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Quantum Mechanics of One- and Two-Electron Atoms

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CONTENTS:

Introduction: Units

The Hydrogen Atom without External Fields
- Nonrelativistic Theory
- DIRAC Theory
- Radiative and other Corrections

The Helium Atom without External Fields
- Nonrelativistic Theory
- Relative Theory

Atoms in External Fields
- ZEEMAN Effect
- STARK Effect in Hydrogen
- STARK Effect in Helium

Interaction with Radiation
- Discrete Spectrum
- The Photoeffect
- Bremsstrahlung

Appendix on Spherical Harmonics
Bibliography
Addenda and Errata
AUTHOR INDEX—SUBJECT INDEX
INDEX OF TABLES.

REVIEW:

. . . One has the immediate impression that "this is for the ages" (or what passes for the ages in physics). Its progenitor, written by Bethe for the famous Volume 24/1 of the original *Handbuch,* served as *the* standard reference in this field for 24 years; the new version may well hold up as long. . . . It is a tribute to the original version that so much of the new article bears such a close resemblance to the old one, even after a quarter of a century and notwithstanding the accumulation of a vast quantity of additional measurement, particularly of the fine- and hyperfine structure of H- and He-like atoms, and of the development of the new techniques of quantum electrodynamics for their interpretation. . . . With its 350 pages crammed with practically all that is known about the subject, it is a monumental work."

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Motion of Magnetic Lines of Force*

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It is often said that magnetic lines of force in a conducting fluid move with the fluid. In the case of a plasma, this means that the lines of force move with the particle drift velocity[1] $\mathbf{v}_p = (\mathbf{E} \times \mathbf{H})/H^2$. Such statements are not directly verifiable, since the velocity of a line of force is not a measurable quantity. However, the statement that the lines of force move with a certain velocity $\mathbf{v}(\mathbf{r}, t)$ does have verifiable consequences, such as: (1) The flux through a closed curve moving with velocity $\mathbf{v}$ is constant. (2) A line, moving with velocity $\mathbf{v}$, which is initially a line of force, remains a line of force in the course of its motion. Statement (2), as well as all other verifiable consequences of the original hypothesis, follows from statement (1). A velocity is said to be flux-preserving if it satisfies (1), line-preserving if it satisfies (2).

It is permissible to ascribe a velocity $\mathbf{v}$ to the lines of force if and only if $\nabla \times (\mathbf{E} + \mathbf{v} \times \mathbf{H})$ vanishes identically. It is always possible to choose a $\mathbf{v}$ satisfying this relation, although it is not generally possible to do this uniquely. To say that a certain velocity $\mathbf{v}$ is permissible means that all the verifiable consequences of ascribing this velocity to the lines of force are valid, i.e., that $\mathbf{v}$ is flux-preserving. In the case of the particle drift velocity the condition for flux-preservation reduces to $\nabla \times [\mathbf{H}(\mathbf{E}\cdot\mathbf{H})/H^2] = 0$.

Even if $\mathbf{v}_p$ is not flux-preserving, there may be some closed curves moving with velocity $\mathbf{v}_p$ which have constant flux. A semiexhaustive enumeration of such curves is given for a general electromagnetic field. Among these curves are those which lie in a surface everywhere perpendicular to $\mathbf{H}$, if this surface is independent of time. A family of such surfaces will exist if and only if $\mathbf{H}\cdot\nabla \times \mathbf{H}$ and $\mathbf{H} \times \dot{\mathbf{H}}$ both vanish identically.

A velocity may be line-preserving without being flux-preserving, but not vice versa. The necessary and sufficient condition for line-preservation is that $\mathbf{H} \times [\nabla \times (\mathbf{E} + \mathbf{v} \times \mathbf{H})]$ should vanish identically.

The motion of the lines of force in a plasma is related only to the transverse motion of the charged particles. The latter is separable from the longitudinal motion if and only if $\mathbf{v}_p$ is line-preserving. A necessary and sufficient condition is also given for the separability of only one component of the transverse motion.

The concept of a line of force is not relativistically covariant, because each point of a line of force has the same time coordinate. A curve in space-time

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[1] We are using a system of units in which $c = 1$, where c is the velocity of light.

which appears as a line of force in one frame of reference will therefore not be a line of force in another frame of reference. However, a moving line of force will trace out a two-dimensional surface in space-time, and it may be that this surface will intersect every space-like hyperplane in a line of force. In that case the surface will appear as the path of a moving line of force in every frame of reference, thus defining a moving line of force as a covariant concept. It is shown that a family of such surfaces exists if and only if $\mathbf{E}\cdot\mathbf{H}$ vanishes identically, in which case they will be generated by lines of force moving with the particle drift velocity $\mathbf{v}_p$.

I. INTRODUCTION

In general, a conducting fluid in a magnetic field will not move relative to the lines of force (*1, 2*). In the case of a magnetohydrodynamic wave, this implies that the lines of force oscillate like stretched strings, with a tension derived from the electromagnetic stress tensor, and with a mass per unit length derived from the density of the fluid, which is stuck to the lines (*1*). If the fluid is a plasma, and if the frequency of oscillation of the magnetic field is small compared to any of the particle gyrofrequencies, then the particles move with the drift velocity $\mathbf{v}_p = (\mathbf{E} \times \mathbf{H})/H^2$, so that the lines of force must also move with this velocity.

In some situations, however, the particles do not move with the field. We shall see later on that plasma oscillations, both of the electronic and of the ionic types, in an external magnetic field are examples of such situations. It is therefore of interest to investigate the general relationships which hold between the motions of the lines of force and of the particles.

A difficulty which arises immediately is that there does not seem to be any simple way to define the velocity of the magnetic field unambiguously. If the magnetic field had a definite velocity, it would be specified as follows: Let l be a line of force at time t, and l' the same line of force at time $t + dt$. Let $\mathbf{v}dt$ be the infinitesimal vector drawn from a point P on l to the line l'. Then $\mathbf{v}$ is the velocity of the magnetic field at the point P. This definition is ambiguous, however, since no method is given for identifying l and l' as positions of the same line of force for different times. Some other line of force l'' at time $t + dt$ could be chosen just as well as l', provided that l'' is only infinitesimally distant from l.

We therefore conclude that the velocity of the magnetic field is not a measurable quantity, so that it is, strictly speaking, meaningless to talk about it. We can, however, ask for the verifiable consequences of ascribing a certain velocity $\mathbf{v}$ to the magnetic lines of force. These include the following:

(1) The magnetic flux through any cycle is constant, where a cycle is defined as a closed curve, every point of which moves[2] with velocity $\mathbf{v}$.

(2) If a line l, moving with velocity $\mathbf{v}$, coincides with a line of force at some

[2] We will hereafter refer to a geometric figure, every point of which moves with velocity $\mathbf{v}$, as simply moving with velocity $\mathbf{v}$. It should, however, be remembered that $\mathbf{v}$ is a function of position, so that different points of the figure will have different velocities.

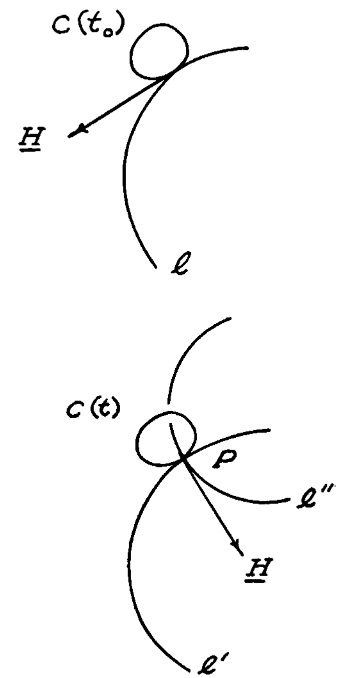

FIG. 1

instant t_0 , it will also coincide with some line of force at every subsequent time t. In other words, if l is initially tangent[3] to $\mathbf{H}$, it will, in the course of its motion, remain tangent to $\mathbf{H}$.

