Electrodynamics of moving dipoles: The case of the missing torque

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In a recent article, Bedford and Krumm ["On the Origin of Magnetic Dynamics," Am. J. Phys. **54**, 1036 (1986)], examine in detail the interaction between a moving line of charges and a magnetic dipole consisting of a conducting ring of current from the point of view of different frames of reference. One particular orientation of the magnetic dipole leads to an apparently paradoxical situation of a torque acting in one inertial frame but not in another. These authors explain the situation by considering the rate of change of mass of the charge carriers and a wall force that prevents them from accelerating in the direction of their motion. They also hint at an intriguing analogy with some kind of inertial or fictitious force such as occurs in noninertial frames. First, the general expression is derived for the torque on a moving magnetic dipole in any orientation and the term representing the missing torque is explicitly revealed. Then the physical origin of this torque is investigated in the particular case considered by Bedford and Krumm, where the magnetic dipole consists of a small current-carrying conducting loop. It appears that, in this case, the elusive torque arises from the interaction between the current carriers in the magnetic dipole loop and the magnetic field due to the surface current generated by the motion of the induced charges on the surface of the conducting loop.

I. INTRODUCTION

In a recent issue of this Journal, Bedford and Krumm[1] examine, from the point of view of different inertial observers, the origin of the torque acting on oriented magnetic dipoles consisting of current-carrying conducting loops above a moving line of charges.

In the rest frame of the magnetic dipole, the answer is, of course, straightforward, the torque being given by $\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}$, where $\mathbf{m}$ is the magnetic dipole moment and $\mathbf{B}$ is the magnetic induction due to the moving line of charges at the dipole location, which are assumed to be very small.

In the rest frame of the moving charges, only an electric field apparently exists, and the torque on the loop cannot be explained as a simple direct magnetic interaction. It turns out that the explanation for the existence of a torque depends on the orientation of the dipole. When the loop is oriented with its dipole moment perpendicular to the line shown in Fig. 1 of Ref. 1, the authors correctly find that the torque arises from the asymmetry in the average line charge density of the two relevant loop sides due to the relativity of simultaneity. On the other hand, when the magnetic dipole moment vector is parallel to the line, the relativity of simultaneity cannot be invoked and the authors interpret the phenomenon in terms of a very different mechanism. Their explanation calls for an additional term in Newton's second law in the form $\dot{m}\mathbf{v}$, where $\dot{m} = d/dt\,[m_0/\sqrt{1-(v^2/c^2)}\,]$ is the rate at which the mass of the charge carriers is increased when they accelerate in the y direction (axle of the dipole). See Fig. 2 of Ref. 1. As the charges in the dipole accelerate in the y direction, their mass is increased and a transversely directed force is required to keep the z component of their velocity constant. This force is exerted by the wall of the loop and, by Newton's third law, a force of equal magnitude is exerted in the $-z$ direction by the charges in the loop. According to these authors, this is the way that the torque can be accounted for. In addition, they seem to attach a particular significance to what they refer to as a curious interplay between geometry and dynamics, going as far as drawing some intriguing analogy between this type of interaction and the fictitious or inertial forces that arise in noninertial frames.

Electromagnetism is a fully relativistic, self-consistent theory, whose point of contact with dynamics is basically through the Lorentz force. Phenomena that are explainable in one inertial frame by means of interactions between charges and currents, i.e., sources and fields, should also be amenable to a similar explanation in another inertial frame without requiring the introduction of the mass of the charge carriers, even though, of course, relativistic mechanics is entirely compatible with electromagnetic theory. As we shall show, the missing torque can indeed be explained within the confines of a model consisting of a conducting loop, by means of the interaction between fields and sources through the familiar Lorentz expression.

II. TORQUE ON MOVING DIPOLES IN ELECTRIC AND MAGNETIC FIELDS

Consider an electric dipole with dipole moment $\mathbf{p} = q\,\boldsymbol{\delta}\mathbf{l}$ and moving with a translational constant velocity $\mathbf{v}$ in the lab frame S. If this dipole lies in a uniform combination of electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, it will be subjected to the Lorentz forces $\pm q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$, which will give rise to a total torque $q\,\boldsymbol{\delta}\mathbf{l}\times(\mathbf{E}+\mathbf{v}\times\mathbf{B})$. In the limit of a point dipole, we have a torque $\boldsymbol{\tau}$ given by

$$\boldsymbol{\tau} = \mathbf{p}\times\mathbf{E} + \mathbf{p}\times(\mathbf{v}\times\mathbf{B})\,, \qquad (1)$$

where the electric dipole moment is measured in the laboratory frame. Now, if we consider a magnetic dipole consisting of two magnetic charges q^* and $-q^*$ a distance $\boldsymbol{\delta}\mathbf{l}$ apart, the dipole moment of this configuration is $\mathbf{m} = q^*\,\boldsymbol{\delta}\mathbf{l}$. If placed in the $\mathbf{E}$ and $\mathbf{B}$ fields, the magnetic charges are subjected to the analog of the Lorentz force[2,3] $\mathbf{F}^* = q^*\,[\mathbf{B} - (1/c^2)\mathbf{v}\times\mathbf{E}]$ and correspondingly a torque arises that consists of two parts,

$$\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B} - \mathbf{m}\times[\mathbf{v}\times(\mathbf{E}/c^2)]\,. \qquad (2)$$

This result could have been obtained directly from (1) by

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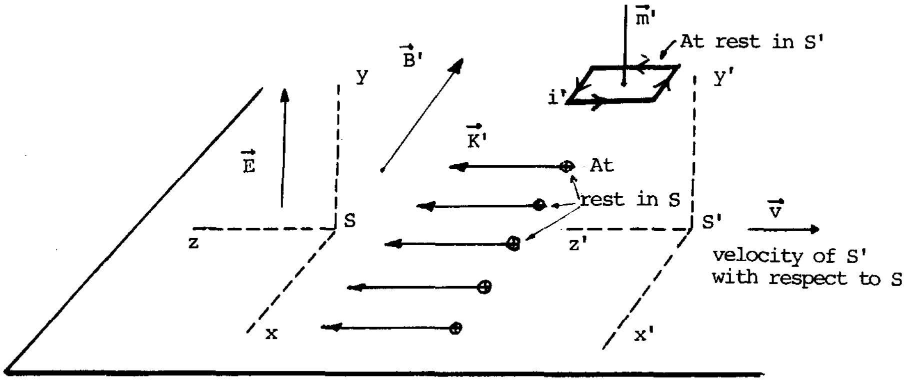

Fig. 1. A plane of uniformly distributed charges (x-z) at rest in the S frame. This plane is moving with velocity $-\mathbf{v}$ with respect to S', the rest frame of a magnetic dipole whose magnetic moment is normal to the plane of charges.

the substitution $\mathbf{p}\rightarrow\mathbf{m}$, $\mathbf{E}\rightarrow\mathbf{B}$, and $\mathbf{B}\rightarrow-\mathbf{E}/c^2$.

