7.3 Green's Function for the Scalar Wave Equation

The Green's function for the scalar Helmholtz equation, just discussed in Sec. 7.2, is particularly useful in solving inhomogeneous problems, *i.e.* problems which arise whenever sources are present within the volume or on the bounding surface. The Green's function for the scalar wave equation must perform a similar function; thus it should be possible to solve the scalar wave equation, with sources present, in terms of a Green's function. To obtain some notion as to the equation this function must satisfy let us consider a typical inhomogeneous problem. Let ψ satisfy

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi q(r,t) \tag{7.3.1}$$

The function $q(r,t)$ describes the source density, giving not only the distribution of sources in space but also the time dependence of the sources at each point in space. In addition to Eq. (7.3.1) it is necessary to state boundary and initial conditions in order to obtain a unique solution (7.3.1). The condition on the boundary surface may be either Dirichlet or Neumann or a linear combination of both. The conditions in time dimension must be Cauchy (see page 685, Chap. 6). Hence it is necessary to specify the value of ψ and $(\partial\psi/\partial t)$ at $t = t_0$ for every point of the region under consideration. Let these values be $\psi_0(r)$ and $v_0(r)$, respectively.

Inspection of (7.3.1) suggests that the equation determining the Green's function $G(\mathbf{r},t|\mathbf{r}_0,t_0)$ is

$$\nabla^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0) \tag{7.3.2}$$

We see that the source is an impulse at $t = t_0$, located at $\mathbf{r} = \mathbf{r}_0$. G then gives the description of the effect of this impulse as it propagates away from $\mathbf{r} = \mathbf{r}_0$ in the course of time. As in the scalar Helmholtz case, G satisfies the homogeneous form of the boundary conditions satisfied by ψ on the boundary. For initial conditions, it seems reasonable to assume that G and $\partial G/\partial t$ should be zero for $t < t_0$; that is if an impulse occurs at t_0, *no effects of the impulse should be present at an earlier time.*

It should not be thought that this cause-and-effect relation, employed here, is obvious. The unidirectionality of the flow of time is apparent for macroscopic events, but it is not clear that one can extrapolate this experience to microscopic phenomena. Indeed the equations of motion in mechanics and the Maxwell equations, both of which may lead to a wave equation, do not have any asymmetry in time. It may thus be possible, for microscopic events, for "effects" to propagate backward in time; theories have been formulated in recent years which employ such solutions of the wave equation. It would take us too far afield, however, to discuss how such solutions can still lead to a cause-effect time relation for macroscopic events.

For the present we shall be primarily concerned with the initial conditions $G(\mathbf{r},t|\mathbf{r}_0,t_0)$ and $\partial G/\partial t$ zero for $t < t_0$, though the existence of other possibilities should not be forgotten.

The Reciprocity Relation. The directionality in time imposed by the Cauchy conditions, as noted above, means that the generalization of the reciprocity relation $G_k(\mathbf{r}|\mathbf{r}_0) = G_k(\mathbf{r}_0|\mathbf{r})$ to include time is not $G(\mathbf{r},t|\mathbf{r}_0,t_0) = G(\mathbf{r}_0 t_0|\mathbf{r},t)$. Indeed if $t > t_0$, the second of these is zero. In order to obtain a reciprocity relation it is necessary to reverse the direction of the flow of time, so that the reciprocity relation becomes

$$G(\mathbf{r},t|\mathbf{r}_0,t_0) = G(\mathbf{r}_0,-t_0|\mathbf{r},-t) \tag{7.3.3}$$

To interpret (7.3.3) it is convenient to place $t_0 = 0$. Then $G(\mathbf{r},t|\mathbf{r}_0,0) = G(\mathbf{r}_0,0|\mathbf{r},-t)$. We see that the effect, at $\mathbf{r}$ at a time t later than an impulse started at $\mathbf{r}_0$, equals the effect, at $\mathbf{r}_0$ at a time 0, of an impulse started at $\mathbf{r}$ at a time $-t$, that is t earlier.

To prove (7.3.3) let us write the equations satisfied by both of the Green's functions:

$$\nabla^2 G(\mathbf{r},t|\mathbf{r}_0,t_0) - \frac{1}{c^2}\left[\frac{\partial^2 G(\mathbf{r},t|\mathbf{r}_0,t_0)}{\partial t^2}\right] = -4\pi\delta(\mathbf{r}-\mathbf{r}_0)\delta(t-t_0)$$

$$\nabla^2 G(\mathbf{r},-t|\mathbf{r}_1,-t_1) - \frac{1}{c^2}\left[\frac{\partial^2 G(\mathbf{r},-t|\mathbf{r}_1,-t_1)}{\partial t^2}\right] = -4\pi\delta(\mathbf{r}-\mathbf{r}_1)\delta(t-t_1)$$

Multiplying the first of these by $G(\mathbf{r},-t|\mathbf{r}_1,-t_1)$ and the second by $G(\mathbf{r},t|\mathbf{r}_0,t_0)$, subtracting, and integrating over the region under investigation and over time t from $-\infty$ to t' where $t' > t_0$ and $t' > t_1$, then

$$\int_{-\infty}^{t'} dt \int dV \Big\{ G(\mathbf{r},t|\mathbf{r}_0,t_0)\nabla^2 G(\mathbf{r},-t|\mathbf{r}_1,-t_1) - G(\mathbf{r},-t|\mathbf{r}_1,-t_1)\nabla^2 G(\mathbf{r},t|\mathbf{r}_0,t_0)$$
$$+ \frac{1}{c^2} G(\mathbf{r},t|\mathbf{r}_0,t_0)\frac{\partial^2}{\partial t^2} G(\mathbf{r},-t|\mathbf{r}_1,-t_1) - \frac{1}{c^2} G(\mathbf{r},-t|\mathbf{r}_1,-t_1)\frac{\partial^2}{\partial t^2} G(\mathbf{r},t|\mathbf{r}_0,t_0)\Big\}$$
$$= 4\pi\{G(\mathbf{r}_0,-t_0|\mathbf{r}_1,-t_1) - G(\mathbf{r}_1,t_1|\mathbf{r}_0,t_0)\} \tag{7.3.4}$$

The left-hand side of the above equation may be transformed by use of Green's theorem and by the identity

$$\frac{\partial}{\partial t}\left[G(\mathbf{r},t|\mathbf{r}_0,t_0)\frac{\partial}{\partial t}G(\mathbf{r},-t|\mathbf{r}_1,-t_1) - G(\mathbf{r},-t|\mathbf{r}_1,-t_1)\frac{\partial}{\partial t}G(\mathbf{r},t|\mathbf{r}_0,t_0)\right]$$
$$= G(\mathbf{r},t|\mathbf{r}_0,t_0)\frac{\partial^2}{\partial t^2}G(\mathbf{r},-t|\mathbf{r}_1,-t_1) - G(\mathbf{r},-t|\mathbf{r}_1,-t_1)\frac{\partial^2}{\partial t^2}G(\mathbf{r},t|\mathbf{r}_0,t_0)$$

We obtain for the left-hand side

$$\int_{-\infty}^{t'} dt \int d\mathbf{S} \cdot [G(\mathbf{r},t|\mathbf{r}_0,t_0) \operatorname{grad} G(\mathbf{r},-t|\mathbf{r}_1,-t_1) - G(\mathbf{r},-t|\mathbf{r}_1,-t_1) \operatorname{grad} G(\mathbf{r},t|\mathbf{r}_0,t_0)]$$
$$+ \frac{1}{c^2} \int dV \left[G(\mathbf{r},t|\mathbf{r}_0,t_0) \frac{\partial G(\mathbf{r},-t|\mathbf{r}_1,-t_1)}{\partial t} - G(\mathbf{r},-t|\mathbf{r}_1,-t_1) \frac{\partial G(\mathbf{r},t|\mathbf{r}_0,t_0)}{\partial t} \right]_{t=-\infty}^{t=t'}$$

The first of these integrals vanishes, for both Green's functions satisfy the same homogeneous boundary conditions on S. The second also vanishes, as we shall now see. At the lower limit both $G(\mathbf{r},-\infty|\mathbf{r}_0,t_0)$ and its time derivative vanish in virtue of the causality condition. At the time $t = t'$, $G(\mathbf{r},-t'|\mathbf{r}_1,-t_1)$ and its time derivative vanish, since $-t'$ is earlier than $-t_1$. Thus the left-hand side of (7.3.4) is zero, yielding reciprocity theorem (7.3.3).