If (1) is satisfied, we say that the velocity $\mathbf{v}$ is flux-preserving, and if (2) is satisfied, that $\mathbf{v}$ is line-preserving. The flux-preserving property is the more fundamental one, since line-preservation is a consequence of flux-preservation, but not vice versa. To see that flux-preservation implies line-preservation, let us suppose that $\mathbf{v}$ is not line-preserving. Let l be a line of force at time t_0 . Ascribing the velocity $\mathbf{v}$ to each point of l, let l' be the line at time t resulting from the motion of l. The line l' will not, in general, be a line of force, since $\mathbf{v}$ is not line-preserving. Let P be a point on l' such that l' is not tangent to $\mathbf{H}$ at P, and let l'' be the line of force through P. Let C be an infinitely small cycle which, at time t, is a circle tangent to l' at P, lying in a plane which is perpendicular to the plane of l' and l''. (See Fig. 1.) Clearly, the flux through C cannot vanish at time

[3] When we say that a vector field such as $\mathbf{H}$ is tangent to a line or surface, we mean that it is tangent at every point of the line or surface, and at every instant of time, unless otherwise specified. A similar convention will hold for other properties and relations of vector fields, such as orthogonality and collinearity.

t, since the normal to the plane of C has a component along the line of force l''. On the other hand, since C is tangent to l' at time t, and since C and l' both move with velocity $\mathbf{v}$, C will be tangent to the line of force l at time t_0, so that the flux through C at time t_0 vanishes. Thus, $\mathbf{v}$ cannot be flux-preserving.

We shall see in Section III that all verifiable consequences of ascribing a velocity $\mathbf{v}$ to the lines of force follow from flux-preservation. Therefore, if $\mathbf{v}$ is flux-preserving, it is permissible to picture the lines of force as moving with velocity $\mathbf{v}$, since all the verifiable consequences of this picture will be true. In particular, if the particle drift velocity $\mathbf{v}_p$ is flux-preserving, it is permissible to say that the lines of force move with the particles. It should be borne in mind, however, that any other flux-preserving velocity could have been chosen for the lines of force without contradicting experimental results in any way.

Necessary and sufficient conditions are derived for flux-preservation and for line-preservation in Sections III and VI, respectively. We shall find that the condition for flux-preservation determines $\mathbf{v}$ completely, except for a boundary condition. Thus, if the symmetry of a problem imposes a natural boundary condition, the field velocity will be uniquely determined. In general, however, there will be no natural boundary condition, so that the assignment of a flux-preserving velocity to the magnetic field will be a matter of arbitrary choice.

One of the results of Section III is that the particle drift velocity is flux-preserving whenever $\mathbf{E}\cdot\mathbf{H}$ vanishes. This condition will almost always be approximately satisfied in a plasma, since an electric field component E_{long} parallel to $\mathbf{H}$ will tend to drive an electric current along $\mathbf{H}$ until the electrostatic field generated by the resulting charge separation cancels E_{long}. More precisely, if we assume infinite conductivity for the plasma, Ohm's law may be written:

$$\mathbf{E} + \mathbf{v} \times \mathbf{H} = 0, \tag{1}$$

in which case $\mathbf{E}$ is certainly perpendicular to $\mathbf{H}$.

However, it is not always correct to use Ohm's law, even with a finite conductivity, since the complete transport equation for the current density contains terms corresponding to physical effects, other than finite conductivity, which limit the current produced by an electric field, but which are not included in Ohm's law. When these effects cannot be neglected, it is possible for $\mathbf{E}$ to have a component along $\mathbf{H}$, and for the particle drift velocity not to be flux-preserving. For example, the current in an electron oscillation is limited by the inertia of the electrons rather than by finite conductivity, while in an ion oscillation it is limited by electron pressure gradients.

If the particle drift velocity is flux-preserving, we can use the constancy of flux through a particular P-cycle, i.e., a cycle with velocity $\mathbf{v}_p$, to yield information concerning the particle motion if the time variation of the magnetic field is known, and vice versa. If $\mathbf{v}_p$ is not flux-preserving, there will be P-cycles for which the magnetic flux is not constant, but there may also be some P-cycles for which it is constant. In certain cases, it will still be possible to use these constant flux P-cycles to make inferences of the type mentioned above, relating particle motions to the time variation of the magnetic field, even though it is no longer permissible to regard the lines of force as moving with velocity $\mathbf{v}_p$. It is therefore of interest to determine which P-cycles have constant flux in the general case where $\mathbf{v}_p$ may or may not be flux-preserving. In Section V, we derive a necessary and sufficient condition for an infinitely small plane cycle to have constant flux, and a sufficient condition for cycles of finite size.

In order to derive the results of Sections IV to VII, it is necessary to use the geometrical concept of the commutator product of two vector fields. This concept is defined in Section II, and its main properties explained.

II. THE COMMUTATOR PRODUCT OF TWO VECTOR FIELDS[4]

Consider two vector fields[5] $\tilde{A}$ and $\tilde{B}$ in an arbitrary n-dimensional space, with components A_i and B_i in some coordinate system. The commutator product of $\tilde{A}$ and $\tilde{B}$ is defined by[6]

$$[\tilde{A}, \tilde{B}]_i = B_j \frac{\partial A_i}{\partial x_j} - A_j \frac{\partial B_i}{\partial x_j}. \tag{2}$$

In Euclidean three-space this definition may be written

$$\begin{aligned}[\mathbf{A}, \mathbf{B}] &= (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \\ &= \nabla \times (\mathbf{A} \times \mathbf{B}) + \mathbf{B}(\nabla\cdot\mathbf{A}) - \mathbf{A}(\nabla\cdot\mathbf{B}).\end{aligned} \tag{2a}$$

If $\tilde{A}$ and $\tilde{B}$ are regarded as contravariant vectors, then $[\tilde{A}, \tilde{B}]$ will also transform as a contravariant vector. It is not necessary to use covariant derivatives, since we have, in the conventional tensor notation:

$$\begin{aligned}[\tilde{A}, \tilde{B}]^i &= B^j \frac{DA^i}{Dx^j} - A^j \frac{DB^i}{Dx^j} \\ &= B^j \frac{\partial A^i}{\partial x^j} + B^j \Gamma_{jk}{}^i A^k - A^j \frac{\partial B^i}{\partial x^j} - A^j \Gamma_{jk}{}^i B^k \\ &= B^j \frac{\partial A^i}{\partial x^j} - A^j \frac{\partial B^i}{\partial x^j}.\end{aligned} \tag{3}$$

The terms involving the Christoffel symbol $\Gamma_{jk}{}^i$ cancel because of the symmetry relation $\Gamma_{jk}{}^i = \Gamma_{kj}{}^i$.

[4] The material in this section and in the appendix is a simplified and unrigorous version of a part of Chapter III of ref. *3*.

[5] We will designate vector fields in conventional three-space by **A**, **B**, etc., and in arbitrary spaces by $\tilde{A}$, $\tilde{B}$, etc.

[6] We shall consistently use the summation convention for repeated indices.

