Thus $\langle V \rangle$ depends on the time (as does V for a classical elliptical orbit). If we now average over a sufficiently long time, the time-dependent off-diagonal contributions to Eq. (19) average to zero and we get Eq. (4), as promised. [Note that the superposition (12) is assumed not to include any continuum states.]

For definiteness we considered the system to be a hydrogenlike atom. More generally, although it is true that for a single stationary bound state the quantum virial theorem gives $\langle 2T \rangle = \langle \mathbf{r}\cdot\mathbf{grad}\, V \rangle$, this result does not hold at each instant for a time-dependent superposition of stationary bound states. Instead, one has $\langle 2T \rangle_{av} = \langle \mathbf{r}\cdot\mathbf{grad}\, V \rangle_{av}$, where, again, av means time average; this is the result that should be compared with the classical virial theorem result $2T_{av} = (\mathbf{r}\cdot\mathbf{grad}\, V)_{av}$.

[1]See, for example, J. B. Marion, *Classical Dynamics* (Academic, New York, 1970), 2nd ed., p. 233.

[2]See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968), 3rd ed., p. 180.

Comment on "Displacement current—A direct derivation," by T. Biswas [Am. J. Phys. 56, 373–374 (1988)]

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In a recent issue of the *American Journal of Physics*, T. Biswas presents a derivation of the Ampère–Maxwell equation which includes the displacement current, starting from the Biot–Savart law.

The author first cites the standard argument that in order to conform to the continuity equation, Ampère's law must have an additional term representing the displacement current. He then argues that "... it often gives the student a feeling that the displacement current is a correction term that does not directly result from the laws of electrodynamics but is mathematically necessary to *fix* Ampère's law." The author then proceeds to give a "direct" derivation of Ampère's law as generalized by Maxwell, in the vacuum:

$$\nabla\times\mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{1}$$

from the Biot–Savart law.

Apparently, the author feels that the Biot–Savart law is a fundamental law of electromagnetism and that the student would rather see the Ampère–Maxwell law (1) derived from it.

The Biot–Savart law is far from being a fundamental law of electromagnetism. In fact, the Biot–Savart law is so restricted that it cannot, in all rigor, be applied to a single moving point charge. The application of the law to this case constitutes the so-called quasistatic approximation that is acceptable so long as the velocity of the charged particle is much smaller than the speed of light.

Certainly, it would not be logical to expect to obtain from the Biot–Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}_1)\times(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^3}\, dv_1, \tag{2}$$

which was originally based on observations of the static magnetic fields produced by steady currents, a result of far greater generality, valid for any time-dependent situation, such as expressed by the Ampère–Maxwell law (1). It would be just as unrealistic to expect to derive the laws of the induced electric field (Faraday's law) from the property of the electrostatic field that its circulation is zero.

Now, *all* the mathematical manipulations following Eq. (3) of Biswas' article are correct, and the important question is to discover what in his derivation is basically at fault.

As we shall show, Eq. (7) in Biswas' article,

$$\frac{\partial}{\partial t}\int \frac{(\mathbf{r}-\mathbf{r}_1)\rho}{|\mathbf{r}-\mathbf{r}_1|^3}\, dv_1 = 4\pi\frac{\partial \mathbf{D}}{\partial t}, \tag{3}$$

is the fly in the ointment.

First, we note that the above result will be zero unless ρ depends on time. Thus we shall rewrite Eq. (3) as

$$\frac{\partial}{\partial t}\int \frac{\rho(\mathbf{r}_1,t)(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^3}\, dv_1 = 4\pi\frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t}. \tag{4}$$

The author, in writing the right-hand side of his Eq. (7), argues that

$$\int \frac{(\mathbf{r}-\mathbf{r}_1)\rho}{|\mathbf{r}-\mathbf{r}_1|^3}\, dv_1 \tag{5}$$

is equal to $(4\pi\epsilon_0)\mathbf{E} = 4\pi\mathbf{D}$, and considers the above integral as an expression of Coulomb's law for distributed charges.

Unfortunately, Coulomb's law is only exactly valid under static conditions, and thus for a ρ that does not depend on time. Hence,

$$\frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}_1,t)(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^3}\, dv_1 = \mathbf{E} \tag{6}$$

is incorrect.

Indeed, taking the curl on both sides of (6) yields $\nabla\times\mathbf{E} = 0$, which is only the equation for electrostatics. In a situation in which the distribution $\rho(\mathbf{r}_1,t)$ varies with time, one should properly find $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$, as required by Faraday's law.

The correct expression of the dynamic electric field is considerably more complicated. It can be obtained from

$$E = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{7}$$

where

$$\phi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}_1, t^*)}{|\mathbf{r} - \mathbf{r}_1|} dv_1, \quad (8)$$

$$A = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}_1, t^*)}{|\mathbf{r} - \mathbf{r}_1|} dv_1, \quad (9)$$

and $t^* = t - (1/c)|\mathbf{r} - \mathbf{r}_1|$ is the retarded time.