We shall demonstrate that it is possible to express the solution (including initial conditions) of the inhomogeneous problem for the scalar wave equation in terms of known inhomogeneities in the Green's function. We shall need Eq. (7.3.1):

$$\nabla_0^2 \psi(\mathbf{r}_0,t_0) - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t_0^2} = -4\pi q(\mathbf{r}_0,t_0)$$

also $$\nabla_0^2 G(\mathbf{r},t|\mathbf{r}_0,t_0) - \frac{1}{c^2} \frac{\partial^2 G(\mathbf{r},t|\mathbf{r}_0,t_0)}{\partial t_0^2} = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0)$$

This last equation may be obtained from (7.3.2) by the use of the reciprocity relation. As is usual, multiply the first equation by G and the second by ψ and subtract. Integrate over the volume of interest and over t_0 from 0 to t^+. By the symbol t^+ we shall mean $t + \epsilon$ where ϵ is arbitrarily small. This limit is employed in order to avoid ending the integration exactly at the peak of a delta function. When employing the final formulas, it is important to keep in mind the fact that the limit is t^+ rather than just t. One obtains

$$\int_0^{t^+} dt_0 \int dV_0 \left\{ G\nabla_0^2\psi - \psi\nabla_0^2 G + \frac{1}{c^2}\left(\frac{\partial^2 G}{\partial t_0^2}\psi - G\frac{\partial^2 \psi}{\partial t_0^2} \right) \right\} = 4\pi \left\{ \psi(\mathbf{r},t) - \int_0^{t^+} dt_0 \int dV_0 q(\mathbf{r}_0,t_0) G \right\}$$

Again employing Green's theorem, etc., we obtain

$$\int_0^{t^+} dt_0 \oint d\mathbf{S}_0 \cdot (G \operatorname{grad}_0 \psi - \psi \operatorname{grad}_0 G) + \frac{1}{c^2} \int dV_0 \left[\frac{\partial G}{\partial t_0}\psi - G\frac{\partial \psi}{\partial t_0} \right]_0^{t^+} + 4\pi \int_0^{t^+} dt_0 \int dV_0 q(\mathbf{r}_0,t_0) G = 4\pi\psi(\mathbf{r},t)$$

The integrand in the first integral is specified by boundary conditions. In the second integral, the integrand vanishes when $t = t^+$ is introduced by virtue of the initial conditions on G. The remaining limit involves only initial conditions. Hence,

$$\begin{aligned}4\pi\psi(\mathbf{r},t) = 4\pi \int_0^{t^+} dt_0 \int dV_0 G(\mathbf{r},t|\mathbf{r}_0,t_0) q(\mathbf{r}_0,t_0) \\ + \int_0^{t^+} dt_0 \oint d\mathbf{S}_0 \cdot (G \operatorname{grad}_0 \psi - \psi \operatorname{grad}_0 G) \\ - \frac{1}{c^2} \int dV_0 \left[\left(\frac{\partial G}{\partial t_0}\right)_{t_0=0} \psi_0(\mathbf{r}_0) - G_{t_0=0} v_0(\mathbf{r}_0) \right] \quad (7.3.5)\end{aligned}$$

where $\psi_0(\mathbf{r}_0)$ and $v_0(\mathbf{r}_0)$ are the initial values of ψ and $\partial\psi/\partial t$.

Equation (7.3.5) gives the complete solution of the inhomogeneous problem including the satisfaction of initial conditions. The surface integrals, as in the Helmholtz case, must be carefully defined. As in that case we shall take a surface value to be the limit of the value of the function as the surface is approached from the interior.

The first two integrals on the right side of the above Eq. (7.3.5) are much the same sort as those appearing in the analogous equation for the case of the Helmholtz equation. The first represents the effect of sources; the second the effect of the boundary conditions on the space boundaries. The last term involves the initial conditions. We may interpret it by asking what sort of source q is needed in order to start the function ψ at $t = 0$ in the manner desired. We may expect that this will require an impulsive type force at a time $t = 0^+$. From (7.3.5) we can show that the source term required to duplicate the initial conditions is

$$(1/c^2)[\psi_0(\mathbf{r}_0)\delta'(t_0) + v_0(\mathbf{r}_0)\delta(t_0)]$$

where by $\delta'(t_0)$ we mean the derivative of the δ function. It has the property

$$\int_a^b f(x)\delta'(x)\,dx = \begin{cases} -f'(0); & \text{if } x = 0 \text{ is within interval } (a,b) \\ 0; & \text{if } x = 0 \text{ is outside interval } (a,b) \end{cases}$$

The physical significance of these terms may be understood. A term of type $v_0\delta(t_0)$ is required to represent an impulsive force, which gives each point of the medium an initial velocity $v_0(\mathbf{r}_0)$. To obtain an initial displacement, an impulse delivered at $t_0 = 0$ must be allowed to develop for a short time until the required displacement is achieved. At this time a second impulse is applied to reduce the velocity to zero but leave the displacement unchanged. It may be seen that the first term $\psi(\mathbf{r}_0,t_0)\delta'(t_0)$ has this form if it is written

$$\lim_{\epsilon\to 0} \left\{ \psi(\mathbf{r}_0,t_0) \left[\frac{\delta(t_0+\epsilon) - \delta(t_0-\epsilon)}{2\epsilon} \right] \right\}$$

Form of the Green's Function. Knowledge of G is necessary to make (7.3.5) usable. As in the case of the scalar Helmholtz equation we shall first find G for the infinite domain. Let us call this function g. The method employed in the scalar Helmholtz case involves assessing the relative strength of the singularities in the functions $\nabla^2 g$ and $\partial^2 g/\partial t^2$ in the equation

$$\nabla^2 g - \frac{1}{c^2}\left(\frac{\partial^2 g}{\partial t^2}\right) = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0)$$

It may be argued that $\nabla^2 g$ is the more singular, since it involves the second derivative of a three-dimensional δ function $\delta(\mathbf{r} - \mathbf{r}') = \delta(x - x') \cdot \cdot \delta(y - y')\delta(z - z')$. Such an argument is not very satisfying. However, for the moment, let us assume it to be true. We shall return to the above equation later and derive the result we shall obtain in a more rigorous manner.

Integrating both sides of the equation over a small spherical volume surrounding the point $\mathbf{r} = \mathbf{r}_0$, that is, $R = 0$, and neglecting the time derivative term, one obtains as in the previous section

$$g \xrightarrow[R\to 0]{} \delta(t - t_0)/R \tag{7.3.6}$$

As before we now proceed to find a solution of the homogeneous equation satisfying this condition, for it is clear that g satisfies the equation

$$\nabla^2 g - \frac{1}{c^2}\frac{\partial^2 g}{\partial t^2} = 0; \quad R \text{ and } t - t_0 \text{ not equal to zero}$$

At $R = 0$ condition (7.3.6) must be employed. Since we are dealing with point sources in an infinite medium g is a function of R rather than of $\mathbf{r}$ and $\mathbf{r}_0$ separately. Hence

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial g}{\partial R}\right) - \frac{1}{c^2}\left(\frac{\partial^2 g}{\partial t^2}\right) = 0 \quad \text{or} \quad \frac{\partial^2(gR)}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2(gR)}{\partial t^2} = 0 \tag{7.3.7}$$

The solutions of this equation are

$$g = \frac{h[(R/c) - (t - t_0)] + k[(R/c) + (t - t_0)]}{R}$$

where h and k are any functions. Comparing with condition (7.3.6) we see that two possibilities (or any linear combination of these) occur, $\delta[(R/c) - (t - t_0)]/R$ or $\delta[(R/c) + (t - t_0)]/R$. The second of these must be eliminated, for it does not satisfy the condition imposed earlier, which requires that the effect of an impulse at a time t_0 be felt at a distance R away at a time $t > t_0$. Therefore

$$g = \frac{\delta[(R/c) - (t - t_0)]}{R}; \quad R, t - t_0 > 0 \tag{7.3.8}$$

representing a spherical shell about the source, expanding with a radial velocity c.

We may now make an a posteriori check of our initial assumption, that the singularity of $\nabla^2 g$ was greater than that for $\partial^2 g/\partial t^2$. This is indicated by the presence of the $1/R$ factor, but to prove it requires a rather nice balancing of infinities. We shall therefore stop to put (7.3.8) on a more firm footing and only then return to discuss the deductions which follow from this formula. Using spherical coordinates for $\delta(\mathbf{R}) = \delta(\mathbf{r} - \mathbf{r}_0)$ and defining

$$\tau = t - t_0$$

it is immediately possible to retrace the steps leading to (7.3.7) and obtain the more general equation, valid also for R and τ equal to zero,

$$\frac{\partial^2(Rg)}{\partial R^2} - \frac{1}{c^2}\frac{\partial^2(Rg)}{\partial \tau^2} = -\frac{2\delta(R)}{R}\,\delta(\tau)$$

The numerical factor 2 enters because the variable R can never be negative. Hence $\int_0^\infty \delta(R)\,dR = \frac{1}{2}$.