It is easy to verify that the commutator product has the following algebraic properties:

$$[\tilde{A}, \tilde{B}] = -[\tilde{B}, \tilde{A}] \tag{4}$$

$$[[\tilde{A}, \tilde{B}], \tilde{C}] + [[\tilde{B}, \tilde{C}], \tilde{A}] + [[\tilde{C}, \tilde{A}], \tilde{B}] = 0. \tag{5}$$

Equation (5) is called the Jacobi identity. It should be observed that the associative law does not hold for the commutator product, i.e.,

$$[[\tilde{A}, \tilde{B}], \tilde{C}] \neq [\tilde{A}, [\tilde{B}, \tilde{C}]]. \tag{6}$$

Every vector field $\tilde{A}$ determines a family of integral curves or $\tilde{A}$-lines, as we shall call them, i.e., curves everywhere tangent to $\tilde{A}$. Through every point there will pass exactly one such curve, except possibly for singular points where $\tilde{A} = 0$. If we have two linearly independent vector fields $\tilde{A}$ and $\tilde{B}$, we may now ask whether there exists a family of integral surfaces, i.e., surfaces tangent to both $\tilde{A}$ and $\tilde{B}$. This will not always be the case, as is shown by the following example in three-space:

$$\mathbf{A} = \mathbf{i}_z , \tag{7a}$$

$$\mathbf{B} = \mathbf{i}_x \cos z + \mathbf{i}_y \sin z, \tag{7b}$$

where $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ are unit vectors in the x, y, and z directions, respectively.

If, in this example, **A** and **B** determined an integral surface Σ, the line of intersection of Σ with any plane parallel to the z axis would be tangent to **A**, and would therefore be a line parallel to the z axis. The surface Σ would then be generated by a family of lines parallel to the z axis, and its line of intersection with the plane $z = z_0$ would be independent of z_0. But since this line of intersection must be tangent to **B**, it will have an equation of the form $y = C + x \tan z_0$, which is not independent of z_0. Since we are led to a contradiction, there can be no integral surface.

To derive a general condition for the existence of an integral surface, we examine the path shown in Fig. 2. The curves joining 1 to 2 and 3 to 4 are segments of $\tilde{A}$-lines, while the curves joining 2 to 3 and 4 to 5 are segments of $\tilde{B}$-lines. These segments are chosen in such a way as to satisfy the relations

$$x_i(2) - x_i(1) = \tfrac{1}{2}[A_i(1) + A_i(2)]\delta, \tag{8a}$$

$$x_i(3) - x_i(2) = \tfrac{1}{2}[B_i(2) + B_i(3)]\epsilon, \tag{8b}$$

$$x_i(4) - x_i(3) = -\tfrac{1}{2}[A_i(3) + A_i(4)]\delta, \tag{8c}$$

$$x_i(5) - x_i(4) = -\tfrac{1}{2}[B_i(4) + B_i(5)]\epsilon, \tag{8d}$$

to second order in δ and ϵ, where δ and ϵ are infinitesimals.

Clearly, $\tilde{A}$ and $\tilde{B}$ will determine an integral surface only if the vector from

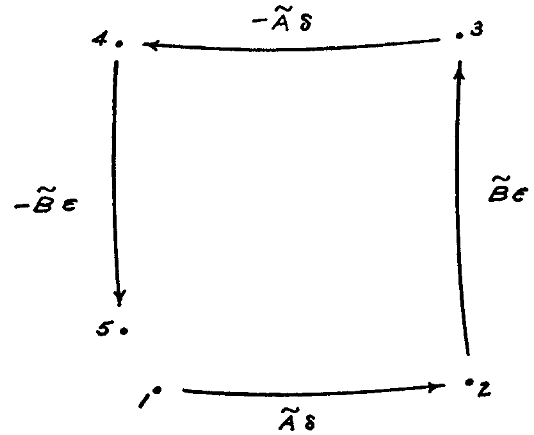

FIG. 2

1 to 5 is in the plane determined by $\tilde{A}(1)$ and $\tilde{B}(1)$, i.e., only if it is a linear combination of $\tilde{A}(1)$ and $\tilde{B}(1)$[7]. The components of this vector are, of course, infinitesimals of second order. Using Eqs. (8), they are given by

$$x_i(5) - x_i(1) = -\frac{\delta}{2}[A_i(3) - A_i(2) + A_i(4) - A_i(1)] + \frac{\epsilon}{2}[B_i(2) - B_i(5) + B_i(3) - B_i(4)]. \tag{9}$$

Since $x_i(5) - x_i(1)$ is of second order, we may replace $A_i(1)$ and $B_i(5)$ by $A_i(5)$ and $B_i(1)$, respectively. Using the relations

$$A_i(3) - A_i(2) = A_i(4) - A_i(5) = \epsilon B_j \frac{\partial A_i}{\partial x_j}, \tag{10a}$$

and

$$B_i(2) - B_i(1) = B_i(3) - B_i(4) = \delta A_j \frac{\partial B_i}{\partial x_j}, \tag{10b}$$

Eq. (9) reduces to

$$x_i(5) - x_i(1) = \epsilon\delta\left[A_j \frac{\partial B_i}{\partial x_j} - B_j \frac{\partial A_i}{\partial x_j}\right] = -\epsilon\delta[\tilde{A}, \tilde{B}]_i . \tag{11}$$

This proves the "only if" part of the following theorem, a more complete proof of which is given in the appendix: Two linearly independent vector fields $\tilde{A}$ and $\tilde{B}$

[7] The converse of this statement is also true, although not obvious.

will generate a family of integral surfaces if and only if the commutator product $[\tilde{A}, \tilde{B}]$ is a linear combination of $\tilde{A}$ and $\tilde{B}$.

The proof of this theorem may easily be modified to yield a proof of the following more general theorem by induction on m: A linearly independent set of m vector fields $\tilde{A}^1, \cdots \tilde{A}^m$ will generate a family of integral m-manifolds if and only if $[\tilde{A}^\alpha, \tilde{A}^\beta]$ is a linear combination of $\tilde{A}^1 \cdots \tilde{A}^m$ for every α and β.

If $\tilde{A}$ and $\tilde{B}$ are collinear, there will certainly exist a family of surfaces tangent to both $\tilde{A}$ and $\tilde{B}$, i.e., any family of surfaces generated by $\tilde{A}$-lines. This family will, of course, be far from unique. Also $[\tilde{A}, \tilde{B}]$ will be collinear with $\tilde{A}$, therefore a linear combination of $\tilde{A}$ and $\tilde{B}$. To see this, we let $\tilde{B} = f\tilde{A}$, where f is a scalar function of position, and write

$$\begin{aligned}[\tilde{A}, \tilde{B}]_i &= fA_j \frac{\partial A_i}{\partial x_j} - A_j \frac{\partial}{\partial x_j}(fA_i) \\ &= -A_i A_j \frac{\partial f}{\partial x_j},\end{aligned} \tag{12}$$

which is collinear with $\tilde{A}$. We may therefore conclude that there exists a family of surfaces tangent to both $\tilde{A}$ and $\tilde{B}$ if and only if $[\tilde{A}, \tilde{B}]$ is a linear combination of $\tilde{A}$ and $\tilde{B}$, whether or not $\tilde{A}$ and $\tilde{B}$ are collinear. However, these surfaces are called integral surfaces only when $\tilde{A}$ and $\tilde{B}$ are linearly independent.

It is interesting to note that the properties of vector fields here discussed are nonmetric, since neither the definition of the commutator product nor the proof of the theorem on existence of integral surfaces made any reference to a metric. In fact, the concepts involved will be applied in Section VI to a space in which no metric has been defined.