As far as forces and torques due to *external* fields are concerned,[4-8] a system of magnetic charges is totally equivalent to a small current loop, thus Eq. (2) also applies to a current loop point dipole. The approach we have taken here is merely a shortcut for quickly arriving at the result (2).

Using the identity $\mathbf{m}\times(\mathbf{v}\times\mathbf{E})=\mathbf{v}\times(\mathbf{m}\times\mathbf{E})+(\mathbf{m}\times\mathbf{v})\times\mathbf{E}$, we can write the magnetic torque on the dipole in the form

$$\boldsymbol{\tau}=\mathbf{m}\times\mathbf{B}-(\mathbf{m}\times\mathbf{v})\times(\mathbf{E}/c^2)-\mathbf{v}\times[\mathbf{m}\times(\mathbf{E}/c^2)]\,. \tag{3}$$

The second term $(\mathbf{v}\times\mathbf{m})\times\mathbf{E}/c^2$ has the form of a torque on an equivalent electric dipole moment $\mathbf{p}=(\mathbf{v}\times\mathbf{m})/c^2$, and it represents a truly relativistic effect whose physical origin can be shown to be a consequence of the relativistic definition of simultaneity.

The third term is truly the "missing torque" for it represents the only contribution to the torque when $\mathbf{B}=0$ and $\mathbf{m}$ is parallel to $\mathbf{v}$. Expressions (2) and (3) are general results in the sense that they are applicable to any point dipole, no matter how it is internally constructed. This important equivalence is mentioned in a number of treatises on electromagnetism. The most extensive discussion on the subject can be found in the text by Fano *et al.*[2]

The point dipole could then be constructed from two true magnetic charges (were they to exist), or from a current running in a small conducting loop, or from charges on rotating dielectrics. It could also be due to the orbital or spinning motion of electrons. The physical origin of the torque, i.e., the exact mechanism by which the torque arises may not be easy to determine and may be different in any of the above examples. Regardless, the final result must always conform to Eqs. (2) or (3), which are dictated by more general arguments. Indeed, these equations are basically the result of the transformation properties of the electromagnetic fields and torques and could have been obtained directly starting from the rest system of the dipole and using the appropriate relativistic transformations.

The real challenge is then to discover the physical origin of the different terms in the torque equation (3). For the case of a dipole consisting of a current in a small conducting loop, which is the case considered by Bedford and Krumm, we already know that the second term is related to the notion of relativistic simultaneity, as shown by these authors and as discussed in some standard treatises on electromagnetism.[9-11] We shall present in Sec. III a physical explanation for the origin of the additional torque term $-(\mathbf{v}/c^2)\times(\mathbf{m}\times\mathbf{E})$, which plays a major role when the magnetic field vanishes and the magnetic moment is parallel to the velocity of the dipole.

Before engaging in this discussion, we need to relate results from one inertial frame to another and, in particular, we shall require the appropriate transformations for the electric and magnetic field as well as for the electric and magnetic dipole moments. As is well known, the transformation equations for the fields in passing from the laboratory frame S to the rest frame of the moving dipole S' are

$$\mathbf{E}'=\gamma(\mathbf{E}+\mathbf{v}\times\mathbf{B}),\quad \mathbf{E}\perp\mathbf{v}\,, \tag{4}$$

$$\mathbf{E}'=\mathbf{E},\quad \mathbf{E}\|\mathbf{v}\,, \tag{5}$$

and

$$\mathbf{B}'=\gamma[\mathbf{B}-(\mathbf{v}/c^2)\times\mathbf{E}],\quad \mathbf{B}\perp\mathbf{v}\,, \tag{6}$$

$$\mathbf{B}'=\mathbf{B},\quad \mathbf{B}\|\mathbf{v}\,, \tag{7}$$

where we now assume that the velocity $\mathbf{v}$ is constant and, as usual, $\gamma=1/\sqrt{1-(v/c)^2}$.

A similar set of equations relating the unprimed to the primed quantities is obtained by changing $\mathbf{v}$ into $-\mathbf{v}$. With regard to the electric and magnetic dipole moments, there are two different but equivalent formulations. In the first formulation,[2] or four-vector representation,[12] the electric dipole is viewed as a system of two charges q' and $-q'$ separated by a distance $\boldsymbol{\delta}\mathbf{l}'$ in the rest frame of the dipole. The dipole moment in the rest frame of the dipole is defined by

$$\mathbf{p}'=\lim_{q'\rightarrow\infty,\ \delta l'\rightarrow 0} q'\,\boldsymbol{\delta}\mathbf{l}'\,. \tag{8}$$

Since charge is invariant, we have $q=q'$. On the other hand, the length $\boldsymbol{\delta}\mathbf{l}'$ must be transferred to frame S by the appropriate contraction factors. We have $\boldsymbol{\delta}\mathbf{l}=\boldsymbol{\delta}\mathbf{l}'$ if $\boldsymbol{\delta}\mathbf{l}'\perp\mathbf{v}$ and $\boldsymbol{\delta}\mathbf{l}=(1/\gamma)\boldsymbol{\delta}\mathbf{l}'$ if $\boldsymbol{\delta}\mathbf{l}'\|\mathbf{v}$. Hence, the transformation equations for $\mathbf{p}$ are

$$\mathbf{p}=\mathbf{p}'\quad(\mathbf{p}\perp\mathbf{v})\,, \tag{9}$$

$$\mathbf{p}=\mathbf{p}'/\gamma\quad(\mathbf{p}\|\mathbf{v})\,. \tag{10}$$

It is, of course, possible to combine (9) and (10) into a single, more general, transformation equation, but we shall not need this more cumbersome form. Similarly, the transformation of the magnetic dipole moment is most easily obtained by considering the model of two magnetic charges

$q^{*\prime}$ and $-q^{*\prime}$ a distance $\delta\mathbf{l}'$ apart, in their rest frame and define

$$\mathbf{m}' = \lim_{q^{*\prime}\to\infty,\ \delta l'\to 0} q^{*\prime}\,\delta\mathbf{l}' . \tag{11}$$