To proceed further it is desirable to employ the relation

$$\delta(R)/R = -\delta'(R) \tag{7.3.9}$$

To demonstrate this multiply $\delta(R)/R$ by a differentiable function $f(R)$ and integrate over R. Let $f(R) = f(0) + f'(0)R + f''(0)(R^2/2!) + \cdots$. Then

$$\int_{-\infty}^{\infty} \frac{f(R)\delta(R)}{R}\,dR = f(0)\int_{-\infty}^{\infty}\frac{\delta(R)}{R}\,dR + f'(0)\int_{-\infty}^{\infty}\delta(R)\,dR + \frac{f''(0)}{2!}\int_{-\infty}^{\infty} R\delta(R)\,dR + \cdots$$

The first of these terms is an integral over an odd function, so that it has a Cauchy principal value of zero; the second one gives $f'(0)$; the third and all higher terms give zero. Hence

$$\int_{-\infty}^{\infty}\frac{f(R)\delta(R)}{R}\,dR = f'(0) = \int_{-\infty}^{\infty} f'(R)\delta(R)\,dR = -\int_{-\infty}^{\infty} f(R)\delta'(R)\,dR$$

This equation may also be derived more directly from the definition of derivative as follows:

$$\delta'(R) = \lim_{\epsilon\to 0}\left[\frac{\delta(R+\epsilon) - \delta(R-\epsilon)}{2\epsilon}\right] = \lim_{\epsilon\to 0}\tfrac{1}{2}\left[\frac{\delta(R+\epsilon)}{-R} - \frac{\delta(R-\epsilon)}{R}\right] = -\frac{\delta(R)}{R}$$

Returning to the equation, we may now write

$$\frac{\partial^2}{\partial R^2}(Rg) - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}(Rg) = 2\delta'(R)\,\delta(\tau)$$

It is clearly appropriate to introduce the variables

$$\xi = R - c\tau; \quad \eta = R + c\tau \tag{7.3.10}$$

The meaning of $\delta'(R)\,\delta(\tau)$ in the new variables must also be determined. To do this note that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau\, dR\, f(\tau,R)\delta'(R)\,\delta(\tau) = -\left(\frac{\partial f}{\partial R}\right)_{R,\tau=0}$$

This may be rewritten $-[(\partial f/\partial \xi) + (\partial f/\partial \eta)]_{\xi,\eta=0}$. Hence under transformation (7.3.10)

$$\delta'(R)\,\delta(\tau) = 2c[\delta'(\xi)\,\delta(\eta) + \delta'(\eta)\,\delta(\xi)] \tag{7.3.11}$$

The factor $2c$ is just the inverse of the change of volume element under the transformation from variables (R,τ) to (ξ,η).

In the new variables ξ and η the equation satisfied by Rg becomes, therefore,

$$\partial^2(Rg)/\partial\xi\,\partial\eta = c[\delta'(\xi)\,\delta(\eta) + \delta'(\eta)\,\delta(\xi)]$$

or $$Rg = c\int_{-\infty}^{\xi} d\xi \int_{-\infty}^{\eta} d\eta\, \delta'(\xi)\,\delta(\eta) + c\int_{\xi}^{\infty} d\xi \int_{\eta}^{\infty} d\eta\, \delta(\xi)\,\delta'(\eta)$$

The limits of integration have been chosen in such a fashion as to yield a solution satisfying the required initial conditions. Upon integration,

$$Rg = c\,\delta(\xi)u(\eta) - c\,\delta(\eta)[1 - u(\xi)] \tag{7.3.12}$$

where $u(\eta) = \begin{cases} 0; & \eta < 0 \\ 1; & \eta > 0 \end{cases}$; therefore $1 - u(\xi) = \begin{cases} 1; & \xi < 0 \\ 0; & \xi > 0 \end{cases}$

The second term of (7.3.12) may be dropped; we can show that one or both of its factors are zero everywhere. The function $\delta(\eta)$ differs from zero only when $\eta = 0$ (*i.e.*, when $c\tau = -R$), but at that point $\xi = 2R$ so that $1 - u(\xi) = 0$. On the other hand, in the first term $\delta(\xi)$ differs from zero when $c\tau = R$, where $\eta = 2R$ and $u(\eta) = 1$. Solving for g in (7.3.12), replacing ξ by $[R - c(t - t_0)]$ and $u(\eta)$ by 1 (for R, $t - t_0 > 1$), Eq. (7.3.12) becomes (7.3.8) as desired.

In order to obtain some concept as to the meaning of (7.3.8), consider the infinite-domain case with initial conditions $\psi = \partial\psi/\partial t = 0$ at $t = 0$. Then

$$\psi(\mathbf{r},t) = \frac{1}{4\pi}\int_0^{t^+} dt_0 \int dV_0 \left\{\frac{\delta[(R/c) - (t - t_0)]}{R}\right\} q(\mathbf{r}_0,t_0); \quad t^+ = t + \epsilon$$

and finally, $$\psi(\mathbf{r},t) = \frac{1}{4\pi}\int dV_0 \left[\frac{q(\mathbf{r}_0, t - R/c)}{R}\right] \tag{7.3.13}$$

We see that the effect at $\mathbf{r}$ at a time t is "caused" by the value of the source function q at $\mathbf{r}_0$ at a time $t - (1/c)|\mathbf{r} - \mathbf{r}_0|$. This is just the statement that the velocity of propagation of the disturbance is c. When the velocity of propagation becomes infinite, the solution reduces to the familiar solution of the Poisson equation, a potential as it should, since the inhomogeneous scalar wave equation becomes the Poisson equation in this limit. As a consequence (7.3.13) is often referred to as the *retarded potential* solution.

Field of a Moving Source. As a simple example consider a point source traveling in an infinite medium with a velocity $\mathbf{v}$. Then $q = q_0\,\delta(\mathbf{r} - \mathbf{v}t)$, where q_0 is related to the strength of the source. From (7.3.5)

$$\psi(\mathbf{r},t) = \frac{q_0}{4\pi}\int_0^{t^+} dt_0 \int dV_0 \left\{\frac{\delta[(R/c) - (t - t_0)]\,\delta(\mathbf{r}_0 - \mathbf{v}t_0)}{R}\right\}$$
$$= \frac{q_0}{4\pi}\int_0^{t^+} dt_0 \left\{\frac{\delta[(1/c)|\mathbf{r} - \mathbf{v}t_0| - (t - t_0)]}{|\mathbf{r} - \mathbf{v}t_0|}\right\}$$

Let
$$p = (1/c)|\mathbf{r} - \mathbf{v}t_0| + t_0$$

Then
$$dp = dt_0\left[\frac{v^2t_0 - \mathbf{v}\cdot\mathbf{r}}{c|\mathbf{r} - \mathbf{v}t_0|} + 1\right]$$

so that
$$\psi(\mathbf{r},t) = \frac{q_0}{4\pi}\int_{r/c}^{t^+ + |r - vt^+|/c} \frac{dp\,\delta(p - t)}{(1/c)(v^2t_0 - \mathbf{v}\cdot\mathbf{r}) + |\mathbf{r} - \mathbf{v}t_0|}$$

The singularity of the δ function occurs when $p = t$. This must define a time t_0 such that a signal emitted by the particle at a time t_0 will arrive at $\mathbf{r}$ at a time t (see Fig. 7.9). The time $(t - t_0)$ equals therefore the distance traveled, $|\mathbf{r} - \mathbf{v}t_0|$ divided by c; $t - t_0 = |\mathbf{r} - \mathbf{v}t_0|/c$, which is just $p = t$. The quantity $\mathbf{r} - \mathbf{v}t_0 = \boldsymbol{\varrho}$ is thus the distance of the source from the observation point $\mathbf{r}$, at a time ρ/c earlier than time t. The time t_0, called the *retarded time*, is the solution of the equation $p = t$, giving the time and therefore the source position $\mathbf{v}t_0$ at the retarded time.

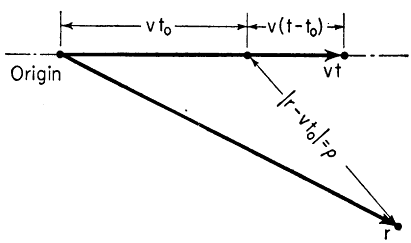

Fig. 7.9 Retarded potential of a moving source.

Integrating over p

$$\psi(r,t) = \frac{q_0}{4\pi}\left[\frac{1}{(1/c)(v^2t_0 - \mathbf{v}\cdot\mathbf{r}) + |\mathbf{r} - \mathbf{v}t_0|}\right]_{p=t}$$

Introducing $\boldsymbol{\varrho} = \mathbf{r} - \mathbf{v}t_0$, we have

$$\psi(\mathbf{r},t) = \frac{q_0}{4\pi}\frac{1}{\rho - (\mathbf{v}\cdot\boldsymbol{\varrho}/c)} \tag{7.3.14}$$

where $\boldsymbol{\varrho}$ is a vector drawn from the position of the source to the observation point at a time $t_0 = t - \rho/c$. We have obtained this solution in Chap. 2 by another method, where it was shown that the factor $1 - (\mathbf{v} \cdot \boldsymbol{\varrho}/c\rho)$ had to be introduced in order to take into account the fact that the source moves in the course of the integration, the factor being required for normalization of the source (see page 214).