III. FLUX-PRESERVATION

Let us assign a velocity vector $\mathbf{v}$ to every point of a magnetic field $\mathbf{H}$, requiring that $\mathbf{v}$ be perpendicular to $\mathbf{H}$. To derive the condition for $\mathbf{v}$ to be flux-preserving, we compute the time rate of change of the flux Φ through an arbitrary cycle C. This quantity consists of two terms, a surface integral arising from the time variation of $\mathbf{H}$ at a fixed point, and a line integral arising from the motion of the boundary C. Thus we find that

$$\begin{aligned}\frac{d\Phi}{dt} &= \int \dot{\mathbf{H}}\cdot\mathbf{n}\, dA + \oint \mathbf{H}\cdot(\mathbf{v}\times d\mathbf{l}) \\ &= -\int [\nabla\times\mathbf{E}]\cdot\mathbf{n}\, dA - \oint (\mathbf{v}\times\mathbf{H})\cdot d\mathbf{l} \\ &= -\int [\nabla\times(\mathbf{E}+\mathbf{v}\times\mathbf{H})]\cdot\mathbf{n}\, dA,\end{aligned} \tag{13}$$

which vanishes for every cycle if and only if $\nabla \times (\mathbf{E} + \mathbf{v} \times \mathbf{H})$ vanishes identically. This is the necessary and sufficient condition for flux preservation. It may also be expressed in the form

$$\mathbf{E} + \mathbf{v} \times \mathbf{H} = -\nabla\phi, \tag{14}$$

where ϕ is some scalar field.

Taking the cross product of $\mathbf{H}$ with Eq. (14), and using the fact that $\mathbf{v}\cdot\mathbf{H} = 0$, we find that

$$\mathbf{v} = \frac{(\mathbf{E} + \nabla\phi)\times\mathbf{H}}{H^2}. \tag{15}$$

The scalar product of $\mathbf{H}$ with Eq. (14) yields

$$\mathbf{H}\cdot\mathbf{E} = -\mathbf{H}\cdot\nabla\phi. \tag{16}$$

If we specify as a boundary condition the value of ϕ at every point of some surface nowhere tangent to $\mathbf{H}$, Eq. (16) will determine a unique solution for ϕ at all points. The corresponding $\mathbf{v}$, given by Eq. (15), will then be a flux-preserving velocity. Thus, we have proved that for every electromagnetic field there exists a flux-preserving velocity, and that this velocity is determined only to within a scalar function defined on a surface. In general, therefore, there will be many flux-preserving velocities for a given electromagnetic field, any one of which may be ascribed to the lines of force without contradicting experimental results.

In certain problems, we can impose a natural boundary condition which, in conjunction with Eqs. (15) and (16), will determine a unique flux-preserving velocity. For example, let $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_1$, where $\mathbf{H}_0$ is a uniform field, and where $\mathbf{E}$ and $\mathbf{H}_1$ are the electric and magnetic fields generated by a plane wave $\exp i(\mathbf{K}\cdot\mathbf{x} - \omega t)$ of small amplitude with $\mathbf{K}$ not perpendicular to $\mathbf{H}_0$. This wave can be a plasma oscillation, a magnetohydrodynamic wave, or any other type of wave with an associated electromagnetic field. In this case, we can specify a unique flux-preserving velocity $\mathbf{v}_F$ by requiring it to be proportional to $\exp i(\mathbf{K}\cdot\mathbf{x} - \omega t)$, working to lowest order in the amplitude. The condition for $\mathbf{v}_F$ to be flux-preserving is

$$\nabla \times (\mathbf{E} + \mathbf{v}_F \times \mathbf{H}) \cong i\mathbf{K} \times (\mathbf{E} + \mathbf{v}_F \times \mathbf{H}_0) = 0. \tag{17}$$

Taking the cross product of $\mathbf{H}_0$ with Eq. (17), and using the fact that $\mathbf{v}_F\cdot\mathbf{H}_0$ vanishes to lowest order, we can easily solve for $\mathbf{v}_F$ to obtain

$$\mathbf{v}_F = \frac{(\mathbf{E}\times\mathbf{K})_\perp}{\mathbf{H}_0\cdot\mathbf{K}}, \tag{18}$$

where $(\mathbf{E} \times \mathbf{K})_\perp$ is the component of $(\mathbf{E} \times \mathbf{K})$ perpendicular to $\mathbf{H}_0$. The restriction that **K** should not be perpendicular to $\mathbf{H}_0$ is necessary to ensure that $\mathbf{H}_0 \cdot \mathbf{K}$ will not vanish. For the special case where $\mathbf{H}_0$ and **K** are perpendicular, Eq. (18) is indeterminate and there may be many flux-preserving velocities.

We shall now justify our claim that any flux-preserving velocity may be assigned to the lines of force, i.e., that every verifiable consequence of picturing the lines of force as moving with velocity **v** is also a consequence of the statement that **v** is flux-preserving. The lines-of-force picture has verifiable consequences only because it is ordinarily connected with observables (the electric and magnetic field vectors) in the following way: The density of magnetic lines and the number of lines cut by a moving circuit are computed on the assumption that the lines move with velocity **v** and are never created or destroyed. The magnetic field intensity is then identified with the density of lines, and the emf around a moving circuit with the negative of the number of lines cut by that circuit per unit time. The verifiable content of this picture is therefore given by the following statement, which will be designated as VC. There exists a family of lines l, moving with velocity **v**, with the following properties:

(1) Through every point of space passes exactly one of the lines l.
(2) The lines l remain tangent to **H** in the course of their motion.
(3) The density of lines l is equal to the intensity of the magnetic field.
(4) The emf around a closed curve moving in an arbitrary manner is equal to minus the total number of lines l cut by the circuit per unit time.

We must show that VC is equivalent to the statement that **v** is flux-preserving. Suppose first that VC is true, and let C be a cycle. Since C and the lines l both move with velocity **v**, the number of lines l linking C is constant, and is equal to the magnetic flux through C, according to VC2 and 3. Thus, VC implies flux-preservation. Now suppose that **v** is flux-preserving, and consider the family of lines l, moving with velocity **v**, which coincide at some particular instant t_0 with the lines of force. This family of lines clearly satisfies VC1. As pointed out in the introduction, **v** must be line-preserving if it is flux-preserving, so that VC2 is satisfied, and VC3 is clearly implied by flux-preservation. According to VC4, the emf about a closed loop C, moving with velocity $\mathbf{v}_c$, is given by

$$\mathcal{E} = -\oint_C \mathbf{H} \cdot [(\mathbf{v}_c - \mathbf{v}) \times d\mathbf{l}], \tag{19}$$

since $(\mathbf{v}_c - \mathbf{v})$ is the relative velocity of C with respect to the lines of force. To show that VC4 follows from flux-preservation, we write

$$\begin{aligned} \mathcal{E} &= \oint_C (\mathbf{E} + \mathbf{v}_c \times \mathbf{H}) \cdot d\mathbf{l} \\ &= \oint_C (\mathbf{E} + \mathbf{v} \times \mathbf{H}) \cdot d\mathbf{l} - \oint_C \mathbf{H} \cdot [(\mathbf{v}_c - \mathbf{v}) \times d\mathbf{l}]. \end{aligned} \tag{20}$$

If **v** is flux-preserving, the first term of the right-hand side vanishes, so that Eqs. (19) and (20) are identical. This completes the proof that VC, the verifiable content of the lines-of-force picture, is equivalent to the statement that **v** is flux-preserving[8]. Hence, if **v** is known to be flux-preserving, it will be permissible to picture the lines of force as moving with velocity **v**, since this assumption will never lead to any consequences contradicting experimental results.

For a given electromagnetic field, the above argument applies equally well to any flux-preserving velocity. It is therefore impossible to distinguish experimentally between two different flux-preserving velocities, since they will impose the same restrictions on the electric and magnetic fields. The choice between such velocities is a matter of arbitrary convention, although it will sometimes be convenient to make this choice in such a way as to satisfy some boundary condition.