We obtain in full analogy with the electric case,

$$\mathbf{m} = \mathbf{m}' \quad (\mathbf{m}\perp\mathbf{v}) , \tag{12}$$

$$\mathbf{m} = \mathbf{m}'/\gamma \quad (\mathbf{m}\|\mathbf{v}) . \tag{13}$$

It is also possible to derive the transformation equation starting from the magnetic moment of an elementary current loop of area $\delta\mathbf{A}'$, using the corresponding definition

$$\mathbf{m}' = \lim_{i'\to\infty,\ \delta A'\to 0} i'\,\delta\mathbf{A}',$$

where i' is the current in the loop rest frame. Now we can simply obtain the torques in both the S and S' frames in terms of the magnetic moments defined in the rest frame. We have, of course, in the rest frame

$$\boldsymbol{\tau}' = \mathbf{m}'\times\mathbf{B}' . \tag{14}$$

In the S frame, Eqs. (2) and (3) together with (12) and (13) yield

$$\boldsymbol{\tau} = \mathbf{m}'\times\mathbf{B} - \mathbf{m}'\times[\mathbf{v}\times(\mathbf{E}/c^2)], \quad \mathbf{B}\perp\mathbf{v} \text{ and } \mathbf{m}\perp\mathbf{v} , \tag{15}$$

$$\boldsymbol{\tau} = (1/\gamma)(\mathbf{m}'\times\mathbf{B}) - (1/\gamma)\mathbf{v}\times[\mathbf{m}'\times(\mathbf{E}/c^2)], \quad \mathbf{B}\perp\mathbf{v} \text{ and } \mathbf{m}\|\mathbf{v} . \tag{16}$$

Here, again, a more general and unique formula for any orientation of **B** and **v** could have easily been obtained, but it is much more cumbersome and will not be needed for our purpose. It is important to note that in all the torque expressions, **E** and **B** represent the *external* electric and magnetic fields in which the dipole is located.

III. MAGNETIC DIPOLES IN THE FIELD OF MOVING ELECTRIC CHARGES

Instead of a magnetic dipole in the field of a moving line of charges considered by Bedford and Krumm, we shall analyze a closely related but somewhat simpler system that displays the same physical characteristics. We shall take as a source an infinitely extended plane of charges (x-z plane), moving at constant speed v along the positive z direction. See Fig. 1. We chose an infinite plane instead of a line so that we would not have to contend with the nonuniformity of the field of a line. The nonuniformity plays no role in the discusssion, it only complicates the issue and can be avoided by making the dipole small enough (even then, forces may appear due to the nonuniformity of the fields). We shall introduce two frames of reference: Frame S' is the frame in which the dipole is at rest. In this frame, the charges in the plane are moving with speed v in the positive z' direction and gives rise to a surface current $\mathbf{K}'$. Frame S is the frame in which the plane charges are at rest and, with respect to it, the dipole moves with constant speed v in the $-z$ direction. Primed and nonprimed quantities refer to these two frames, respectively. The initial specification is merely a matter of convenience. We shall assume the following are given: the magnetic dipole moment $\mathbf{m}'$ in its own frame S', and the charge density σ in its own rest frame S.

A. Dipole moment normal to the plane of moving charges

To establish the notation and check the validity of the formulation expressed by Eq. (15), we briefly consider this case even though it is *not* controversial.

(1) In the S' frame, we have $m_y' = i'A'$, where i' is the current in the dipole whose area is A'. In this frame, the source appears as a combination of electric surface charges with density σ' and a surface current $\mathbf{K}'$ in the z' direction. They form a four-vector and values in different frames are connected by the Lorentz transformation with $\mathbf{v} = -v\mathbf{k}$,

$$K_z' = K' = \gamma(0 + \sigma v) = \gamma\sigma v = \sigma' v , \tag{17}$$

$$\sigma' = \gamma(\sigma + 0) = \gamma\sigma . \tag{18}$$

The magnetic field due to $\mathbf{K}'$ is readily obtained by Ampere's law and only has one component (see Fig. 1), $B_x' = -\frac{1}{2}\mu_0 K'$.

Applying Eq. (14), $\boldsymbol{\tau}' = \mathbf{m}'\times\mathbf{B}'$, we find at once

$$\tau_z' = \tfrac{1}{2}\mu_0 m' K' = \tfrac{1}{2}\mu_0 m'\sigma' v = \tfrac{1}{2}\gamma\mu_0\sigma m' v . \tag{19}$$

(2) We now view the same situation in the rest frame S of the moving charges. Here, the dipole moves with speed v along the $-z$ direction. Clearly, there is no magnetic field B due to the sheet of charges. Thus $\mathbf{B} = 0$. Using Eq. (15) we now find for the torque,

$$\boldsymbol{\tau} = -\mathbf{m}'\times\left(\mathbf{v}\times\frac{\mathbf{E}}{c^2}\right) = -\mathbf{v}\left(\mathbf{m}'\cdot\frac{\mathbf{E}}{c^2}\right) + (\mathbf{m}'\cdot\mathbf{v})\,\frac{\mathbf{E}}{c^2} = -\mathbf{v}\left(\mathbf{m}'\cdot\frac{\mathbf{E}}{c^2}\right). \tag{20}$$

Here, $\mathbf{v} = -v\mathbf{k}$, $\mathbf{E} = E_y\,\mathbf{j} = (\sigma/2\epsilon_0)\,\mathbf{j}$. Hence, we obtain at once

$$\tau_z = \tfrac{1}{2}\mu_0\sigma m' v . \tag{21}$$

Comparing this result with that in the S' frame, i.e., Eq. (19), we find

$$\tau_z' = \gamma\tau_z . \tag{22}$$

The transformation of the torque can be obtained from the transformation of the force and thus depends on the orientation of the force. In this case, whether we view the magnetic dipole as a system of magnetic charges or as a small current loop, the forces as well as the moment arms are perpendicular to the direction of motion of the dipole. See Fig. 2(a). The transverse forces in the S frame are related to those in the S' frame by[13–15] $F = F'/\gamma$, whereas the moment arms are unchanged, hence,

$$\tau_z = \tau_z'/\gamma$$

is the expected torque transformation and is in accordance with the results expressed by Eq. (22). We shall not discuss the physical origin of this torque, which, as already stated, is fairly well explained in the literature on the subject.

B. Dipole moment parallel to the velocity of the moving charges

This is the controversial case that prompted the new investigation.

(1) In the frame S' of the dipole, the torque is simply given by $\boldsymbol{\tau}' = \mathbf{m}'\times\mathbf{B}'$. See Fig. 3.

