Answer to Question #26 ["Electromagnetic field momentum," Robert H. Romer, Am. J. Phys. 63(9), 777–779 (1995)]

(1) The difficulty in interpreting Poynting's vector as proportional to momentum for a system that includes sources as well as fields was first pointed out by Poincaré in 1906. A relativistically consistent formalism can only be achieved by adding terms that include stresses in the sources that arise when the fields are generated.

The usual relativistic argument begins by recasting the Lorentz-force 4-vector,

$$f_\mu = F_{\mu\nu} j^\nu = \left(\frac{\mathbf{j}\cdot\mathbf{E}}{c}, \rho\mathbf{E} + \frac{\mathbf{j}}{c} \times \mathbf{B} \right)$$

(in cgs units and with metric $\eta_{00}=1$, $\eta_{11}=\eta_{22}=\eta_{33}=-1$; Greek indices run over 0,1,2,3, while Latin indices run over 1,2,3) as the derivative of a stress tensor:

$$f_\mu = -\frac{\partial T_{\mu\nu}}{\partial x_\nu} = -\partial^\nu T_{\mu\nu}.$$

This leads to the result

$$T_{\mu\nu} = \frac{1}{4\pi} F_{\mu\alpha} F^\alpha_\nu + \frac{1}{16\pi} \eta_{\mu\nu} F^\beta_\alpha F^\beta_\alpha$$

$$= \begin{pmatrix} \dfrac{E^2+B^2}{8\pi} & \dfrac{\mathbf{E}\times\mathbf{B}}{4\pi} = \dfrac{\mathbf{S}}{c} \\ \dfrac{\mathbf{E}\times\mathbf{B}}{4\pi} & -\dfrac{1}{4\pi}(E_iE_j + B_iB_j - \delta_{ij}) \dfrac{E^2+B^2}{2} \end{pmatrix}.$$

Next, one makes a trial definition of an energy-momentum 4-vector for the fields as

$$P_\mu = \int T_{0\mu}\, dvol,$$

so that

$$P_0 = \int T_{00}\, dvol = \frac{1}{8\pi} \int (E^2+B^2) dvol = U_f,$$

$$P_i = \int T_{0i}\, dvol = \frac{1}{4\pi} \int \mathbf{E}\times\mathbf{B}\, dvol = c\mathbf{P}_f,$$

where

$$\mathbf{P}_f = \frac{1}{4\pi c} \int \mathbf{E}\times\mathbf{B}\, dvol$$

is the field 3-momentum that is the subject of Question #26. Then, $P_\mu = (U, c\mathbf{P}_f)$ has the appearance of a familiar 4-vector.

(2) If there are no sources present (free-field case), then the Lorentz-force 4-vector vanishes, the 4-divergence of $T_{\mu\nu}$ vanishes also, and one can verify that P_μ really transforms like a 4-vector.

The argument thus far is seconded in the books of Rohrlich and of Jackson, who do not advocate carrying it further.

(3) Poincaré suggests we proceed to the case where sources of the fields are present. By direct application of a Lorentz transformation to the stress tensor $T^\star_{\mu\nu}$, where the $\star$ indicates the rest frame of the sources, one deduces that P_μ fails to transform like a 4-vector if there are nonzero spatial components to the stress tensor, i.e., if some $\int T^\star_{ij} \neq 0$.

Poincaré noted that if some $\int T^\star_{ij}$ are nonzero, then the system of sources is not in mechanical equilibrium until mechanical stresses $\int P^\star_{ij} = -\int T^\star_{ij}$ are developed to counter the electromagnetic stresses. The $P^\star_{ij}$ can be embedded in a 4-tensor $P_{\mu\nu}$ that includes the mechanical rest energy $m_{\mathrm{mech}}c^2 = \int P^\star_{00}$ and the mechanical momentum $c\mathbf{P}_{\mathrm{mech}} = \int P^\star_{0i} = \int P^\star_{i0}$. Then, when one defines

$$P_\mu = \int (T_{0\mu} + P_{0\mu}) dvol,$$

one has a true 4-vector, with

$$P_0 = U + m_{\mathrm{mech}}c^2, \quad P_i = c(\mathbf{P}_f + \mathbf{P}_{\mathrm{mech}}).$$

This formalism does not quite succeed in providing an independent interpretation of the "field momentum" $\mathbf{P}_f$ when sources are present. That is, only the sum $\mathbf{P}_f + \mathbf{P}_{\mathrm{mech}}$ has a dynamical meaning, where $\mathbf{P}_{\mathrm{mech}}$ includes a contribution associated with the mechanical stresses that arise in response to electromagnetic forces.