We might inquire whether it is feasible to specify the velocity of the magnetic field more closely by modifying VC in such a way as to impose stronger restrictions on the electric and magnetic fields. In view of the following considerations, this does not seem likely. Considering first the magnetic field, we may write the condition for flux-preservation in the form[9]

$$\dot{\mathbf{H}} = \nabla \times (\mathbf{v} \times \mathbf{H}), \tag{21}$$

since $\dot{\mathbf{H}} = -\nabla \times \mathbf{E}$. Equation (21) is linear and homogeneous in **H**, and of first order in t; it therefore determines **H** completely if the initial values are known. Thus, the statement that a given vector function **v** of space and time is a flux-preserving velocity already determines the magnetic field completely, except for an initial condition, and it is hard to see how any additional assumption concerning the motion of the lines of force could restrict the initial values of the magnetic field.

The electric field is restricted by flux-preservation to the extent that its curl is determined, as is easily seen from VC4 for closed curves which are fixed in position. The electric field itself is therefore determined modulo the gradient of an arbitrary scalar. To determine **E** more closely, we might strengthen VC4 to the following:

VC4′. The emf across an element of arc $d\mathbf{l}$, moving with arbitrary velocity $\mathbf{v}_c$, is given by the negative of the number of lines of force cut by $d\mathbf{l}$ per unit time.

If this is correct, we have

$$\mathcal{E} = [\mathbf{E} + \mathbf{v}_c \times \mathbf{H}] \cdot d\mathbf{l} = -\mathbf{H} \cdot [(\mathbf{v}_c - \mathbf{v}) \times d\mathbf{l}], \tag{22}$$

or

$$\mathbf{E} + \mathbf{v} \times \mathbf{H} = 0, \tag{23}$$

which can be satisfied if and only if **E** is perpendicular to **H**. In that case, Eq.

[8] It also proves that VC4 is not independent of VC1-3, since the latter imply flux-preservation, which in turn implies VC4.

[9] Equation (21) is also derived by Cowling in ref. *4*.

(23) implies that

$$\mathbf{v} = \frac{\mathbf{E} \times \mathbf{H}}{H^2} = \mathbf{v}_p, \tag{24}$$

since $\mathbf{v} \cdot \mathbf{H}$ vanishes. Since it is possible to assign a velocity to the magnetic field in such a way as to satisfy VC4′ only in the speical case where $\mathbf{E} \cdot \mathbf{H}$ vanishes, it seems preferable to keep the weaker assumption VC4. We conclude that there is probably no simple way to strengthen the verifiable content of the lines-of-force picture, since **E** and **H** already seem to be determined by VC (i.e., by flux-preservation) as closely as they can be determined by any reasonable assumptions concerning the motion of lines of force.

The guiding center of a charged particle in an electromagnetic field will move with a velocity component perpendicular to **H** of $\mathbf{v}_p$, given by Eq. (24), provided that the magnetic moment of the particle is small enough to make the drifts due to inhomogeneities in the magnetic field negligible. The velocity $\mathbf{v}_p$ at a point P may also be defined as the velocity of a Lorentz frame in which the electric field at P is parallel to **H**, neglecting v_p^2 and v_pE/H in comparison to one. Since a Lorentz frame may be given an additional velocity parallel to the magnetic field without changing the electric field, this definition contains an ambiguity, which we remove by specifying that $\mathbf{H} \cdot \mathbf{v}_p = 0$.

When $\mathbf{v}_p$ is flux-preserving, it is permissible to picture the lines of force as moving with the particles. Clearly, a necessary and sufficient condition for this is:

$$\nabla \times (\mathbf{E} + \mathbf{v}_p \times \mathbf{H}) = \nabla \times [\mathbf{H}(\mathbf{E} \cdot \mathbf{H})/H^2] = 0. \tag{25}$$

In particular, this condition will hold whenever **E** is perpendicular to **H**.

In the case of the plane wave discussed above, Eq. (25) reduces to

$$i(\mathbf{E} \cdot \mathbf{H}_0)(\mathbf{K} \times \mathbf{H}_0)/H_0^2 = 0, \tag{26}$$

which holds if and only if one of the two following conditions is satisfied: (1) **E** is perpendicular to $\mathbf{H}_0$, or (2) the wave is travelling along the magnetic field. Under these conditions, Eq. (18) for the unique flux-preserving velocity reduces to $\mathbf{v}_F = \mathbf{v}_p$. For a plasma oscillation travelling at an angle to the magnetic field, **E** and $\mathbf{H}_0$ are not perpendicular (*5, 6*), so that $\mathbf{v}_F$ is necessarily different from $\mathbf{v}_p$. In this case, therefore,we cannot consider the field as moving with the particles.

IV. *P*-CYCLES LYING IN ORTHOGONAL SURFACES

Let us consider an electromagnetic field for which the particle drift velocity is not flux-preserving. In this case, we must not picture the lines of force as fixed to the particles. (If we wish to treat the magnetic field as a moving entity, we must use some flux-preserving velocity $\mathbf{v}_F$, different from $\mathbf{v}_p$.)

Since $\mathbf{v}_p$ is not flux-preserving, there will exist P-cycles through which the flux is not constant. However, there may still be some P-cycles having constant flux, and it is of interest to determine which P-cycles have this property, since these special P-cycles can be used in certain cases to relate the particle motions to the time variation of the magnetic field, just as any P-cycle can be used in the special case where $\mathbf{v}_p$ is flux-preserving.

The P-cycles which remain perpendicular to **H** in the course of their motion constitute a special class of constant flux P-cycles. To see this, we write

$$\begin{aligned} \frac{d\Phi}{dt} &= -\int [\nabla \times (\mathbf{E} + \mathbf{v}_p \times \mathbf{H})] \cdot \mathbf{n}\, dA \\ &= -\int \{\nabla \times [\mathbf{H}(\mathbf{E} \cdot \mathbf{H})/H^2]\} \cdot \mathbf{n}\, dA \\ &= -\oint \frac{(\mathbf{E} \cdot \mathbf{H})(\mathbf{H} \cdot d\mathbf{l})}{H^2}, \end{aligned} \tag{27}$$

which vanishes whenever $\mathbf{H} \cdot d\mathbf{l} = 0$. If there exists a family of orthogonal surfaces, i.e., of surfaces perpendicular to **H**, then $d\Phi/dt$ will vanish at time t_0 for any P-cycle which lies in such a surface at time t_0. We shall now prove that a family of orthogonal surfaces will exist at a particular instant for a nowhere vanishing magnetic field **H** if and only if $\mathbf{H} \cdot \nabla \times \mathbf{H}$ vanishes at that instant.

Let **A** and **B** be linearly independent vector fields orthogonal to **H**. The orthogonality conditions are

$$H_iA_i = H_iB_i = 0, \tag{28}$$

which, when differentiated with respect to x_j, give

$$H_i \frac{\partial A_i}{\partial x_j} + \frac{\partial H_i}{\partial x_j} A_i = 0, \tag{29a}$$

$$H_i \frac{\partial B_i}{\partial x_j} + \frac{\partial H_i}{\partial x_j} B_i = 0. \tag{29b}$$

Using Eqs. (29), we find that

$$\begin{aligned} \mathbf{H} \cdot [\mathbf{A}, \mathbf{B}] &= H_iB_j \frac{\partial A_i}{\partial x_j} - H_iA_j \frac{\partial B_i}{\partial x_j} \\ &= \frac{\partial H_i}{\partial x_j} (A_jB_i - A_iB_j) \\ &= (\mathbf{A} \times \mathbf{B}) \cdot (\nabla \times \mathbf{H}). \end{aligned} \tag{30}$$

Since **H** and $(\mathbf{A} \times \mathbf{B})$ are parallel, this expression will vanish if and only if

$\mathbf{H}\cdot\nabla\times\mathbf{H}$ vanishes. But the vanishing of $\mathbf{H}\cdot[\mathbf{A},\mathbf{B}]$ is equivalent to the linear dependence of $[\mathbf{A},\mathbf{B}]$ on $\mathbf{A}$ and $\mathbf{B}$, which is in turn equivalent to the existence of a family of integral surfaces of $\mathbf{A}$ and $\mathbf{B}$. These surfaces will be perpendicular to $\mathbf{H}$.