Since $\mathbf{m}' = m'\mathbf{k}$ and $\mathbf{B}' = -\frac{1}{2}\mu_0 K'\mathbf{i}$, we find

$$\tau_y' = -\tfrac{1}{2}\mu_0 m' K' = -\tfrac{1}{2}\mu_0 m'\sigma' v = -\tfrac{1}{2}\mu_0\gamma\sigma m' v . \tag{23}$$

(2) In the S frame, there is no magnetic field due to the plane of charges and thus it would seem that we cannot account for a torque in this situation. Formally, the answer is simple. Introducing $\mathbf{B} = 0$, the external magnetic field in Eq. (16), we find at once the "missing torque":

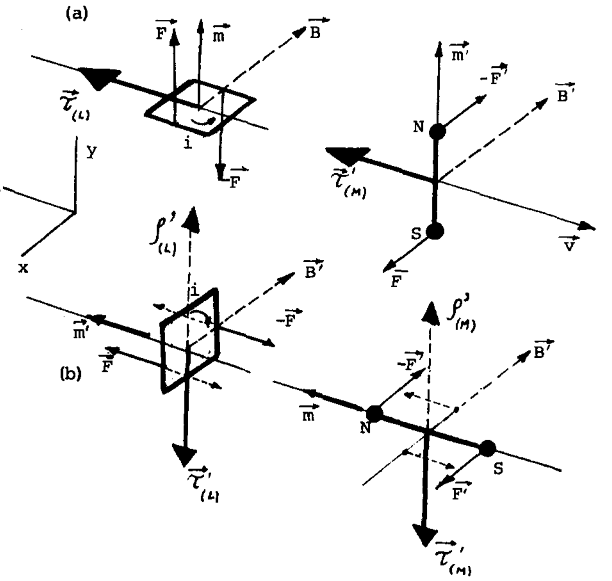

Fig. 2. (a) When the magnetic moment is perpendicular to the velocity, the forces responsible for the torque are transverse for both the magnetic charge dipole and the current loop. (b) When the magnetic moment is along or opposite the velocity, the forces responsible for the torque are still transverse for the magnetic charge dipole but parallel and antiparallel to the velocity for the current loop. A reaction torque from the support exactly counterbalances the magnetic torque in the current loop by supplying the reaction forces $\mathbf{F}_R$ and $-\mathbf{F}_R$. The same reaction forces, when applied to the system of magnetic charges at right, give rise to a rate of angular momentum in the laboratory frame.

$$\boldsymbol{\tau} = -\frac{1}{\gamma}\mathbf{v}\times\left(\mathbf{m}'\times\frac{\mathbf{E}}{c^2}\right) = \frac{1}{\gamma}(\mathbf{m}'\cdot\mathbf{v})\frac{\mathbf{E}}{c^2} - \frac{1}{\gamma}\mathbf{m}'\left(\mathbf{v}\cdot\frac{\mathbf{E}}{c^2}\right). \tag{24}$$

Here, we have $\mathbf{m}' = m'\mathbf{k}$, $\mathbf{v} = -v\mathbf{k}$, and $\mathbf{E} = (\sigma/2\epsilon_0)\mathbf{j}$. Hence,

$$\tau_y = -(1/\gamma)m'(v/c^2)E_y = -(1/2\gamma)\mu_0 m' v\sigma. \tag{25}$$

The forces giving rise to a torque in a magnetic charge system dipole are transverse to the velocity. See Fig. 2(b). Hence, the forces in the S and S' frames are related by $F = F'/\gamma$. The moment arms are along the velocity and also suffer a contraction: $d = d'/\gamma$. The torque Fd obeys the transformation

$$\tau_y = \tau'_y/\gamma^2. \tag{26}$$

This result is confirmed by expressions (23) and (25) for the torques in different frames.

Thus the torque is fully accounted for by Eq. (16), which holds exactly for the monopole model. The origin of this torque directly comes from the velocity-dependent term in the analog of the Lorentz force for magnetic charges.

In the case of a magnetic dipole consisting of a current in a conducting loop, as investigated by Bedford and Krumm, the physical origin of the "missing torque" $-(1/\gamma c^2)\mathbf{v}\times(\mathbf{m}'\times\mathbf{E})$ is far less obvious. A careful analysis of both external and internal fields will shed some light on the mechanism responsible for this torque in the current loop model.

We observe that in S, as well as in S', the electric field $\mathbf{E}_s$ of the plane of charges causes charges to be induced at the surface of the loop. As these induced charges move with speed v along $-z$, they generate a magnetic field $\mathbf{B}_i$ in S (but not in S'), and an electric field $\mathbf{E}_i$. As we shall see, it is the interaction between the magnetic field created by the induced charges and the current carriers of the magnetic dipole that is responsible for the torque in the S frame.

Consider a thin conducting loop of arbitrary shape lying in the $x'y'$ plane of its rest frame S'. See Fig. 4. Let σ'_i be the induced surface charge density on the outer surface of the loop. These induced charges create their own electric field $\mathbf{E}'_i$ that distorts the original electric field $\mathbf{E}'_s$ due to the plane of charges. The actual electric field in S' is the resultant of these two and its line of force will be normal to the outer surface of the loop. Now, at all points *within* the conducting loop, in its own rest frame, we must have on account of the electrostatic equilibrium condition, $\mathbf{E}'_i + \mathbf{E}'_s = 0$. This condition determines the distribution of induced charges on the surface of the loop.[16]

These induced charges σ'_i are at rest in S' and do not give rise to a magnetic field in that frame: $\mathbf{B}'_i = 0$. From the point of view of S, however, the electric field $\mathbf{E}_i$ certainly exists and there is also a magnetic field $\mathbf{B}_i$ due to the surface current $\mathbf{K}_i$ created by the motion of the induced charges on the dipole. We shall show that this magnetic field $\mathbf{B}_i$ in the S frame is uniform at all points *within* the loop. If these induced charges are indeed responsible for the torque in the S frame, then it should be possible to deduce its value by considering these charges as the sources of the magnetic field in the S frame.

Recall that in the calculation for the torque through Eq. (3) or (16), only the external fields are to be considered. In this interpretation, the induced charges are part of the external sources, and the external fields here consist of two parts, that due to the induced charges and that due to the plane of charges. We write in the S frame,

$$\mathbf{B}_{\text{ext}} = \mathbf{B}_i + \mathbf{B}_s \tag{27}$$

and since the charges in the plane do not move in S, we have $\mathbf{B}_s = 0$. Thus

$$\mathbf{B}_{\text{ext}} = \mathbf{B}_i. \tag{28}$$

We also have

$$\mathbf{E}_{\text{ext}} = \mathbf{E}_i + \mathbf{E}_s, \tag{29}$$

where $\mathbf{E}_s$, the field of the plane of charge, is known and equal to $\sigma/2\epsilon_0\,\mathbf{j}$.