Two-dimensional Solution. If the source distribution q is independent of z, we have a problem in which the dependence on space is only two-dimensional; that is, ψ depends only upon x and y. The "two-dimensional point source" for such a problem is a line source, a uniform source extending from $z_0 = -\infty$ to $z_0 = +\infty$ along a line parallel to the z axis passing through (x_0,y_0). The Green's function for two-dimensional problems may be therefore found by integrating the three-dimensional point source from $z_0 = -\infty$ to $z_0 = +\infty$:

$$g(\boldsymbol{\varrho},t|\boldsymbol{\varrho}_0,t_0) = \int_{-\infty}^{\infty} \frac{\delta[(R/c) - (t - t_0)]}{R}\, dz_0$$

where $\boldsymbol{\varrho} = x\mathbf{i} + y\mathbf{j}$ is the radius vector in the x, y plane. We may expect that g is a function of $|\boldsymbol{\varrho} - \boldsymbol{\varrho}_0| = P$ and of $t - t_0 = \tau$. Indeed, the above equation may be rewritten

$$g(P,\tau) = \int_{-\infty}^{\infty} \frac{\delta[(R/c) - \tau]}{R}\, d\zeta$$

where $\zeta = z_0 - z$ and where

$$R^2 = \zeta^2 + P^2; \quad d\zeta/dR = R/\zeta$$

Hence

$$g(P,\tau) = 2\int_0^{\infty} \frac{\delta[(R/c) - \tau]}{\sqrt{R^2 - P^2}}\, dR = \frac{2}{\sqrt{c^2\tau^2 - P^2}} \int_0^{\infty} \delta[(R/c) - \tau]\, dR$$

or finally

$$g(P,\tau) = \begin{cases} \dfrac{2c}{\sqrt{c^2\tau^2 - P^2}}; & P < c\tau \\ 0; & P > c\tau \end{cases} \tag{7.3.15}$$

We see, from Eq. (7.3.15), a striking difference between the two- and the three-dimensional cases. In three dimensions, the effect of an impulse, after a time τ has elapsed, will be found concentrated on a sphere of radius $R = c\tau$ whose center is at the source point. This is a virtue of the function $\delta[(R/c) - \tau]$ which occurs in (7.3.8). In two dimensions, the effect at a time τ due to an impulsive source is spread over the entire region $P < c\tau$. To be sure, there is a singularity at $P = c\tau$, but this is very weak indeed when compared with the δ function singularity in the three-dimensional case. The explanation of the difference may be readily seen by examining the line source in three dimensions. The effects, after a time τ has elapsed, of each point source constituting the line source will be found in a different region of the xy plane. Thus we infer that an impulsive line source does not emit a cylindrical shell wave with the disturbance present only at the wave surface $P = c\tau$. Rather there is a "wake," which trails off behind this wave surface. This wake is characteristic of two-dimensional problems and is not encountered in either the three- or one-dimensional problems as we shall see. This has been mentioned in pages 145 and 687.

One-dimensional Solutions. The three-dimensional source distribution corresponding to a Green's function which depends only upon $(x - x_0)$ (not upon $y - y_0$ and $z - z_0$) is the plane source upon which point sources (or equivalent line sources) are uniformly distributed. Such Green's functions are useful for problems in which there is no space dependence upon y or z. It may be obtained from (7.3.15) by integrating $g(P,\tau)$ over y_0 keeping x and x_0 fixed. Let $\xi = x - x_0$, $\eta = y - y_0$; then

$$\begin{aligned} g(\xi,\tau) &= 2c \int_{-\gamma}^{\gamma} \frac{d\eta}{\sqrt{c^2\tau^2 - \xi^2 - \eta^2}}; \quad \gamma = \sqrt{c^2\tau^2 - \xi^2}; \quad |\xi| < c\tau \\ &= 0; \quad |\xi| > c\tau \end{aligned}$$

The integration over η may be easily performed to yield $2c\pi$, so that

$$g(\xi,\tau) = 2c\pi \left[1 - u \left(\frac{|\xi|}{c} - \tau \right) \right] \tag{7.3.16}$$

Again we note that the effect of an impulse delivered at a time t_0, at the point x_0, for one-dimensional situations (or on the entire plane $x = x_0$ in three dimensions) is not concentrated at the point $|x - x_0| = \pm c(t - t_0)$ but rather exists throughout the region of extent $2c(t - t_0)$ with the source point x_0 at the mid-point.

Initial Conditions. To obtain a better grasp of the significance of the various expressions (7.3.8), (7.3.15), and (7.3.16), let us discuss the initial-value problem. Suppose that v_0 and ψ_0, the initial velocity and displacement, are known at every point in space; what are the velocity and displacement at a time t, assuming no sources present, that is $q = 0$? The solution of this problem may be obtained from (7.3.5):

$$\psi(r,t) = \frac{1}{4\pi c^2} \int dV_0 \left\{ g_{t_0=0} v_0(\mathbf{r}_0) - \left(\frac{\partial g}{\partial t_0} \right)_{t_0=0} \psi_0(\mathbf{r}_0) \right\} \tag{7.3.17}$$

The integration extends over all of space [we have also assumed that the surface integral in (7.3.5) vanishes at infinity].

Let us consider the one-dimensional case first, where the evaluation of (7.3.17) is simple and the interpretation of the result is straightforward. For one dimension, (7.3.17) becomes

$$\psi(x,t) = \frac{1}{4\pi c^2} \int dx_0 \left\{ g_{t_0=0} v_0(x_0) - \left(\frac{\partial g}{\partial t_0} \right)_{t_0=0} \psi_0(x_0) \right\}$$

where g is given by Eq. (7.3.16). The functions $g_{t_0=0}$ and $(\partial g/\partial t_0)_{t_0=0}$ have the values $2c\pi\{1 - u[(|\xi|/c) - t]\}$ and $-2c\pi\delta[(|\xi|/c) - t]$, respectively, where $|\xi| = |x - x_0|$. The integration may be easily performed to yield

$$\psi(x,t) = \tfrac{1}{2}\left\{\frac{1}{c}\int_{x-ct}^{x+ct} v_0(x_0)\,dx_0 + \psi_0(x + ct) + \psi_0(x - ct)\right\} \qquad (7.3.18)$$

This is the familiar d'Alembert solution of the initial problem in one dimension [see Eq. (11.1.58)]. It may be also obtained directly from the differential equation itself (see page 685).

From Eq. (7.3.18) we see that, if the medium, say a string, is given an initial displacement with no velocity, the initial pulse breaks up into

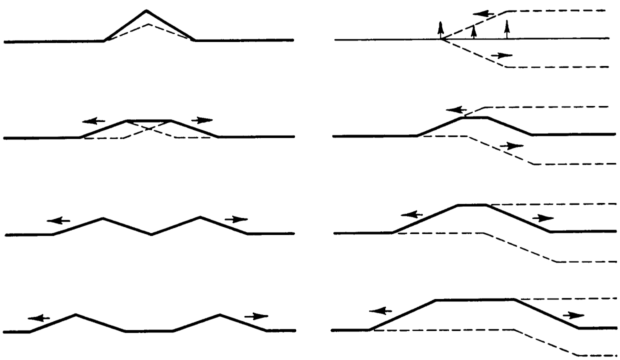

Fig. 7.10 Motions of plucked and struck strings. The solid lines give the shapes of the strings at successive times and the dotted lines give the shapes of the two "partial waves," traveling in opposite directions, whose sum is the shape of the string.

two identical waves, one traveling in the positive x direction and the other in the negative x direction. The sum of the two waves at $t = 0$ adds up to the initial displacement ψ_0. For some time thereafter, while they partially overlap, the composite shape will be rather complicated, until they finally separate. This behavior is illustrated in Fig. 7.10 above. Note that there is no wake trailing off behind each wave. For two-dimensional problems we shall find that such a wake is developed.

The initial problem in two dimensions is solved by (7.3.17). Here it is convenient to place the origin at the point of observation. Then

$$\psi(0,t) = \frac{1}{4\pi c^2}\int dS_0 \left\{g_{t_0=0}v_0(\varrho_0) - \left(\frac{\partial g}{\partial t_0}\right)_{t_0=0}\psi_0(\varrho_0)\right\}$$

where g is given by Eq. (7.3.15). Introducing polar coordinates and observing that $\partial g/\partial t_0 = -(\partial g/\partial t)$, then

$$\psi(0,t) = \frac{1}{2\pi c}\left\{\int_0^{2\pi} d\phi_0 \int_0^{ct} \rho_0\, d\rho_0 \frac{v_0(\varrho_0)}{\sqrt{c^2t^2 - \rho_0^2}} + \frac{\partial}{\partial t}\left[\int_0^{2\pi} d\phi_0 \int_0^{ct} \rho_0\, d\rho_0 \frac{\psi_0(\varrho_0)}{\sqrt{c^2t^2 - \rho_0^2}}\right]\right\} \quad (7.3.19)$$

This equation shows that the value of ψ at a point depends upon the original values of $\partial\psi/\partial t$ and ψ within a circle of radius ct about the observation point.