So far, we have proved only that $d\Phi/dt$ vanishes instantaneously for a P-cycle lying in an orthogonal surface. It may be, however, that such a P-cycle will not remain perpendicular to $\mathbf{H}$ in the course of its motion. In order to show that $d\Phi/dt$ vanishes identically, we also need the following result: If $\mathbf{H}\cdot\nabla\times\mathbf{H}$ vanishes identically, (i.e., if, at each instant of time, there exists a family of surfaces perpendicular to $\mathbf{H}$) then a necessary and sufficient condition that any P-cycle initially perpendicular to $\mathbf{H}$ will remain perpendicular to $\mathbf{H}$ in the course of its motion is that $\mathbf{H}\times\dot{\mathbf{H}}$ should vanish identically.

To prove sufficiency, let $d\mathbf{l} = \mathbf{r}_2 - \mathbf{r}_1$ be an element of arc belonging to a P-cycle which is initially perpendicular to $\mathbf{H}$. Because of the relations

$$\frac{d\mathbf{H}}{dt} = \dot{\mathbf{H}} + (\mathbf{v}_p\cdot\nabla)\mathbf{H}, \tag{31a}$$

and

$$\frac{d}{dt}(d\mathbf{l}) = \frac{d\mathbf{r}_2}{dt} - \frac{d\mathbf{r}_1}{dt} = \mathbf{v}_{p_2} - \mathbf{v}_{p_1} = (d\mathbf{l}\cdot\nabla)\mathbf{v}_p, \tag{31b}$$

we may write

$$\begin{aligned}\frac{d}{dt}(\mathbf{H}\cdot d\mathbf{l}) &= \frac{d\mathbf{H}}{dt}\cdot d\mathbf{l} + \mathbf{H}\cdot\frac{d}{dt}(d\mathbf{l})\\ &= \left[\dot{H}_i + v_{p_j}\frac{\partial H_i}{\partial x_j} + H_j\frac{\partial v_{p_j}}{\partial x_i}\right]dl_i\\ &= \left[\dot{H}_i + v_{p_j}\left(\frac{\partial H_i}{\partial x_j} - \frac{\partial H_j}{\partial x_i}\right) + \frac{\partial}{\partial x_i}(v_{p_j}H_j)\right]dl_i\\ &= \left[\dot{H}_i + v_{p_j}\left(\frac{\partial H_i}{\partial x_j} - \frac{\partial H_j}{\partial x_i}\right)\right]dl_i,\end{aligned} \tag{32a}$$

using the fact that $\mathbf{v}_p$ is perpendicular to $\mathbf{H}$. Equation (32a) may be rewritten in vector notation as

$$\frac{d}{dt}(\mathbf{H}\cdot d\mathbf{l}) = [\dot{\mathbf{H}} - \mathbf{v}_p\times(\nabla\times\mathbf{H})]\cdot d\mathbf{l}. \tag{32b}$$

Since $\dot{\mathbf{H}}$ is parallel to $\mathbf{H}$, while $\mathbf{v}_p$ and $\nabla\times\mathbf{H}$ are both perpendicular to $\mathbf{H}$, and since $\mathbf{H}$ never vanishes, Eq. (32b) reduces to

$$\frac{d}{dt}(\mathbf{H}\cdot d\mathbf{l}) = f\,\mathbf{H}\cdot d\mathbf{l}, \tag{33}$$

where f is a scalar function of position. Equation (33), in conjunction with the fact that $\mathbf{H}\cdot d\mathbf{l}$ vanishes initially, implies that $\mathbf{H}\cdot d\mathbf{l}$ will vanish at any subsequent time. Thus, if $d\mathbf{l}$ is initially perpendicular to $\mathbf{H}$, it will remain so. This proves sufficiency.

To prove necessity, we observe that since $(d/dt)(\mathbf{H}\cdot d\mathbf{l})$ must vanish for *every* $d\mathbf{l}$ perpendicular to $\mathbf{H}$, Eq. (32b), which may be derived as before, implies that $\dot{\mathbf{H}} - \mathbf{v}_p\times(\nabla\times\mathbf{H})$ is parallel to $\mathbf{H}$. But since $\mathbf{v}_p\times(\nabla\times\mathbf{H})$ is itself parallel to $\mathbf{H}$, so is $\dot{\mathbf{H}}$. Thus we have $\mathbf{H}\times\dot{\mathbf{H}} = 0$, which completes the proof of necessity.

The condition $\mathbf{H}\times\dot{\mathbf{H}} = 0$ means that the direction of $\mathbf{H}$ at any point is independent of time. This will be the case if and only if the family of orthogonal surfaces is also independent of time.

The condition that $\mathbf{H}\cdot\nabla\times\mathbf{H}$ must vanish at all times may be weakened to the vanishing of $\mathbf{H}\cdot\nabla\times\mathbf{H}$ at one particular instant. If $\mathbf{H}\cdot\nabla\times\mathbf{H}$ vanishes at some instant t_0, there will be a family of orthogonal surfaces at time t_0. But since the direction of $\mathbf{H}$ is constant, these surfaces will remain orthogonal to $\mathbf{H}$, so that $\mathbf{H}\cdot\nabla\times\mathbf{H}$ must remain equal to zero.

Examples of magnetic fields which do not have a family of orthogonal surfaces are:

(a) The magnetic field of a circularly polarized plane electromagnetic wave in free space. At any given instant, this field has the form

$$\mathbf{H} = \mathbf{i}_x\cos z + \mathbf{i}_y\sin z. \tag{34}$$

(b) The total field inside an infinitely long cylindrical wire placed within a uniform external field with its axis parallel to the field. If the current density is uniform throughout the wire, the total field is of the form

$$\mathbf{H} = -\mathbf{i}_x y + \mathbf{i}_y x + \mathbf{i}_z\,. \tag{35}$$

Examples of fields which have a time-independent family of orthogonal surfaces are:

(c) The magnetic field of a plane polarized electromagnetic wave in free space.

(d) The magnetic field produced by an ac circuit, provided that the wavelength is large compared with the dimensions of the circuit. Under these circumstances, the displacement current is negligible, and the magnetic field at a given time t may be approximated by the static field which would be produced if the current were to remain fixed at the value which it has at time t. In that case, $\nabla\times\mathbf{H}$ will vanish in the space external to the circuit, and $\dot{\mathbf{H}}$ will not only be parallel to $\mathbf{H}$, but even proportional to it in magnitude, with a proportionality factor which depends on the time but is independent of position.

(e) $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_1$, where $\mathbf{H}_0$ satisfies $\mathbf{H}_0\cdot(\nabla\times\mathbf{H}_0) = 0$ and is independent of time, and where $\mathbf{H}_1$ is a small oscillatory disturbance. The orthogonal surfaces of $\mathbf{H}_0$ are approximately perpendicular to $\mathbf{H}$, even though $\mathbf{H}$ itself may not have a time-independent family of orthogonal surfaces. If both **E** and $\mathbf{H}_1$ are regarded as infinitesimals of the first order, $d\Phi/dt$ for a P-cycle lying in an orthogonal surface of $\mathbf{H}_0$ will be an infinitesimal of the second order. The orthogonal surfaces of $\mathbf{H}_0$ may therefore be used as orthogonal surfaces for **H** to first order in the amplitude of the oscillation. This situation arises in the case of the plane wave discussed in Section III, where the orthogonal surfaces of $\mathbf{H}_0$ are simply planes perpendicular to $\mathbf{H}_0$.