(4) There remains the specific topic of Question #26: What interpretation should be given when $\mathbf{P}^\star_f \neq 0$ in the "rest frame" of the sources? In view of the difficulty of giving any independent meaning to $\mathbf{P}_f$ when sources are present, this issue is secondary.

It is not very satisfactory to note that one can always find a frame in which $\mathbf{P}_f$ vanishes, since, in general, the center of mass of the sources will be moving in this frame.

Instead, we advocate a fairly trivial solution to the problem. Simply regard the value $\mathbf{P}^\star_f$ as a constant of the system without an interpretation of anything being in motion. This is a consistent view because the dynamical significance of momentum is in its derivative,

$$f_\mu = \frac{dP_\mu}{d\tau},$$

where τ is the proper time, and in conservation laws, both of which are unaffected by an additive constant. In this sense no dynamical meaning can be assigned to the value of $\mathbf{P}^\star_f$, and one can consistently choose not to give it any further interpretation.

We can amplify this point by recalling the Lorentz transformation of the 4-momentum $P_\mu = (U_f, c\mathbf{P}_f)$ in a boost by $\boldsymbol{\beta} = \mathbf{v}/c$ from the rest ($\star$) frame:

$$\mathbf{P}_f = \gamma \left(\mathbf{P}^\star_f + \frac{U^\star_f}{c^2} \mathbf{v} \right),$$

where $\gamma = 1/\sqrt{1-(v/c)^2}$. Thus, in a frame where the system moves with the velocity $\mathbf{v}$, the part of the momentum that is proportional to velocity depends on the effective mass $U^\star_f/c^2$ in the rest frame and not on the momentum $\mathbf{P}^\star_f$ in the rest frame. A nonzero value of $\mathbf{P}^\star_f$ in the rest frame has no dynamical effect on the momentum.

We have gotten used to electrons and photons having spin without being able to identify anything that rotates. So I propose that we not worry too much about a nonzero static value for the "field momentum" that has no dynamical consequence. Foregoing any interpretation of $\mathbf{P}^\star_f$ is even easier than for electron spin since that latter has dynamical significance.

(5) I append a further argument (which perhaps has an error) that shows how the "field momentum" $\mathbf{P}_f$ by itself

does not consistently behave like a nonrelativistic momentum, whether or not its value in the rest frame of the sources is zero.

We consider a system that, when at rest, produces fields $\mathbf{E}_0$ and $\mathbf{B}_0$. The corresponding "field momentum" $\mathbf{P}_0$ may or may not be zero, but, in any case, is a constant vector. Only the velocity-dependent part of the "field momentum" will have relevance to $\mathbf{F}=d\mathbf{P}/dt$.

Next, consider the system when it is moving with center-of-mass velocity $\mathbf{v}$, where $v \ll c$. We suppose that there is no change in the state of the system relative to its center of mass, so fields $\mathbf{E}_0$ and $\mathbf{B}_0$ still hold in the rest frame of the system. Then the nonrelativistic limit of the transformation of the electromagnetic field tells us that

$$\mathbf{E}=\mathbf{E}_0-\frac{\mathbf{v}}{c}\times\mathbf{B}_0, \quad \mathbf{B}=\mathbf{B}_0+\frac{\mathbf{v}}{c}\times\mathbf{E}_0,$$

and so the "field momentum" associated with the moving system is

$$\mathbf{P}_f=\mathbf{P}_0+\frac{1}{4\pi c^2}\int [(E_0^2+B_0^2)\mathbf{v}+(\mathbf{E}_0\cdot\mathbf{v})\mathbf{E}_0 + (\mathbf{B}_0\cdot\mathbf{v})\mathbf{B}_0]dvol,$$