As in the discussion of the one-dimensional case, consider the initial condition $v_0 = 0$. Moreover, let $\psi_0(\varrho_0) = \delta(\varrho_0 - \varrho)$; that is, let the initial motion be an impulse delivered at point ϱ. Then

$$\psi(0,t) = \frac{1}{2\pi c}\frac{\partial}{\partial t}\left[\int_0^{2\pi} d\phi_0 \int_0^{ct} \frac{\rho_0\, d\rho_0 \delta(\varrho - \varrho_0)}{\sqrt{c^2t^2 - \rho_0^2}}\right]$$

$$= \begin{cases} \dfrac{1}{2\pi c}\dfrac{\partial}{\partial t}\left[\dfrac{1}{\sqrt{c^2t^2 - \rho^2}}\right]; & \text{if } \rho < ct \\ 0; & \text{if } \rho > ct \end{cases}$$

Combining and differentiating result in

$$\psi(0,t) = \frac{1}{2\pi c}\frac{\partial}{\partial t}\left[\frac{u(ct - \rho)}{\sqrt{c^2t^2 - \rho^2}}\right]$$

$$= \frac{1}{2\pi}\left\{-\frac{ct\, u(ct - \rho)}{(c^2t^2 - \rho^2)^{\frac{3}{2}}} + \frac{\delta(ct - \rho)}{\sqrt{c^2t^2 - \rho^2}}\right\}$$

Thus the signal at the origin is zero until $ct = \rho$. At this time, a pulse (the second term in the above expression) arrives at the origin. This is, however, followed by a wake trailing off behind the pulse as described by the first term, which, for $ct \gg \rho$, decreases with time as $1/c^2t^2$. Contrast this to the results obtained for a similar pulse at x_0 in the one-dimensional case. There the signal will arrive at a point x at a time $|x - x_0|/c$. It will be an exact duplicate of the original signal except for a reduction in amplitude by $\frac{1}{2}$; there will be no wake. This difference between one- and two-dimensional cases is illustrated in Fig. 7.11.

The presence of a wake is characteristic of propagation in uniform media in two dimensions. In three as well as in one dimension, the shape of pulse remains unchanged in propagating away from its initial position. This is an immediate consequence of the form of the Green's function for three dimensions as given by Eq. (7.3.8). The presence of the δ function prevents the formation of a wake. It does not necessarily guarantee that the pulse is unchanged in shape. In one-dimensional problems the shape of a pulse is preserved in which a given *displacement* (with zero velocity) is initially established. In three dimensions, the reverse is true. The shape of a pulse is maintained in which the initial displacement is zero and the *velocity* is given initially.

We may see this by deriving the analogue of (7.3.19) and (7.3.18). Introducing the three-dimensional Green's function into (7.3.17) and taking the origin at the observation point, we have

$$\psi(0,t) = \frac{1}{4\pi c^2} \int d\Omega_0 \int dr_0 \left[r_0 \delta\left(\frac{r_0}{c} - t\right) v_0(\mathbf{r}_0) - r_0 \delta'\left(\frac{r_0}{c} - t\right) \psi_0(\mathbf{r}_0) \right]$$

where $d\Omega_0$ is the element of solid angle of the sphere in the "subzero" coordinates (that is, $\sin\theta_0\, d\theta_0\, d\phi_0$). It is immediately clear that the

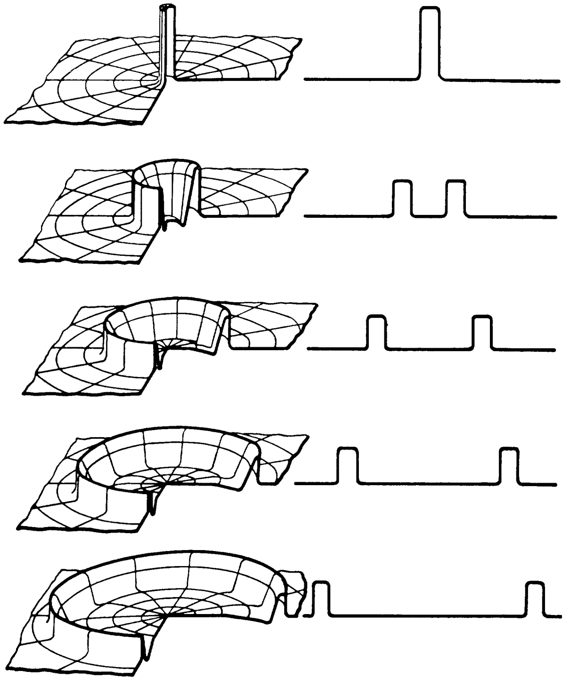

Fig. 7.11 Comparison between the behavior of string and membrane. First sketches show initial shapes, the lower ones the shapes at successive instants later. One-quarter of the membrane has been cut away to show the shape of the membrane, with the "wake" following the initial sharp wave front.

effects which occur at the observation point at a time t have their origin in the conditions initially present on the surface of a sphere of radius ct centered about the observation point. The integration over r_0 may now be performed. We need to write in the dependence of ψ_0 and v_0 on the coordinates θ_0, ϕ_0 on the spherical surface so that $\psi_0(\mathbf{r}_0) = \psi_0(r_0,\theta_0,\phi_0)$. The first integral is evaluated by employing the properties of the δ function. We obtain $[c^2 t v_0(ct,\theta_0,\phi_0)]$.

To evaluate the second term, one may integrate by parts or one may employ the following property of δ' (see page 837):

$$\int_{-\infty}^{\infty} f(x')\delta'(x' - x)\,dx' = -f'(x)$$

Thus the second term is $-c^2\{\partial[t\psi_0(ct,\theta_0,\phi_0)]/\partial t\}$. Collecting these results one finally obtains

$$\psi(0,t) = \frac{1}{4\pi}\int d\Omega_0 \left\{tv_0(ct,\theta_0,\phi_0) + \frac{\partial}{\partial t}[t\psi_0(ct,\theta_0,\phi_0)]\right\} \tag{7.3.20}$$

The direct dependence of ψ on v_0 and the appearance of the derivative acting upon ψ_0 are in accordance with our introductory remarks. Equation (7.3.20) is known as *Poisson's solution.*

Huygens' Principle. The Green's function for the infinite region may also be used to obtain the mathematical expression of *Huygens' principle.* From an elementary point of view Huygens' principle postulates that each point on a wave front acts as a point source emitting a spherical wave which travels with a velocity c. The field at a given point some time later is then the sum of the fields of each of these point sources; the envelope of the wavelets from all the points is the next wave front.

To derive this principle, we turn to the general equation (7.3.5) and consider the situation where there are no sources (that is, $q = 0$ within a surface S) and, in addition, the initial values of ψ and $\partial\psi/\partial t$ are zero. We see that the volume integral in Eq. (7.3.5), involving the initial condition, vanishes, so that all that is left is

$$\psi(\mathbf{r},t) = \frac{1}{4\pi}\int_0^{t^+} dt_0 \oint d\mathbf{S}_0 \cdot \left\{\left[\frac{\delta(t_0 - t + R/c)}{R}\right]\operatorname{grad}_0\psi - \psi\operatorname{grad}_0\left[\frac{\delta(t_0 - t + R/c)}{R}\right]\right\}$$

where we have inserted Eq. (7.3.8) into what is left of (7.3.5). Integrating over t_0 is not too difficult for the first term:

$$\int_0^{t^+}\left(\frac{1}{R}\right)\delta\left(t_0 - t + \frac{R}{c}\right)\operatorname{grad}_0\psi(\mathbf{r}_0,t_0)\,dt_0 = \left(\frac{1}{R}\right)\operatorname{grad}_0\psi\left(\mathbf{r}_0,t - \frac{R}{c}\right)$$

The second term is not much more difficult as long as we watch our δ's and (δ')'s. We have

$$\begin{aligned}
\int_0^{t^+}\psi\operatorname{grad}_0&\left[\frac{\delta(t_0 - t + R/c)}{R}\right]dt_0\\
&= \int_0^{t^+}\psi\frac{\partial}{\partial R}\left[\frac{\delta(t_0 - t + R/c)}{R}\right]\operatorname{grad}_0 R\,dt_0\\
&= \int_0^{t^+}\psi(\mathbf{r}_0,t_0)(\mathbf{R}/R^3)\left[-\delta\left(t_0 - t + \frac{R}{c}\right) + \frac{R}{c}\delta'\left(t_0 - t + \frac{R}{c}\right)\right]dt_0\\
&= -\frac{\mathbf{R}}{R^3}\left\{\psi\left(\mathbf{r}_0, t - \frac{R}{c}\right) + \frac{R}{c}\left[\frac{\partial}{\partial t_0}\psi(\mathbf{r}_0,t_0)\right]_{t_0=t-(R/c)}\right\}
\end{aligned}$$

Consequently the potential at a point **r** and time t, inside a surface S containing no sources and having null initial values internally, is given entirely by the integral of surface values on S:

$$\psi(\mathbf{r},t) = \frac{1}{4\pi} \oint d\mathbf{S}_0 \cdot \left[\left(\frac{1}{R}\right) \operatorname{grad}_0 \psi(\mathbf{r}_0,t_0) + \left(\frac{\mathbf{R}}{R^3}\right) \psi(\mathbf{r}_0,t_0) - (\mathbf{R}/cR^2) \frac{\partial}{\partial t_0} \psi(r_0,t_0) \right]_{t_0=t-(R/c)} \quad (7.3.21)$$

If now part of the surface S_0 is along a wave front and the rest is at infinity or where ψ is zero, we can say that the field value ψ at (r,t) is "caused" by the field ψ in the wave front at a time $t - (R/c)$ earlier. From another point of view, the effect of the wave front on the field in front of it at a later time is equivalent to that of a distribution of sources over the surface of the wave front: an ordinary surface charge proportional to the gradient of ψ normal to the wave front, a double layer proportional to ψ itself, and a curious single sheet, proportional to the time rate of change of ψ on the surface, which is strongest straight ahead of the surface but drops off proportional to the cosine of the angle between the normal to the surface and the direction of propagation (*i.e.*, the direction of **R**).