Correspondingly, in the rest frame of the loop,

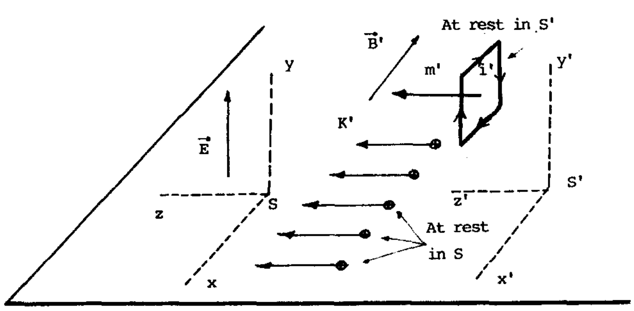

Fig. 3. The magnetic dipole moment is now parallel to the velocity of the electric charges in the (x-z) plane.

$$\mathbf{B}'_{\rm ext} = \mathbf{B}'_i + \mathbf{B}'_s \tag{30}$$

and, since, as stated $\mathbf{B}'_i = 0$, we have

$$\mathbf{B}'_{\rm ext} = \mathbf{B}'_s . \tag{31}$$

Finally,

$$\mathbf{E}'_{\rm ext} = \mathbf{E}'_i + \mathbf{E}'_s = 0 \tag{32}$$

on account of the electrostatic equilibrium condition.

Now all the primed fields are related to unprimed fields by the transformation equations (4)–(7). We now find

$$\mathbf{E}_{\rm ext} = \gamma(\mathbf{E}'_{\rm ext} - \mathbf{v}\times\mathbf{B}'_{\rm ext}) = -\gamma\mathbf{v}\times\mathbf{B}'_s , \tag{33}$$

$$\mathbf{B}_{\rm ext} = \gamma[\mathbf{B}'_{\rm ext} + (\mathbf{v}/c^2)\times\mathbf{E}'_{\rm ext}] = \gamma\mathbf{B}'_{\rm ext} = \gamma\mathbf{B}'_s . \tag{34}$$

In addition,

$$\mathbf{B}'_s = \gamma[\mathbf{B}_s - (\mathbf{v}/c^2)\times\mathbf{E}_s] = -\gamma(\mathbf{v}/c^2)\times\mathbf{E}_s \tag{35}$$

and thus the total external magnetic field in the S frame, which is also equal to $\mathbf{B}_i$ is given by

$$\mathbf{B}_{\rm ext} = \gamma\mathbf{B}'_s = -\gamma^2(\mathbf{v}/c^2)\times\mathbf{E}_s . \tag{36}$$

Since $\mathbf{E}_s$ is uniform, so is $\mathbf{B}_{\rm ext}$, but only within the loop where the electrostatic condition (32) holds. Also, from Eqs. (33) and (35), we obtain

$$\mathbf{E}_{\rm ext} = -\gamma\mathbf{v}\times\left(-\gamma\frac{\mathbf{v}}{c^2}\times\mathbf{E}_s\right)$$

$$= \frac{\gamma^2}{c^2}[\mathbf{v}(\mathbf{v}\cdot\mathbf{E}_s) - \mathbf{E}_s v^2] = -\frac{\gamma^2 v^2}{c^2}\mathbf{E}_s . \tag{37}$$

Inserting $\mathbf{B}_{\rm ext}$ and $\mathbf{E}_{\rm ext}$ into the general expression (16) for the torque in the S frame yields after some algebraic reduction

$$\boldsymbol{\tau} = \frac{\gamma}{c^2}\left[-\mathbf{v}(\mathbf{m}'\cdot\mathbf{E}_s) + (\mathbf{m}'\cdot\mathbf{v})\mathbf{E}_s\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\mathbf{m}'(\mathbf{v}\cdot\mathbf{E}_s)\right]$$

$$= \frac{1}{\gamma c^2}(\mathbf{v}\cdot\mathbf{m}')\mathbf{E}_s .$$

Since $\mathbf{v} = -v\mathbf{k}$ and $\mathbf{E}_s = (\sigma/2\epsilon_0)\mathbf{j}$, we find $\tau_y = -\frac{1}{2\gamma}\mu_0\sigma m' v$, which is the same result as Eq. (25).

Thus this interpretation, which attributes the origin of the magnetic field responsible for the torque in the S frame to the motion of the induced charges and their surface currents, seems justified. It is illuminating to calculate the torque in the S frame directly using the Lorentz force. Con-