neglecting a term in $(v/c)^2$. The rate of change of this momentum is

$$\frac{d\mathbf{P}_f}{dt}=\frac{2U_0}{c^2}\mathbf{a}+\frac{1}{2\pi c^2}\int (\mathbf{E}_0\cdot\mathbf{a})\mathbf{E}_0+(\mathbf{B}_0\cdot\mathbf{a})\mathbf{B}_0]dvol,$$

where $\mathbf{a}=d\mathbf{v}/dt$ is the acceleration of the system and U_0 is the rest-frame field energy:

$$U_0=\frac{1}{8\pi}\int (E_0^2+B_0^2)dvol.$$

While, as expected, the constant value P_0 does not appear in the expression for the rate of change of "field momentum," this expression does not quite have the desired form, $m_{\text{eff}}\mathbf{a}$. I infer that this is another demonstration of the view of Poincaré that the "field momentum" $\mathbf{P}_f$ cannot be interpreted by itself when sources are present.

(6) Regarding the specific example of nested electric and magnetic dipoles, it is easy to see that the diagonal elements of the electromagnetic stress tensor, T_{ii}, are nonvanishing. The sphere of charge and sphere of current-carrying coils would fly apart without some kind of glue. The resulting mechanical stresses change the rest mass of the system and, when it is in motion, its momentum by an amount comparable to the electromagnetic "mass" and momentum contributions. Trying to interpret the electromagnetic momentum without considering the corresponding stress-induced changes in the mechanical momentum is counterproductive.

But the bottom line is that no meaningful interpretation can be given to the nonzero $\mathbf{P}_f^*$ for that system in its rest frame.

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Answer to Question #26 ["Electromagnetic field momentum," Robert H. Romer, Am. J. Phys. 63(9), 777–779 (1995)]

Here are two well-known facts from special relativity: (1) At the location of sources the electromagnetic energy tensor T_{elm} is not divergence-free; (2) the necessary and sufficient condition for the hyperplane integral of a symmetric tensor of second rank T to be independent of the orientation of that hyperplane is that T be divergence-free; and if that integral is orientation-independent, then it is a four-vector. Fact (2) is sometimes called "von Laue's theorem."

From these two facts it follows that the electromagnetic four-momentum when defined as the hyperplane integral of T_{elm} [see, e.g., CCP[1] (Eq. 6–18)] is not a four-vector if there are sources present. This result is physically obvious because the presence of sources implies that the system is not closed unless other (nonelectromagnetic) forces are included. In the particular system considered in Question #26 this is especially evident: When considered in its *rest* frame, the electromagnetic interactions by themselves yield a nonvanishing momentum which must be held in equilibrium by other interactions. (The realization of the system involves current carrying wires, etc.) But when the system is closed, the total energy tensor will be divergence-free and by (2) the total four-momentum will be a four-vector. A well-known special case of this is the extended charged particle with Poincaré stresses.

It would seem that one must deal with a *closed* system if one wants an energy-momentum *four-vector*. However, there is a way around it: Independence of the plane orientation is a sufficient but not a necessary condition for the integral to be a four-vector. One can define the electromagnetic energy-momentum P_{elm} for a system in uniform motion as the Lorentz boosted rest frame [see CCP (Eq. 6–24)]. That integral is not independent of the orientation of the hyperplane: the plane is specified to be the plane in which the system is at rest. So defined, P_{elm} is a four-vector even though there are sources present! In the rest frame, it is just the conventional definition as given in Question #26. Romer's equation (3) is therefore correct and consistent with (Eq. 6–24).

The advantage of this definition lies in its separating the electromagnetic fields from the rest of the system in a covariant way: $P_{\text{total}}=P_{\text{elm}}+P_{\text{other}}$. These are all four-vectors.

Finally, one should note that radiation fields form a T_{elm} that is always divergence-free, so that all the above is of interest only for "bound" fields.

The answer I have just given is contained in abbreviated form in the paper by Griffiths and Owen.[2] But since the issue has been raised again, I have presented it in a more general and explicit form.

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[1]CCP refers to my book *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1965 and 1990).

[2]David J. Griffiths and Russell E. Owen, " Mass renormalization in classical electrodynamics," Am. J. Phys. **51** (12), 1120–1126 (1983).