies that $\omega^2 \approx \omega_0^2$. Also, (7) and $Q \geq 1/2$ are sufficient to ensure that the circuit is underdamped, and when $d < c/2\omega_0$, A6) is satisfied. Thus, if one assumes a circuit with

A9) $d < c/2\omega_0$,

A10) $Q \gg 1$,

then $\omega \approx \omega_0$.

III. EXAMPLE

In this section, an example is provided which illustrates the behavior of a two-capacitor circuit based on the preceding analysis.

Under assumptions A4) and A5), a wire with cross-sectional radius, r, in the shape of a circular loop has a self-inductance given by [8]

$$L = \left(\mu_0\frac{d}{2}\right)\left(\ln\left(\frac{4d}{r}\right) - 1.75\right). \tag{9}$$

Here μ_0 is the absolute permeability of free space. Following [5], the capacitors have a capacitance of 100 μF each and d and r for the wire are $1\times10^{-1}m$ and $5\times10^{-4}m$, respectively. Then $C = 50\ \mu F$, $L = 3.1 \times 10^{-7}H$ and A9) and A10) are satisfied, so $\omega \approx \omega_0 = 2.54\times10^5s^{-1}$ and $R_{\text{rad}} = 6.3\times10^{-16}$ ohm. The current is

$$I(t) = (1.27 \times 10^5)Q_0e^{-1.0\times10^{-9}t}\sin((2.54 \times 10^5)t), \qquad t \geq 0. \tag{10}$$

The power radiated by the circuit at time t is

$$I^2(t)R_{\text{rad}} = (1 \times 10^{-5})Q_0^2e^{-2\times10^{-9}t}\sin^2((2.54 \times 10^5)t). \tag{11}$$

Integrating the expression in (11) over time accounts for the difference in the initial and equilibrium energies, a standard result for LCR circuits [4] which is necessarily true for LCR_{rad} circuits by virtue of the definition of R_{rad} elements. Note, however, the extremely slow process of decay for this example. The time, $T_{1/2}$, for the amplitude of the current to decay to one-half of its maximum value is approximately

$$T_{1/2} = 2Lln2/R_{\text{rad}}. \tag{12}$$

With the above values for L and R_{rad}, $T_{1/2} = 6.8 \times 10^8s$ or about 22 years. Similarly, one can show that it takes about 11 years for the circuit to radiate away half of the energy difference between the initial and equilibrium states. Table I provides a brief survey of $T_{1/2}$ values for various two-capacitor circuits which satisfy the assumptions made earlier.

IV. Conclusion

The two-capacitor problem as typically presented is ambiguous. Assumptions A1)–A5) and A7)–A10) allow one to model the problem to account for energy loss through EMR using elementary circuit analysis techniques. These assumptions also represent a certain sacrifice of physical realism. Assumptions A1)–A4) are standard and provide good results in many low frequency, low voltage, low current situations. With regards to A4), however, in zero or low electrical resistance circuits the current may not be uniform throughout the cross section of the wire, being confined largely to the surface even at low frequencies. For example, superconducting wires transport current only on their surface [11]. This does not effect the formula for R_{rad}, but it is necessary to modify the formula for the geometric self inductance [10]:

$$L = \left(\mu_0 \frac{d}{2}\right)\left(\ln\left(\frac{4d}{r}\right) - 2.00\right). \tag{13}$$

In the example, this has the effect of reducing the induction from $3.1 \times 10^{-7} H$ to $2.9 \times 10^{-7} H$.

Considering the other assumptions, A5) and A7) are technical but not unreasonable; they are used in deriving and approximating integral expressions that lead to formulas (2), (9), and (13). Assumption A9) is analogous to a Hertzian dipole approximation and hence a reasonable consequence of A4). Assumption A10) is technical for the approximation $\omega \approx \omega_0$ but it can also be viewed as a consequence of A4) when one considers the dependence of Q on d. Assumption A8) is most critical. There is no *a priori* reason to believe that formula (2) for a driven circuit should be valid for the one considered here. However, arguing from Maxwell's equations [10] has demonstrated that A8) with $\omega \approx \omega_0$ does provide a good approximation for this circuit.

Note that in the cases considered here the radiation resistances are many orders of magnitude smaller than the electrical resistances which would be present if the wires consisted of a conductor such as copper. The wire in the first example would have a resistance of about 8×10^{-3} ohms [5]. This reflects the fact that radiation resistance effects are usually negligible in the presence of electrical resistance or other forms of energy loss.

If the scope of the two-capacitor problem is expanded to include circuits in which the above assumptions are not reasonable, then the problem becomes more complex. It should be recognized that the EMR losses presented here make only a small contribution towards the complete understanding of such circuits.

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Raymond P. Mayer received the A.B. degree from Alma College, Alma, MI, in 1955, the M.S. degree in chemistry, in 1957, and the Ph.D. degree in organic chemistry, in 1958, from the University of Michigan, Ann Arbor.

Since 1969, he has been Professor of Science and Mathematics at West Shore Community College, Scottville, MI. His prior positions were as Research Chemist at Michigan Chemical Corporation, St. Louis, MI and Dow Chemical Co., Ludington, MI. Previous papers include organic rearrangement mechanisms, catalytic dehydrogenation kinetics, design of experiments, and statistics. He has received several patents in the industrial chemical area.

Dr. Mayer is a member of the American Chemical Society, the American Association of Physics Teachers, and the Michigan Mathematical Association of Two Year Colleges.

John R. Jeffries received the B.S. degree in mathematics and physical science from Indiana University, Bloomington, in 1960, and the M.A. degree in mathematics from Purdue University, West Lafayette, IN, in 1967, and has done additional graduate study at Ball State University, Muncie, IN, and Michigan State University, East Lansing.

He has been Professor of Mathematics at West Shore Community College, Scottville, MI, since 1970.

From 1960 to 1970, he was a Secondary Education teacher and was the Chicago area Science Education Teacher of the Year in 1962–1963 and 1963–1964. As a charter member of the Michigan Engineering Liaison Committee, he has been active over the past 20 years in engineering curricula and articulation. He also works with Design of Experiments and Quality Control Education.

George F. Paulik received the B.S. degree in mathematics from Washington University, St. Louis, MO, in 1979, and the Ph.D. degree in mathematics from Indiana University, Bloomington, in 1985.

He is currently with the Department of Mathematics at the University of Iowa, Iowa City, IA. He previously taught at the University of Kentucky, Lexington, and Purdue University, West Lafayette, IN. His research and publications are in the area of partial differential equations. He is a member of the American Mathematical Society.