In most cases (except for simple and trivial ones) the exact values of ψ, grad ψ, and $\partial\psi/\partial t$ along a whole wave front are not known exactly. But in a great many cases of interest these quantities are approximately known, so that Eq. (7.3.21) may be used to compute approximate values of ψ at a later time. This will be discussed in detail in Sec. 11.4.

Boundaries in the Finite Region. We turn next to the effects of introducing boundaries upon which Green's functions must satisfy specific boundary conditions. The techniques which may be employed here are very similar to those which we discussed in the steady-state case in the preceding Sec. 7.2. As in that case, there are two methods: the method of images and the method of eigenfunctions. Let us discuss the image method first, in which we shall utilize our knowledge of the Green's function for the infinite domain. The only singularity which occurs is naturally at the source point at the time the impulse is initiated, so that generally

$$G(\mathbf{r},t|\mathbf{r}_0,t_0) = \frac{\delta[t_0 - t - (R/c)]}{R} + F(\mathbf{r},t|\mathbf{r}_0,t_0)$$

where F is a solution of the homogeneous wave equation which is free of singularities in the region under discussion.

A simple example is useful here. In Fig. 7.12 the source is at Q; it is initiated at t_0. An infinite rigid plane is at $x = 0$. To satisfy the boundary conditions on the plane we start an image pulse at I, the image of Q at the same time t_0. To obtain the proper cancellation at $x = 0$, the effects of these must be added so that

$$G(\mathbf{r},t|\mathbf{r}_0,t_0) = \frac{\delta[t_0 - t + (R/c)]}{R} + \frac{\delta[t_0 - t + (R'/c)]}{R'}$$

It may be easily verified that $(\partial G/\partial x)_{x=0}$ is zero for all time t. The effect of the second term is to give a reflection which occurs at $x = 0$ at the correct time. This is the only reflection which takes place, and thus only one term in addition to the Green's function for the infinite domain is required. We note that in the case of the wave equation it is necessary to specify not only the position of the image but also the time t' at which the image pulse is started. Fortunately, in most problems this has a simple solution, in that all the images are started at the same time t' as the source pulse itself. For sufficiently regular geometries, it is possible to employ the image method much as in the case of the Helmholtz equation.

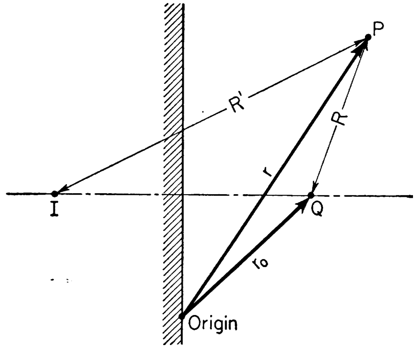

Fig. 7.12 Image at I, of impulsive wave source at Q.

Eigenfunction Expansions. We utilize the Green's function for the Helmholtz equation. The expression $G_k(\mathbf{r}|\mathbf{r}_0)e^{-i\omega t}$ is a solution of the wave equation with a simple-harmonic point source located at $\mathbf{r}_0$. We have, for $\omega = kc$,

$$\nabla^2[G_k e^{-i\omega(t-t_0)}] - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}[G_k e^{-i\omega(t-t_0)}] = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)e^{-i\omega(t-t_0)}$$

By properly superposing these simple harmonic solutions it is possible to obtain the Green's function for a pulse at a point in space, corresponding to Eq. (7.3.2). For this purpose we employ the integral representation of the δ function:

$$\delta(t - t_0) = \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} e^{-i\omega(t-t_0)}\, d\omega$$

From linearity we expect that the Green's function for the pulse is related to the Helmholtz equation solution by the equation

$$G(\mathbf{r},t|\mathbf{r}_0,t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\mathbf{r}|\mathbf{r}_0|k)e^{-i\omega(t-t_0)}\, d\omega \qquad (7.3.22)$$

where $\omega = kc$ and $G(\mathbf{r}|\mathbf{r}_0|k) = G_k(\mathbf{r}|\mathbf{r}_0)$. This relation will be derived in a more painstaking manner in Sec. 11.1. The simplicity of Eq. (7.3.22) is misleading. It may be recalled that for finite regions G_k has singularities whenever $k = k_n$, where k_n is an eigenvalue of the scalar Helmholtz equation for a solution ψ_n satisfying the boundary condition satisfied by G_k. More explicitly, if the ψ_n's are normalized, then from (7.2.39),

$$G_k = 4\pi \sum_n \frac{\bar{\psi}_n(\mathbf{r}_0)\psi_n(\mathbf{r})}{k_n^2 - k^2}$$

Thus integration (7.3.22) cannot proceed along the real axis of ω (or k) but must avoid these singularities in some fashion. *The contour choice is dictated by the postulate of causality* discussed earlier. To see this let us introduce the expansion for G_k into integral (7.3.22):

$$G(\mathbf{r},t|\mathbf{r}_0,t_0) = 2c^2 \sum_n \psi_n(\mathbf{r}_0)\psi_n(\mathbf{r}) \int_{-\infty}^{\infty} \frac{e^{-i\omega(t-t_0)}}{\omega_n^2 - \omega^2}\, d\omega; \quad \omega_n = ck_n$$

The contour must be chosen so that $G(\mathbf{r},t|\mathbf{r}_0,t_0) = 0$ when $t < t_0$. The appropriate contour is shown in Fig. 7.13. It is parallel to the real axis and just *above* it. When $t > t_0$, the contour may be closed in the lower half plane by a semicircle of large ($\rightarrow \infty$) radius without changing the value of the integral. We may now employ the Cauchy integral formula (4.2.9) to obtain $(2\pi/\omega_n)\sin[\omega_n(t - t_0)]$. When $t < t_0$, the contour may be closed by a semicircle in the upper half plane. Since there are no poles in the upper half plane, the value of the integral for $t < t_0$ is zero. Hence

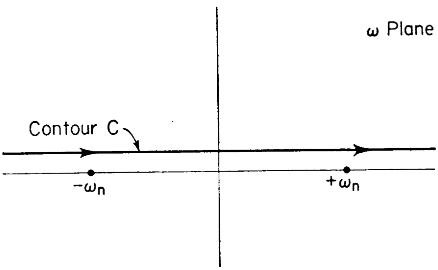

Fig. 7.13 Contour for integral of Eq. (7.3.22).

$$G(\mathbf{r},t|\mathbf{r}_0,t_0) = 4\pi c^2 \sum_n \frac{\sin[\omega_n(t - t_0)]}{\omega_n} u(t - t_0)\bar{\psi}_n(\mathbf{r}_0)\psi_n(\mathbf{r}) \quad (7.3.23)$$

where $u(t - t_0)$ is the unit function vanishing for $t < t_0$ and equaling unity for $t > t_0$.

Employing (7.3.23) we may now obtain an explicit evaluation of the initial- and boundary-value problem as given in (7.3.5). Let us consider each term in (7.3.5) separately. The first term ψ_1 gives the effect of sources distributed throughout the volume.

$$\begin{aligned}\psi_1 &= \frac{1}{4\pi}\int_0^{t^+} dt_0 \int dV_0\, G(\mathbf{r},t|\mathbf{r}_0,t_0)q(\mathbf{r}_0,t_0) \\ &= c^2 \sum_n \frac{1}{\omega_n}\int_0^{t^+} dt_0 \int dV_0 \sin[\omega_n(t - t_0)]\, \bar{\psi}_n(\mathbf{r}_0)\psi_n(\mathbf{r})q(\mathbf{r}_0,t_0)\end{aligned}$$

Let
$$\phi_n(\mathbf{r},t) = \psi_n(\mathbf{r})e^{-i\omega_n t}$$
Then
$$\psi_1 = -c^2 \operatorname{Im} \left\{ \sum_n \frac{\phi_n(\mathbf{r},t)}{\omega_n} \int_0^{t^+} dt_0 \int dV_0\, q(\mathbf{r}_0,t_0)\bar{\phi}_n(\mathbf{r}_0,t_0) \right\} \tag{7.3.24}$$

We see that the amplitude of excitation of the nth mode is proportional to the multiple integral in (7.3.24). The excited amplitude is large when the space dependence of q is very much like that of ϕ_n and if the time dependence is close to $e^{-i\omega_n t}$, a result which agrees with expectations. In the case of exact resonance $q \simeq e^{-i\omega_n t}$ we note that ψ_1 increases linearly with time and no longer oscillates.