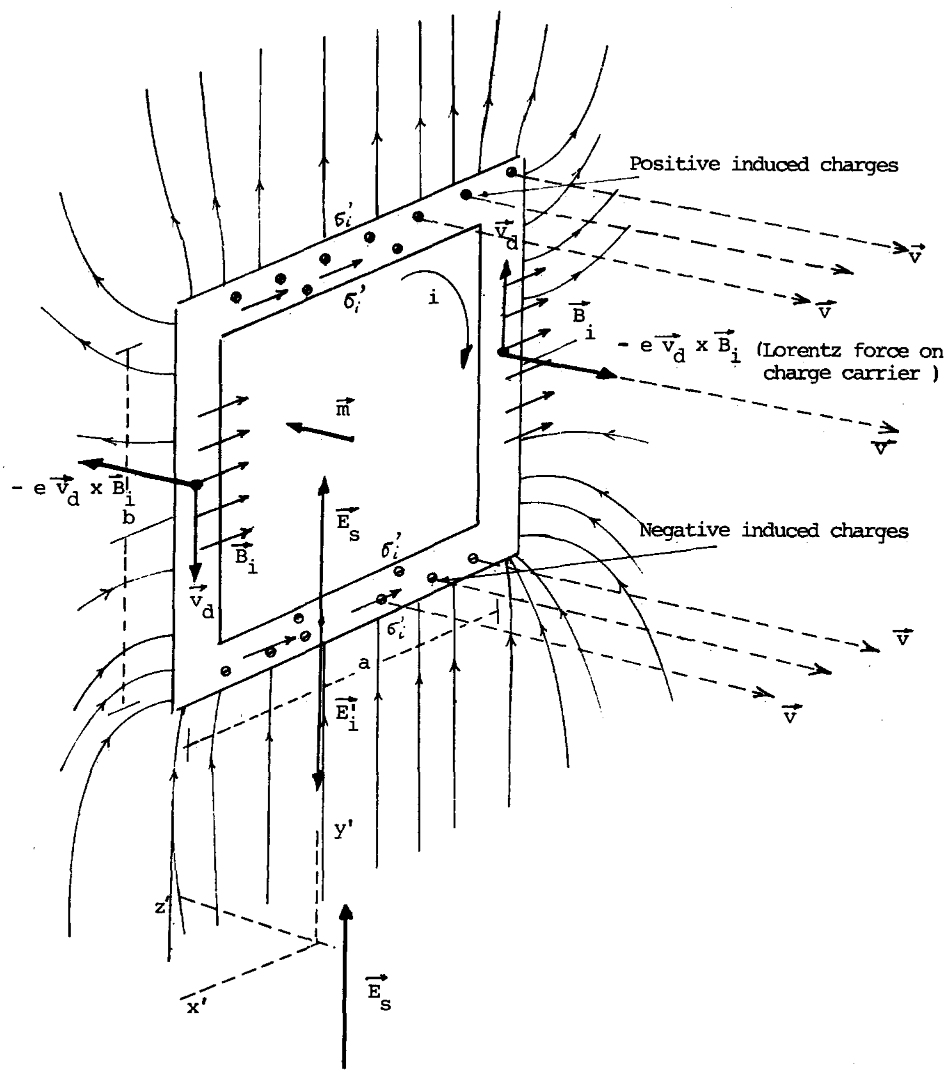

Fig. 4. The magnetic dipole, as a conducting loop with induced charges on its periphery creating an electric field $\mathbf{E}'_i$ in S' and both an electric field $\mathbf{E}_i$ and a magnetic field $\mathbf{B}_i$ in S. In frame S', we have $\mathbf{E}'_i + \mathbf{E}'_s = 0$, at all points within the conducting loop. The field $\mathbf{B}_i$ in the lab frame is uniform inside the loop. The charge carriers of the neutral current in the right and left segments of the loop are subjected to the Lorentz force $-e\mathbf{v}_d\times\mathbf{B}_i$.

sider, for example, a rectangular loop of area $A' = a'b'$ and carrying a current i' in its rest frame. Figure 4 shows such a loop moving with velocity $\mathbf{v}$ in the $-z$ direction. Also shown are the induced charges moving with velocity $\mathbf{v}$ and distributed in such a way as to have $\mathbf{E}'_i + \mathbf{E}'_s = 0$. The magnetic field they create as seen from the S frame is uniform within the loop and is given by Eq. (36): $\mathbf{B}_i = -\gamma^2(\mathbf{v}/c^2)\times\mathbf{E}_s$. The drifting electrons that are the charge carriers of the neutral current move with velocity $\mathbf{v}'_d$ around the loop. The ones in the right segment experience the Lorentz force $\mathbf{F} = -e\mathbf{v}_d\times\mathbf{B}_i$, where $\mathbf{v}_d$ is measured in the S frame. We have $v_d = v'_d/\gamma$ since the velocity is transverse to the motion of the loop. Thus the force on one electron is $(1/\gamma)ev'_dB_i$ and is directed along $-\mathbf{v}$. The left segment will experience the same force per electron in the opposite direction and the carriers on the top and bottom segments will not be subjected to any force. Hence, a net torque appears that is equal to $\tau = Fa = (1/\gamma)ev'_dB_ia$ for each electron. Suppose there are N carriers in the right segment, then the total force on that section is

$$(1/\gamma)Nev'_dB_i = (1/\gamma)(N/V')V'ev'_dB_i = (1/\gamma)V'(n'ev'_d)B_i,$$

where V' is the volume of the segment and $n' = N/V'$ in the rest frame. Now, $n'ev'_d = J' = i'/s'$, where s' is the cross section and $V' = s'b'$. The force is then given by $(1/\gamma)b'i'B_i$. The total torque on the loop is directed along $-y$ and is equal to

$$\tau_y = -(1/\gamma)(a'b')i'B_i = -(1/\gamma)m'B_i = -\gamma(m'/c^2)vE_s = -(\gamma/2)\,\mu_0\sigma m'v\,. \quad (38)$$

This result can easily be generalized to apply to a loop of arbitrary shape.

Comparing the torque in the rest frame [Eq. (23)] to the result in Eq. (38) in the S frame, we find

$$\tau_y = \tau'_y\,. \quad (39)$$

Here, inspection of Fig. 3(b) at left shows that the forces are not transverse to the velocity, hence $F = F'$. The moment arms are transverse and unchanged. Hence the proper torque transformation is $\tau_y = \tau'_y$, and is confirmed by (39).

Upon comparing Eq. (25) for the torque on a system of magnetic charges $\boldsymbol{\tau}_M$ and Eq. (38) for the torque on a current loop $\boldsymbol{\tau}_L$ obtained directly from the Lorentz force, we find that they agree with each other except for a factor of γ^2. Specifically,

$$\boldsymbol{\tau}_L = \gamma^2\boldsymbol{\tau}_M\,. \quad (40)$$

Thus it would seem that the equivalence discussed in Sec. II is only valid up to first order in (v/c) and that it may be possible to distinguish between the two cases if the experimenter could detect second-order effects. A subtle argument will demonstrate that the equivalence is exact in spite of the apparent discrepancy.

In order to assure the rotational equilibrium of the current loop, a reaction torque ρ'_L must be supplied by the support in such a way as to balance the magnetic *forces* exactly. See Fig. 3(b). In the rest frame of the loop we have

$$\boldsymbol{\tau}'_L + \boldsymbol{\rho}'_L = 0 \quad (41)$$

and in the S frame

$$\boldsymbol{\tau}_L + \boldsymbol{\rho}_L = 0\,. \quad (42)$$

Hence,

$$\boldsymbol{\rho}_L = -\boldsymbol{\tau}_L = -\boldsymbol{\tau}'_L \quad (43)$$

on account of (39). If, instead, we have a system of magnetic charges with the same $\mathbf{m}'$, held in equilibrium in their rest frame by the same torque $\boldsymbol{\rho}'_M = \boldsymbol{\rho}'_L$, then the rest frame equilibrium equation is

$$\boldsymbol{\tau}'_M + \boldsymbol{\rho}'_M = 0\,. \quad (44)$$

In the S frame, the rotational equilibrium is *not* given by $\boldsymbol{\tau}_M + \boldsymbol{\rho}_M = 0$, but by the equation

$$\boldsymbol{\tau}_M + \boldsymbol{\rho}_M = \frac{d\mathbf{L}}{dt}, \quad (45)$$

where $d\mathbf{L}/dt$ is the rate of change of the angular momentum of the system. Even though the system does *not* rotate, there is in S a rate of change of angular momentum due to the support forces directed along and against the velocity and giving rise to a corresponding energy flux for each side equal to $\mathbf{F}_R\cdot\mathbf{v}$. The rate of mass increase per side is $(\mathbf{F}_R\cdot\mathbf{v})/c^2$ and the angular momentum of the right segment increases at a rate $d(Mvr)/dt = (dM/dt)vr$, where M is the mass of the right segment and $r = a/2$. Thus the total rate for both sides is

$$\frac{dL}{dt} = \frac{2(\mathbf{F}_R\cdot\mathbf{v})}{c^2}\left(v\,\frac{a}{2}\right) = \rho_M\,(v/c)^2\,. \quad (46)$$