The above potentials satisfy the Lorentz condition $\nabla \cdot \mathbf{A} + (1/c)^2(\partial\phi/\partial t) = 0$, and one can show that this condition is fulfilled if the densities of charge and current satisfy the equation of continuity. Thus we find, in general,

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0} \nabla \int \frac{\rho(\mathbf{r}_1, t^*)}{|\mathbf{r} - \mathbf{r}_1|} dv_1 - \frac{\partial}{\partial t} \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}_1, t^*)}{|\mathbf{r} - \mathbf{r}_1|} dv_1. \quad (10)$$

We observe that even if the retardation effects are neglected (and then of course the derivation cannot claim to be exact), the electric field is still *not* given by Eq. (6). In this case, replacing t^* by t, one finds

$$\mathbf{D} = \epsilon_0 \mathbf{E} = \left(\frac{1}{4\pi}\right) \int \frac{\rho(\mathbf{r}_1, t)(\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3} dv_1$$

$$-\left(\frac{1}{4\pi c^2}\right) \int \frac{[\partial \mathbf{J}(\mathbf{r}_1, t)/\partial t]}{|\mathbf{r} - \mathbf{r}_1|} dv_1.$$

Thus, even the quasistatic electric field is not given by (6). An additional integral over the time derivative of the current distribution must be included.

In conclusion,

$$\frac{1}{4\pi} \int \frac{\rho(\mathbf{r}_1, t)(\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3} dv_1$$

can only represent $\mathbf{D}$ if ρ is strictly constant, and in this case $\partial \mathbf{D}/\partial t = 0$, showing that the Biot–Savart equation, following the steps of Biswas up to his Eq. (7), leads to $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, as it should.

This example shows how very cautious one must be in interpreting certain electrodynamic expressions in their integral form.

[a] Deceased. At the time of Professor Namias' death, this note had been tentatively accepted, pending a few minor modifications; the necessary final revisions were made by the editor. Please address all correspondence concerning this article to: P. D. Gupta, Chemistry and Physics Department, Purdue University Calumet, Hammond, IN 46323.

Note on "Field versus action-at-a-distance in a static situation," by N. L. Sharma [Am. J. Phys. 56, 420–423 (1988)]

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In a recent article[1] N. L. Sharma presents a variation on the "Feynman disk paradox."[2] Sharma considers a uniformly magnetized charged conducting sphere. When the charge is drained off (by touching a grounding wire to the south pole) the sphere begins to rotate, in apparent violation of conservation of angular momentum. The point of the "paradox" is to demonstrate that even *static* electromagnetic fields can carry angular momentum—in this instance, the angular momentum initially stored in the fields is

$$L_{em} = \tfrac{2}{9}\mu_0 Q a^2/M,$$

where M is the magnetization, Q is the charge, and a is the radius of the sphere. Sharma demonstrates that this is precisely the angular momentum picked up by the sphere when it discharges. (As the current flows over the surface to the south pole, it experiences a magnetic force in the azimuthal direction; it is the torque associated with this force that causes the sphere to rotate.)

Now, the angular momentum density stored in the electromagnetic fields,

$$\epsilon_0 \mathbf{r} \times (\mathbf{E} \times \mathbf{B}),$$

can be removed *either* by turning off $\mathbf{E}$ (discharging the sphere) *or* by eliminating $\mathbf{B}$ (demagnetizing the sphere), and my purpose here is to note that you get the same answer either way, although the *mechanisms* are entirely different. Suppose that instead of draining off the charge we heat up the sphere, so that (passing through the Curie temperature) it gradually loses its magnetization. The magnetic field inside the sphere is uniform:

$$\mathbf{B} = \tfrac{2}{3}\mu_0 M \hat{z} \quad (r < a);$$

as $\mathbf{B}$ decreases, it will induce an azimuthal electric field, in accordance with Faraday's law:

$$\mathbf{E} = -\frac{1}{3}\mu_0 r \sin\theta \frac{dM}{dt}\hat{\phi} \quad (r \leqslant a).$$

This field exerts a torque on the surface charge $\sigma = Q/4\pi a^2$:

$$\mathbf{N} = \int (\mathbf{r} \times \mathbf{E})\sigma\, dS = -\frac{1}{6}\mu_0 Q a^2 \frac{dM}{dt}\hat{z} \int \sin^3\theta\, d\theta$$

$$= -\frac{2}{9}\mu_0 Q a^2 \frac{dM}{dt}\hat{z},$$

which causes the sphere to rotate. The final angular momentum is evidently

$$L_{mech} = \tfrac{2}{9}\mu_0 Q a^2 M,$$

the same as the angular momentum originally stored in the fields.

[1] N. L. Sharma, Am. J. Phys. **56**, 420 (1988). A similar model was discussed by E. M. Pugh and G. E. Pugh, Am. J. Phys. **35**, 153 (1967) and R. H. Romer, Am. J. Phys. **35**, 445 (1967). See also R. H. Romer, Am. J. Phys. **53**, 15 (1985). For further references, see T.-C. E. Ma, Am. J. Phys. **54**, 949 (1986).

[2] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1964), Vol. II, p. 17-5.