The second term in Eq. (7.3.5) gives the effect of sources distributed on the boundary. The results are rather similar to those obtained from the first term. The third term involves the satisfaction of initial conditions. We require $(\partial G/\partial t_0)_{t_0=0}$ and $G_{t_0=0}$ before proceeding.

$$G(\mathbf{r},t|\mathbf{r}_0,0) = 4\pi c^2 \sum_n \frac{\sin(\omega_n t)}{\omega_n} u(t)\bar{\psi}_n(\mathbf{r})\psi_n(\mathbf{r}_0) \tag{7.3.25}$$

$$\left[\frac{\partial}{\partial t_0} G(\mathbf{r},t|\mathbf{r}_0,t_0)\right]_{t_0=0} = -4\pi c^2 \sum_n \cos(\omega_n t)\, u(t)\bar{\psi}_n(\mathbf{r})\psi_n(\mathbf{r}_0) \tag{7.3.26}$$

where we have placed $[\sin(\omega_n t)/\omega_n]\delta(t) = 0$. The third term ψ_3 in (7.3.5) becomes

$$\psi_3 = \sum_n \left\{ \frac{\sin \omega_n t}{\omega_n} u(t)\psi_n(\mathbf{r}) \int \bar{\psi}_n(\mathbf{r}_0)v_0(\mathbf{r}_0)\, dV_0 \right.$$
$$\left. + \cos \omega_n t\, \psi_n(\mathbf{r}) \int \bar{\psi}_n(\mathbf{r})\psi_0(\mathbf{r}_0)\, dV_0 \right\} \tag{7.3.27}$$

We may verify at once that $\psi_3(t = 0)$ is just ψ_0 and $(\partial\psi_3/\partial t)_{t=0} = v_0$ as it should. This also shows that we could have obtained Eq. (7.3.27) directly without going through the intermediary of the Green's function. On the other hand it verifies the validity of fundamental formula (7.3.5) for the case of finite regions.

Transient Motion of Circular Membrane. An example will be useful at this point to illustrate the sort of results one obtains for a time-dependent problem. A circular membrane of radius a under a tension T and of mass σ per unit area is given an initial displacement in a small region about its center. The edges of the membrane are fixed, so that the boundary conditions are $\psi(r) = 0$ at $r = a$. We shall represent the initial conditions by means of a δ function:

$$\psi_0(\mathbf{r}) = A\delta(\mathbf{r}); \quad v_0(r) = 0 \tag{7.3.28}$$

Here A is a constant. Introducing (7.3.28) into (7.3.27), the solution of the initial-value problem yields

$$\psi(r,t) = A \sum_n \cos(\omega_n t)\, \psi_n(\mathbf{r})\bar{\psi}_n(0) \tag{7.3.29}$$

To proceed further it is necessary to obtain the eigenfunction $\psi_n(\mathbf{r})$. This will be discussed in great detail in Sec. 11.2. For the present we note that the Helmholtz equation separates in polar coordinates (r,ϕ) and that a general solution, which is finite and single-valued for $r < a$, is a sum of terms $e^{\pm im\phi} J_m(kr)$, where J_m is the Bessel function of the first kind, of order m.

It is now necessary to introduce the boundary conditions at $r = a$. This leads to an equation determining k:

$$J_m(ka) = 0 \tag{7.3.30}$$

Let the values of k satisfying this equation be k_{mp}, the subscript m indicating the order of the Bessel function, the letter p indicating a particular root of (7.3.30). For the purposes of this illustration it will suffice to employ the asymptotic form for $J_m(ka)$ [Eq. (5.3.68)]:

$$J_m(ka) \xrightarrow[ka\to\infty]{} \sqrt{\frac{2}{\pi ka}} \cos\left[ka - \frac{2m+1}{4}\pi\right]$$

Thus $J_m(ka)$ is zero whenever the argument of the cosine is an odd number of $\pi/2$:

$$k_{mp}a \simeq \tfrac{1}{4}(2m+1)\pi + \tfrac{1}{2}(2p+1)\pi; \quad p \text{ integer}$$

We may now return to expression (7.3.29), giving the response of the membrane to an impulse at $r = 0$, $t = 0$. The functions ψ_n are thus

$$\psi_n = N_{mp}J_m(k_{mp}r)e^{\pm im\phi}$$

where to each n we associate a particular couple (m,p) and a particular sign of the exponential. The factor N_{mp} is chosen so that

$$\int \bar{\psi}_n\psi_n \, dA = 1$$

where the area of integration is the membrane. For (7.3.29) we require $J_m(0)$. Since $J_m(z) \xrightarrow[z\to 0]{} 0(z^m)$, we see that $J_m(0) = \delta_{0m}$ [see Eq. (5.3.63)]. Hence the summation (7.3.29) reduces to a sum over zero-order Bessel functions. The absence of any angular dependence is not surprising in view of the circular symmetry of the initiating pulse (7.3.28). The response at any subsequent time t and at a position r is given by

$$\psi(r,t) = A \sum_p \cos\,(k_{0p}ct)N_{0p}^2J_0(k_{0p}r) \tag{7.3.31}$$

Equation (7.3.31) is exact. Note that the set $N_{0p}\cos(k_{0p}ct)\,J_0(k_{0p}r)$ describes the free radial vibrations of the membrane. Generally the response to an initial impulse may be expressed in terms of a superposition of free vibrations, each mode vibrating with its own frequency. This is to be contrasted to the response of the system to a steady driving force of a given frequency. In that case, the response has the same frequency as the driving force and the space dependence involves a superposition of the $\psi_n(r)$'s, all of them vibrating with the frequency of the driving force.

Let us consider the response back at the starting point, $r = 0$. Then (7.3.31) becomes

$$\psi(0,t) = A \sum_p \cos(k_{0p}ct)\, N_{0p}^2$$

We introduce the approximate value of the zeros:

$$\psi(0,t) \simeq A \sum_p \cos\left[\left(\frac{2p+3}{4}\right)\frac{\pi ct}{a}\right] N_{0p}^2 \qquad (7.3.32)$$

When will the original pulse refocus at $r = 0$? On first consideration we might think this would occur when $t = 2a/c$, the time for the pulse to go to the edge of the membrane and back. This is, however, not the case. As may be seen from the asymptotic behavior of $J_0(z) \simeq \sqrt{2/\pi z}\cdot$ $\cdot\cos(z - \frac{1}{4}\pi)$, a phase change of $\pi/4$ occurs in passing from the region $r \simeq 0$ to $r \simeq a$. This is characteristic of propagation in two dimensions. No such phase change occurs in either one or three dimensions.

Because of this phase shift, it is necessary for two traversals from the center out to the edge to occur before a final phase shift of π occurs and the pulse is refocused. Hence we may expect that, when $ct = 4a$, the pulse will reform itself at $r = 0$. This may be readily verified by substitution in (7.3.32) for $\psi(0,4a) \simeq -A\sum_p N_{0p}^2$. (The initial pulse $\psi(0,0)$ is $A\Sigma N_{0p}^2$.) We should like to emphasize again that this phenomenon occurs only in two dimensions; in one and three dimensions it does not occur. The pulse from the center of a sphere of radius a re-forms at the center at a time $t = 2a/c$.

There is one final point which also shows the striking difference in wave propagation in two as compared with one or three dimensions. In the latter the initial pulse re-forms exactly at the proper time. In two dimensions this is not so because of the wake developed as the wave progresses. This may be seen in the present instance as follows. Expression (7.3.32) is approximate, for the approximate values of the roots of the Bessel function J_0 were utilized. If the precise values of the roots had been employed, there would have been no value of ct at which the phase $(k_{0p}ct)$ would be exactly the same for all p. In other words, there would be no value of ct for which the free vibration initiated by the pulse would have all returned to their initial phase. Thus the free vibrations would never interfere in the proper fashion to re-form the initial situation exactly.

As another example of the construction of Green's function for the scalar wave equation let us derive expression (7.3.8), the infinite space Green's function, by direct utilization of the superposition method. In that case

$$G_k(\mathbf{r}|\mathbf{r}_0) = e^{ikR}/R; \quad k = \omega/c$$

so that

$$g(R,\tau) = \frac{1}{2\pi R}\int_{-\infty}^{\infty} e^{i(kR-\omega\tau)}\, d\omega = \frac{1}{2\pi R}\int_{-\infty}^{\infty} e^{i\omega[(R/c)-\tau]}\, d\omega$$

It should be noted that we have carefully chosen the relative sign between the factor kR and ωt to be such that $e^{i(kR-\omega\tau)}/R$ represents a wave *diverging* from the source as time progresses, *i.e.*, as τ increases. This is the manner in which we satisfy the causality principle. We now make use of the integral representation for the δ function, Eq. (7.3.22), to obtain

$$g(R,\tau) = \delta[(R/c) - \tau]/R$$

Klein-Gordon Equation. The Green's function for the time-dependent Klein-Gordon equation satisfies the equation

$$\nabla^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} - \kappa^2 G = -4\pi\delta(t - t_0)\delta(\mathbf{r} - \mathbf{r}_0) \qquad (7.3.33)$$