The equilibrium equation (45) becomes

$$\boldsymbol{\tau}_M + \boldsymbol{\rho}_M = \boldsymbol{\rho}_M v/c^2$$

or

$$\boldsymbol{\rho}_M = -\gamma^2\boldsymbol{\tau}_M\,. \quad (47)$$

On account of (40) and (43), we find

$$\boldsymbol{\rho}_M = -\boldsymbol{\tau}_L = \boldsymbol{\rho}_L\,. \quad (48)$$

Since the experimenters in both frames only measure the reaction torque, they will find, according to (48), no difference between the reaction torques to be supplied in each case.

The above explanation was first put forward by Laue[17,18] in his discussion of the relativistic lever. His argument can be found in the classic treatise by Tolman,[19] as well as in the standard textbook by Panofsky and Phillips.[9]

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SOLUTION TO THE PROBLEM ON PAGE 105

The goof[1] was in taking moments about the point of contact at the floor. There are two "safe" places to take moments: the c.m., and a point *at rest* in the inertial frame. The sliding contact point is neither! For the cylinder, rolling *without slipping*, the contact point is momentarily at rest. So Eq. (3) is valid. But Eq. (3') is wrong—disastrously!

Here is a correct derivation. Eqs. (1') and (2') are correct. Also, we have $z = R\cos\theta$, giving $z' = -\theta' R\sin\theta$ and

$$z'' = -\theta'' R\sin\theta - \theta'^2 R\cos\theta. \tag{7}$$

Insert Eqs. (7) and (2') into (1'). Take the stick to be uniform with $I_{\text{c.m.}} = mR^2/3$. Use "natural" time units $t_1 = (R/g)^{1/2}$, and dimensionless time $\tau = t/t_1$. Let $d/d\tau = '$. That gives

$$\theta'' = [\sin\theta - \theta'^2\sin\theta\cos\theta]/(\tfrac{1}{3} + \sin^2\theta), \tag{8}$$

which can be solved numerically. We want to verify the "intuitively obvious" result that when θ_0 goes to zero, the time to reach any finite θ goes to infinity. Therefore, take $\theta \ll 1$ and $\theta' \ll 1$ in Eq. (8), which then becomes $\theta'' = 3\theta$. For release with initial values, $\theta = \theta_0$ and $\theta' = 0$, this has the solution $\theta = \theta_0\cosh[(3)^{1/2}\tau]$. For large τ with $(3)^{1/2}\tau \gg 1$, this becomes $2\theta = \theta_0\exp[(3)^{1/2}\tau]$, i.e.,

$$\tau = (3)^{-1/2}\ln(2\theta/\theta_0). \tag{9}$$

For example, take $\theta = 0.1$ rad and $\theta_0 = 0.0001$ rad to get $\tau = 4.39$. Keeping θ constant at 0.1 rad, every further decrease in θ_0 by a factor of 10 increases t by adding 1.33 natural units. Thus t goes slowly (logarithmically) to infinity as θ_0 goes to zero. That agrees with intuition. You can now collect your bet.

[1] The erroneous result, Eq. (5') and its derivation, were shown to me by a famous physics professor who shall remain anonymous. I exclaimed, "It has to be wrong!" He insisted, "No, it's right!" So we bet. He lost. We still don't know the original source of this momentous error. Has the reader seen it before? Is it, we fear (or hope?), in some textbook?

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Comment on "Electrodynamics of moving dipoles: The case of the missing torque," by V. Namias [Am. J. Phys. 57, 171 (1989)]

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In the second part of his article,[1] Namias seeks an explanation for the torque on a magnetic dipole moving in the direction of its axis above a charge sheet. He ascribes this torque to magnetic forces on the dipole current, the magnetic field being produced by surface charges on the loop induced by the external electric field. This explanation is certainly not general since it will clearly not apply in cases where the dipole is made of nonconducting materials, e.g., one or more charges fixed to a rotating dielectric. Moreover, even in the case he treats, we believe his explanation to be inadequate.

For Namias, the role of the external electric field in the explanation is solely to induce the required surface charge distribution on the loop (see Ref. 1, Fig. 3). Suppose we reproduce this distribution by fixing charges to a dielectric, and reproduce the neutral current by counterrotating positively and negatively charged dielectric loops. His analysis would imply that such a magnetic dipole arrangement moving in a region of *no external field* would experience a torque, which is clearly false. Indeed, the *magnetic* forces on the currents that he holds to be responsible for the torque exist in this case, and the *lack of torque* is to be accounted for! The mechanism described in our article[2] does this, as a careful consideration of the motion of the current carriers in the *internal* electric field of the dipole assembly shows. Furthermore, the reintroduction of the stationary charged sheet, which now leaves the charge distributions on the dielectrics unaltered, results in a torque, which can be accounted for only by a consideration of the motion of the current carriers in the external electric field.

Finally, in our view, the way to treat the case of a conducting loop is to ignore *internal* force pairs that are equal and opposite by Newton's third law (the magnetic force he describes is one member of such a pair) as is normally done in mechanics, and to consider the forces exerted by the *external* field only. Work is done *by this field* on the current carriers within the conductor, positive work on one side of the loop, negative on the other, and as shown in Ref. 1, Fig. 3; *this* is what gives rise to the torque.

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A discussion of the dielectric model of Bedford and Krumm [Am. J. Phys. 57, 178 (1989)]

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In an attempt to refute my explanation for the origin of the missing torque,[1] Bedford and Krumm[2] have devised an argument that, in my opinion, is not valid. In my discussion, the role played by the external electric field due to the plane of charges is essential. These authors present a model in which the external field is eliminated and the exact same distribution of induced charges is reproduced artificially, so to speak, by fixing them on a dielectric loop. They then also reproduce the neutral current by means of counterrotating positively and negatively charged dielectric loops.

This arrangement, of course, does not give rise to any torque. They conclude that my analysis, which would imply that such a magnetic dipole arrangement moving in a region of no external electric field would experience a torque, must be erroneous. Contrary to this statement, my analysis when applied to their model does yield a zero torque in all cases as expected.