Finally, we consider an electromagnetic field which has a time-independent family of orthogonal surfaces, but for which $\mathbf{v}_p$ is not flux-preserving.

(f) A uniform electric field **E** drives a current through an infinitely long straight wire. As in example (b), the magnetic field consists of a uniform external field parallel to the wire, in addition to the field produced by the current in the wire. Introducing cylindrical coordinates (r, θ, z) with the z axis along the wire, the fields outside the wire may be written in the form

$$\mathbf{H} = \mathbf{i}_\theta/r + \mathbf{i}_z , \tag{36}$$

$$\mathbf{E} = \mathbf{i}_z . \tag{37}$$

The magnetic field lines are helices with axis along the wire. Defining the pitch of a helix as the distance parallel to the axis between adjacent turns, the pitch of a field line is equal to $2\pi r^2$, where r is the radius of the cylinder determined by the field line. Since $\nabla \times \mathbf{H}$ and $\dot{\mathbf{H}}$ both vanish outside the wire, there exists a time-independent family of orthogonal surfaces. It is easy to verify that the orthogonal surfaces are given by equations of the form

$$\theta = -z + \text{constant}. \tag{38}$$

Thus, an orthogonal surface is generated by a semi-infinite straight line perpendicular to the z axis, the motion of which consists of a uniform translation in the z direction combined with a uniform rotation about the z axis.

Since $\nabla \times [\mathbf{H}(\mathbf{E}\cdot\mathbf{H})/H^2]$ has components in the θ and z directions, $\mathbf{v}_p$ will not be flux-preserving. To see this directly, we consider a P-cycle C which does not link the wire, and which is everywhere perpendicular to the z axis. Since $H_z = 1$, the flux through C is equal to its area. The particle drift velocity $\mathbf{v}_p$ is directed radially inward, so that C is pulled in toward the wire. This decreases the area of C, and hence also the flux through C. On the other hand, a P-cycle lying in an orthogonal surface and not linking the wire will be twisted around so as to remain perpendicular to **H** as it moves toward the wire, so that the decrease in area is compensated by the increase in field intensity.

V. GENERAL CHARACTERIZATION OF CONSTANT FLUX CYCLES

In the preceding section we examined a special class of constant flux cycles, i.e., the P-cycles lying in orthogonal surfaces. We shall now attempt to give as complete a characterization as possible of the constant flux cycles in an arbitrary magnetic field with an arbitrary velocity **v** assigned to the cycles. In particular, we shall derive a necessary and sufficient condition for a plane cycle of infinitesimal size, hereafter referred to as an infinitesimal cycle, to have constant flux, and a sufficient condition for cycles of finite size. For this purpose, we define a sequence of derived magnetic fields as follows:

$$\mathbf{H}^{(0)} = \mathbf{H}, \tag{39a}$$

$$\mathbf{H}^{(\alpha+1)} = \dot{\mathbf{H}}^{(\alpha)} + \nabla \times (\mathbf{H}^{(\alpha)} \times \mathbf{v}). \tag{39b}$$

Clearly, each $\mathbf{H}^{(\alpha)}$ satisfies

$$\nabla\cdot\mathbf{H}^{(\alpha)} = 0. \tag{40}$$

We shall make frequent use of the following result, which will be referred to as the collinearity theorem. For any $\alpha \geqq 0$, if $\mathbf{H}^{(\alpha)}$ and $\mathbf{H}^{(\alpha+1)}$ are collinear, then $\mathbf{H}^{(\beta)}$ is collinear with $\mathbf{H}^{(\alpha)}$ for any $\beta \geqq \alpha$. To prove this by induction on β, let $\mathbf{H}^{(\alpha+1)} = f\mathbf{H}^{(\alpha)}$, where f is a scalar function of space and time, and assume that $\mathbf{H}^{(\beta)} = f_\beta \mathbf{H}^{(\alpha)}$. Then,

$$\nabla\cdot\mathbf{H}^{(\beta)} = \nabla\cdot(f_\beta\mathbf{H}^{(\alpha)}) = f_\beta\nabla\cdot\mathbf{H}^{(\alpha)} + \nabla f_\beta\cdot\mathbf{H}^{(\alpha)}. \tag{41}$$

Since $\nabla\cdot\mathbf{H}^{(\alpha)}$ and $\nabla\cdot\mathbf{H}^{(\beta)}$ both vanish, we have

$$\nabla f_\beta\cdot\mathbf{H}^{(\alpha)} = 0. \tag{42}$$

We now show that $\mathbf{H}^{(\beta+1)}$ is collinear with $\mathbf{H}^{(\alpha)}$ if $\mathbf{H}^{(\beta)}$ is, thus completing the proof by induction:

$$\begin{aligned} \mathbf{H}^{(\beta+1)} &= \dot{\mathbf{H}}^{(\beta)} + \nabla \times (\mathbf{H}^{(\beta)} \times \mathbf{v}) \\ &= \frac{\partial}{\partial t}(f_\beta\mathbf{H}^{(\alpha)}) + \nabla \times (f_\beta\mathbf{H}^{(\alpha)} \times \mathbf{v}) \\ &= f_\beta\dot{\mathbf{H}}^{(\alpha)} + \dot{f}_\beta\mathbf{H}^{(\alpha)} + f_\beta\nabla \times (\mathbf{H}^{(\alpha)} \times \mathbf{v}) + \nabla f_\beta \times (\mathbf{H}^{(\alpha)} \times \mathbf{v}) \\ &= f_\beta\mathbf{H}^{(\alpha+1)} + \dot{f}_\beta\mathbf{H}^{(\alpha)} + \mathbf{H}^{(\alpha)}(\nabla f_\beta\cdot\mathbf{v}) - \mathbf{v}(\nabla f_\beta\cdot\mathbf{H}^{(\alpha)}) \\ &= (f_\beta f + \dot{f}_\beta + \mathbf{v}\cdot\nabla f_\beta)\mathbf{H}^{(\alpha)} = f_{\beta+1}\mathbf{H}^{(\alpha)}. \end{aligned} \tag{43}$$

In a similar way, the following "coplanarity theorem" can be proved. For any $\alpha \geqq 0$, if $\mathbf{H}^{(\alpha)}$, $\mathbf{H}^{(\alpha+1)}$, and $\mathbf{H}^{(\alpha+2)}$ are coplanar, then $\mathbf{H}^{(\beta)}$ is coplanar with $\mathbf{H}^{(\alpha)}$ and $\mathbf{H}^{(\alpha+1)}$ for any $\beta \geqq \alpha$.