It is easy to verify that the Green's function for the Klein-Gordon equation may be employed in much the same way as the Green's function for the scalar wave equation. For example, the reciprocity condition (7.3.3) and the general solution (7.3.5) apply as well here. There are, however, important physical differences between the two. These may be best illustrated by considering the Klein-Gordon Green's function for the infinite domain, thus obtaining the analogue of Eq. (7.3.8). The function $g(\mathbf{r},t|\mathbf{r}_0,t_0)$ may be obtained by superposition of the solutions obtained for a simple harmonic time dependence $e^{-i\omega(t-t_0)}$ rather than the impulsive one given by $\delta(t - t_0)$. The necessary superposition is given by Eq. (7.3.22). The individual solutions may be then given by $g(R|\sqrt{\omega^2 - c^2k^2})$, where

$$[\nabla^2 + (\omega/c)^2 - \kappa^2]\, g[R|\sqrt{\omega^2 - (c\kappa)^2}] = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)$$

The solution of this equation is

$$g = \frac{\exp\left[i\sqrt{(\omega/c)^2 - \kappa^2}\,R\right]}{R}; \quad R = |\mathbf{r} - \mathbf{r}_0| \tag{7.3.34}$$

In the limit $\omega/c \gg \kappa$ Eq. (7.3.34) becomes $g = e^{i(\omega/c)R}/R$ as it should. For the opposite case $\omega/c \ll \kappa$

$$g \xrightarrow[(\omega/c)\ll\kappa]{} e^{-\kappa R}/R$$

giving a characteristically "damped" space dependence. This is, of course, not related to any dissipation. From the one-dimensional mechanical analogue (Chap. 2, pages 138 *et seq.*), a string embedded in an elastic medium, we see that it is a consequence of the stiffness of the medium.

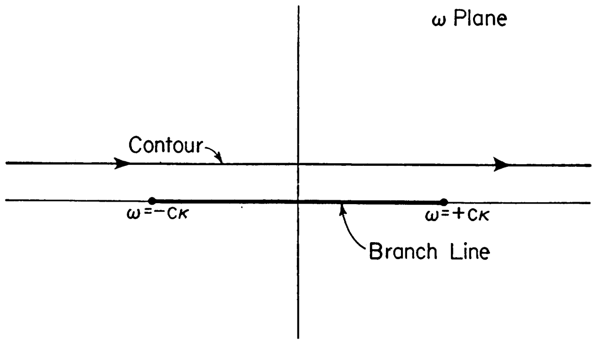

Fig. 7.14 Contour for integral of Eq. (7.3.37) for $R > ct$.

Employing (7.3.22) we may now write as the solution of (7.3.33) valid for an infinite medium

$$g(\mathbf{r},t|\mathbf{r}_0,t_0) = \frac{1}{2\pi R}\int_{-\infty}^{\infty} \exp i\left[\sqrt{(\omega/c)^2 - \kappa^2}\,R - \omega\tau\right] d\omega \tag{7.3.35}$$

where $\tau = t - t_0$. Function g is a function of R and τ only, as expected. We must now specify the path of integration. Before doing so it is convenient, for convergence questions, to introduce the function $h(R,\tau)$ such that

$$\partial h(R,\tau)/\partial R = Rg(\mathbf{r},t|\mathbf{r}_0,t_0) \tag{7.3.36}$$

Hence

$$h(R,\tau) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\exp i\left[\sqrt{(\omega/c)^2 - \kappa^2}\,R - \omega\tau\right]}{\sqrt{(\omega/c)^2 - \kappa^2}}\, d\omega \tag{7.3.37}$$

The integrand has branch points at $\omega = \pm c\kappa$. The relation of the path of integration relative to these branch points is determined by the causality condition. We choose the path and branch line shown in Fig. 7.14. First note that $h = 0$ if $R > c\tau$, as the causality postulate would demand for this case. In the limit of large ω the exponent in (7.3.37) approaches $i\omega[(R/c) - \tau] = i\omega|(R/c) - \tau|$. The path of integration may then be closed in the upper half of the ω plane without changing the value of the integral. Since the integrand has no singularities in the upper half plane, the integral is zero.

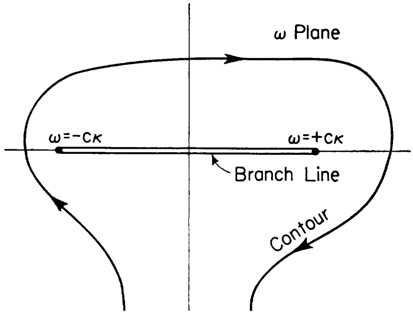

Fig. 7.15 Contour for integral of Eq. (7.3.37) for $R < ct$.

Now consider h for $R < c\tau$. The contour is then deformed to the one shown in Fig. 7.15. It may now be reduced to a more familiar form. We introduce a new variable ϑ, such that

$$\tau = |\sqrt{\tau^2 - (R/c)^2}| \cosh \vartheta \quad \text{and let} \quad \omega = c\kappa \cosh x$$

Then $h(R,\tau) = \dfrac{c}{2\pi i} \displaystyle\int_{\infty + \frac{3}{2}\pi i}^{\infty - \frac{1}{2}\pi i} \exp\,[-i\kappa c|\sqrt{\tau^2 - (R/c)^2}| \cosh (x - \vartheta)]\, dx$

Finally let $(x - \vartheta) = i\xi$; then

$$h(R,\tau) = \frac{c}{2\pi} \int_{+\frac{3}{2}\pi - i\infty}^{-\frac{1}{2}\pi - i\infty} \exp\,[-i\kappa c|\sqrt{\tau^2 - (R/c)^2}| \cos \xi]\, d\xi$$

This is just the integral representation of the Bessel function of zero order [see Eq. (5.3.65)] so that

$$h(R,\tau) = -cJ_0[\kappa c \sqrt{\tau^2 - (R/c)^2}]; \quad R < c\tau$$

Combining this with the expression for $c\tau < R$ we finally obtain

$$h(R,\tau) = -cJ_0[\kappa c \sqrt{\tau^2 - (R/c)^2}]\, u[\tau - (R/c)]$$

The Green's function g is then

$$g(R,\tau) = \frac{1}{R}\frac{\partial h}{\partial R} = \frac{\delta[\tau - (R/c)]}{R} J_0[\kappa c \sqrt{\tau^2 - (R/c)^2}] - \frac{\kappa}{\sqrt{\tau^2 - (R/c)^2}} J_1[\kappa c \sqrt{\tau^2 - (R/c)^2}]\, u[\tau - (R/c)]$$

or

$$g(R,\tau) = \frac{\delta[\tau - (R/c)]}{R} - \frac{\kappa}{\sqrt{\tau^2 - (R/c)^2}} J_1[\kappa c \sqrt{\tau^2 - (R/c)^2}] \cdot u[\tau - (R/c)] \quad (7.3.38)$$

We observe that in the limit $\kappa \to 0$, $g(R,\tau)$ reduces to the Green's function for the scalar wave equation (7.3.8) as it must.

The effect of a pulse at a distance R from its position and at a time τ later vanishes as long as $R > c\tau$, that is as long as the wave initiated by the pulse has not had sufficient time to reach the observation point R. At $R = c\tau$, the original pulse arrives, diminished in amplitude, as it must be, by the geometrical factor $1/R$. It is then followed by a wake which is given by the second term in (7.3.38). This wake, for large values of the time, will decrease in amplitude by the factor $[\tau^2 - (R/c)^2]^{-\frac{3}{4}}$. We may understand this phenomenon by noting that the phase velocity of a plane wave satisfying the Klein-Gordon equation is a function of ω:

$$v = \frac{\omega}{\sqrt{(\omega/c)^2 - \kappa^2}} \quad \text{or} \quad \frac{v}{c} = \frac{\omega}{\sqrt{\omega^2 - (\kappa c)^2}}$$

Since a pulse is made up of many frequencies combined, it is not surprising that the associated plane waves do not arrive at the observation point in the same relative phases as they had at the start. An equivalent description may be obtained from the mechanical realization of the Klein-Gordon equation, as given in Sec. 2.1.

7.4 *Green's Function for Diffusion*

The diffusion equation differs in many qualitative aspects from the scalar wave equation, and of course, the Green's functions will exhibit these differences. The most important single feature is the asymmetry of the diffusion equation with respect to the time variable. For example, if $\psi(\mathbf{r},t)$ is a solution of the scalar wave equation, so is $\psi(\mathbf{r},-t)$. However, if $\psi(\mathbf{r},t)$ is a solution of the diffusion equation

$$\nabla^2\psi = a^2(\partial\psi/\partial t) \tag{7.4.1}$$

the function $\psi(\mathbf{r},-t)$ is not; it is a solution of a quite different equation

$$\nabla^2\psi(\mathbf{r},-t) = -a^2(\partial\psi/\partial t)$$

Thus the equation carries with it a directionality in time; *i.e.*, it differentiates between past and future. The scalar wave equation and indeed all equations which are applicable to microscopic (*e.g.*, atomic) phenomena are symmetric in time. The directionality in time of the diffusion equation is a consequence of the fact that the field which does the diffusing represents the behavior of some average property of an ensemble of many particles. As can be inferred from the theorems of thermodynamics, irregularities in such averages, which may initially exist, will smooth out as time progresses. Looking to the future, entropy increases; looking to the past, entropy was smaller.