Let us examine the situation from both the point of view of the rest frame S' and the S frame, using the general expression for the torque in the respective systems [Eqs. (14)–(16) in Ref. 1]. In the rest frame S', we have $\mathbf{B}' = 0$ and thus $\boldsymbol{\tau}' = 0$. In the S frame, we can attack the problem in two different ways, just as was done in my article.

(1) The dipole system consists of the sprayed charges on the dielectric and the rotating charges giving rise to the neutral current. Such a system is in a zero external source field and, consequently, $\mathbf{E} = 0$ and $\mathbf{B} = 0$ in Eq. (16), where only the fields external to the system must be used. We conclude $\boldsymbol{\tau} = 0$.

(2) The dipole system consists of the rotating charges

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only making up the neutral current. The sprayed charges have been distributed and adjusted exactly so as to recreate the electric field $\mathbf{E}_i'$, the same as that due to the induced surface charges on the surface of the conducting loop. In this point of view, the field $\mathbf{E}_i'$ can be considered external to the dipole. Since there is no plane of charge, we have now $\mathbf{E}_s = 0$ and $\mathbf{B}_s = 0$, as well as $\mathbf{E}_s' = 0$ and $\mathbf{B}_s' = 0$. Here, we also have $\mathbf{B}_i' = 0$.

Thus, in this model, Eqs. (27)–(32) are replaced by

$$\mathbf{B}_{\text{ext}} = \mathbf{B}_i + \mathbf{B}_s = \mathbf{B}_i\,, \quad (1)$$

$$\mathbf{E}_{\text{ext}} = \mathbf{E}_i + \mathbf{E}_s = \mathbf{E}_i\,, \quad (2)$$

$$\mathbf{B}_{\text{ext}}' = \mathbf{B}_i' + \mathbf{B}_s' = \mathbf{B}_i' = 0\,, \quad (3)$$

$$\mathbf{E}_{\text{ext}}' = \mathbf{E}_i' + \mathbf{E}_s' = \mathbf{E}_i'\,. \quad (4)$$

Using the transformation equations for the fields, we find now

$$\mathbf{E}_i = \gamma(\mathbf{E}_i' - \mathbf{v}\times\mathbf{B}_i') = \gamma\mathbf{E}_i', \quad (5)$$

$$\mathbf{B}_i = \gamma[\mathbf{B}_i' + (\mathbf{v}/c^2)\times\mathbf{E}_i'] = \gamma(\mathbf{v}/c^2)\times\mathbf{E}_i'\,. \quad (6)$$

Since the fields in (5) and (6) are now external, we can introduce them in the equation of the torque in the S system (16) and obtain

$$\boldsymbol{\tau} = \mathbf{m}'\times[\mathbf{v}\times\mathbf{E}_i'] - \mathbf{v}\times[\mathbf{m}'\times(\mathbf{E}_i'/c^2)] \quad (7)$$

$$= (1/c^2)[\mathbf{v}(\mathbf{m}'\cdot\mathbf{E}_i') - (\mathbf{m}\cdot\mathbf{v})\mathbf{E}_i' - (\mathbf{v}\cdot\mathbf{E}_i')\mathbf{m}' + (\mathbf{v}\cdot\mathbf{m}')\mathbf{E}_i'] = 0,$$

as expected. Thus the role of the external field is not solely to induce the surface distribution on the loop, as claimed by Bedford and Krumm. Elimination of the original plane of charges by mimicking the charge distribution on the loop is not sufficient; one must, in addition, assure that the total electric field inside the loop is zero in its rest frame. In their dielectric model, the rotating dielectric charges responsible for the dipole moment lie in an electric field $\mathbf{E}_i'$ and there is no way to annul this electric field exactly unless an external source electric field $\mathbf{E}_s'$ is introduced, which is such that $\mathbf{E}_i' + \mathbf{E}_s' = 0$. As a result, their dielectric model is not equivalent to a conducting loop with induced charges due to an external source. Therefore, their argument against the explanation of the missing torque in terms of an interaction between the magnetic field of the surface currents generated by the induced charges and the charge carriers of the dipole does not seem to be valid.

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Experiments in two dimensions using a video camera and microcomputer

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The kinematics and dynamics of two-dimensional motion are commonly investigated in the teaching laboratory using an air table equipped with either a camera for stroboscopic photography or with spark recording equipment. Both these methods of data acquisition have disadvantages as discussed by Decker and Jeffery,[1] who used videotaping techniques to record data. We offer an alternative approach, which utilizes a television camera–microcomputer interface to achieve high precision and has the additional advantage of producing a real-time display of a two-dimensional trajectory. The apparatus is, therefore, suitable for lecture demonstrations as well as regular laboratory use.

A television camera[2] with an 8.5-mm focal length lens is mounted above a standard Ealing–Daw air table.[3] Pucks have been altered to hold a center-mounted 6-V minilamp and a nickel–cadmium battery. The puck's position coordinates are determined by measuring the relationship between the intensity blip on the television camera's video output (produced by the light bulb) and the television camera's horizontal and vertical synch pulses. The x-y positions are determined once each camera frame and the data are displayed by a plot on the computer's screen and is stored in memory. The electronics consists of a set of counters used for recording x position, y position, and time, a master clock, and appropriate logic and signal-conditioning circuitry.[4] Interfacing requires two 8-bit parallel input/output ports; we have used both Apple IIe and S-100 microcomputers. The apparatus determines the vertical coordinate to an accuracy of 2 cm, the horizontal coordinate to

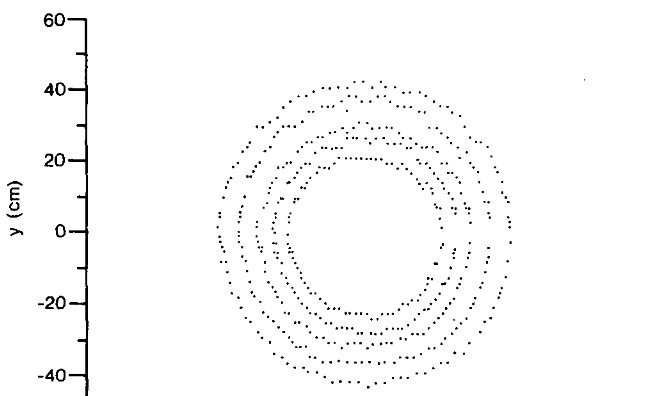

Fig. 1. Circular motion: raw data of the trajectory of a puck moving under the influence of a constant central force.