The derived fields are defined in such a way that the time rate of change of the flux of $\mathbf{H}^{(\alpha)}$ through a cycle C is equal to the flux of $\mathbf{H}^{(\alpha+1)}$ through C:

$$\frac{d}{dt}\int \mathbf{H}^{(\alpha)}\cdot\mathbf{n}\,dA = \int \dot{\mathbf{H}}^{(\alpha)}\cdot\mathbf{n}\,dA + \oint \mathbf{H}^{(\alpha)}\cdot(\mathbf{v}\times d\mathbf{l})$$
$$= \int [\dot{\mathbf{H}}^{(\alpha)} + \nabla\times(\mathbf{H}^{(\alpha)}\times\mathbf{v})]\cdot\mathbf{n}\,dA \qquad (44)$$
$$= \int \mathbf{H}^{(\alpha+1)}\cdot\mathbf{n}\,dA.$$

Thus, the αth derivative of the flux of $\mathbf{H}$ with respect to time is equal to the flux of $\mathbf{H}^{(\alpha)}$. If we assume that the flux of $\mathbf{H}$ through C is an analytic function of time, we may conclude that C has constant flux if and only if the flux of $\mathbf{H}^{(\alpha)}$ through C vanishes for all $\alpha \geqq 1$ at some particular instant t_0. If C is an infinitesimal cycle, this means that $\mathbf{H}^{(\alpha)}\cdot\mathbf{n}$ must vanish for all $\alpha \geqq 1$ when $t = t_0$, where $\mathbf{n}$ is a unit vector perpendicular to the plane of C. If this holds at time t_0, it will of course also hold at any time t. Designating the vector space V determined by the $\mathbf{H}^{(\alpha)}$'s for $\alpha \geqq 1$ as the derivative space, C has constant flux if and only if $\mathbf{n}$ is initially perpendicular to the derivative space, i.e., perpendicular to every vector in the derivative space. Four cases may be distinguished.

Case I. $\mathbf{H}^{(1)}$ vanishes. Since

$$\mathbf{H}^{(1)} = \dot{\mathbf{H}} + \nabla\times(\mathbf{H}\times\mathbf{v})$$
$$= -\nabla\times(\mathbf{E} + \mathbf{v}\times\mathbf{H}), \qquad (45)$$

the vanishing of $\mathbf{H}^{(1)}$ is simply the condition for flux-preservation. Clearly, $\mathbf{H}^{(\alpha)} = 0$ for any $\alpha \geqq 1$, so that $d(V)$, the dimension of the derivative space, is zero, and all cycles have constant flux.

To show that this case applies, it is sufficient to prove either that $\mathbf{H}^{(\alpha)}$ vanishes initially for any $\alpha \geqq 1$, or that $\mathbf{H}^{(1)}$ vanishes identically. It is not, however, sufficient merely to show that $\mathbf{H}^{(1)}$ vanishes initially.

Case II. $\mathbf{H}^{(1)}$ does not vanish, but $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ are collinear. Then, according to the collinearity theorem, all the $\mathbf{H}^{(\alpha)}$'s are collinear with $\mathbf{H}^{(1)}$ for $\alpha \geqq 1$, and $d(V) = 1$. An infinitesimal cycle will have constant flux if and only if its normal vector is initially perpendicular to $\mathbf{H}^{(1)}$, in which case it will remain perpendicular to $\mathbf{H}^{(1)}$ for all time.

Case III. $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ are not collinear, but $\mathbf{H}^{(1)}$, $\mathbf{H}^{(2)}$, and $\mathbf{H}^{(3)}$ are coplanar. The coplanarity theorem then implies that all the $\mathbf{H}^{(\alpha)}$'s with $\alpha \geqq 1$ are coplanar with $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$, so that $d(V) = 2$. An infinitesimal cycle will have constant flux if and only if its normal vector is initially perpendicular to both $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$, i.e., perpendicular to the derivative space, in which case it will remain perpendicular to $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ for all time.

Case IV. $\mathbf{H}^{(1)}$, $\mathbf{H}^{(2)}$, and $\mathbf{H}^{(3)}$ are linearly independent, so that $d(V) = 3$. In this case, it is impossible for $\mathbf{n}$ to be perpendicular to all the $\mathbf{H}^{(\alpha)}$'s, and no infinitesimal cycle will have constant flux.

Having derived necessary and sufficient conditions for an infinitesimal cycle to have constant flux, we turn our attention to the cycles of finite size. Clearly, a finite cycle C will have constant flux if it is the boundary of a surface of which the normal vector is perpendicular to the derivative space, i.e., a surface which is tangent to the derivative space. Such a surface will be decomposable into elements of area bounded by infinitesimal cycles with constant flux, so that C itself must have constant flux.

We define a constant flux surface as one which is tangent to the derivative space. Our first inclination may be to assert that a finite cycle C has constant flux if it lies initially in a constant flux surface. This cannot be valid, however, as we may see by considering the case $d(V) = 1$. In that case, every finite cycle C lies in such a surface, i.e., the surface generated by the $\mathbf{H}^{(1)}$-lines through C. If this surface has the topology of a cylinder linked by C, the flux of $\mathbf{H}^{(1)}$ through C will not necessarily vanish, and then the flux of $\mathbf{H}$ will not be constant. To eliminate this possibility, we stipulate that C must lie contractably in the surface, i.e., that it must be possible to contract C continuously into a point without leaving the surface. Thus, a sufficient condition for a finite cycle C to have constant flux is that, initially, C should lie contractably in a constant flux surface. While this condition is sufficient, it is not necessary, as we shall prove later on.

The class of constant flux surfaces will consist of all surfaces when $d(V) = 0$, of the surfaces tangent to $\mathbf{H}^{(1)}$ when $d(V) = 1$, and of no surfaces when $d(V) = 3$. If $d(V) = 2$, it will consist of the integral surfaces of $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$, provided that such surfaces exist, i.e., provided that the commutator product of $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ lies in the plane of $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$. If this condition is not fulfilled, the infinitesimal cycles with constant flux will not fit together to form finite cycles.

It is sufficient to assume that $[\mathbf{H}^{(1)}, \mathbf{H}^{(2)}]$ lies in the plane of $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ at some particular instant t_0. When this assumption is valid, there is a family of integral surfaces at time t_0. Let Σ be a surface, moving with velocity $\mathbf{v}$, which coincides at time t_0 with an integral surface. The surface Σ may be divided into elements of area bounded by infinitesimal cycles. The normal vectors to these infinitesimal cycles are perpendicular to $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ when $t = t_0$, hence for any t. Therefore, Σ remains tangent to $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ in the course of its motion, i.e., it remains an integral surface. Since $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ have a family of integral surfaces at all times, $[\mathbf{H}^{(1)}, \mathbf{H}^{(2)}]$ remains linearly dependent upon $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$.

We shall now compute the constant flux surfaces for several illustrative fields, in each case using the particle drift velocity. In example (b) of Section IV, we add a uniform electric field parallel to the z axis, driving the current along the wire. The fields in cylindrical coordinates have the form

$$\mathbf{H} = r\mathbf{i}_\theta + \mathbf{i}_z, \qquad (46)$$

$$\mathbf{E} = \mathbf{i}_z. \qquad (47)$$

The lines of force are helices with pitch independent of r. The particle drift velocity is given by

$$\mathbf{v}_p = -\frac{r}{1+r^2}\mathbf{i}_r, \tag{48}$$

and the first derived magnetic field by

$$\mathbf{H}^{(1)} = -\frac{\partial}{\partial r}\left(\frac{r^2}{1+r^2}\right)\mathbf{i}_\theta - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r^2}{1+r^2}\right)\mathbf{i}_z, \tag{49}$$

which is collinear with $\mathbf{H}$. According to the collinearity theorem for $\alpha = 0$, all the $\mathbf{H}^{(\alpha)}$'s are therefore collinear with $\mathbf{H}$. The derivative space has dimension one, and the constant flux surfaces are the surfaces tangent to $\mathbf{H}$, i.e., the magnetic tubes of force.

In example (f) of Section IV,

$$\mathbf{H} = \mathbf{i}_\theta/r + \mathbf{i}_z, \tag{50}$$

$$\mathbf{E} = \mathbf{i}_z, \tag{51}$$

$$\mathbf{v}_p = -\frac{r}{1+r^2}\mathbf{i}_r = v_r\mathbf{i}_r, \tag{52}$$

$$\mathbf{H}^{(1)} = \frac{2r}{(1+r^2)^2}\mathbf{i}_\theta - \frac{2}{(1+r^2)^2}\mathbf{i}_z, \tag{53}$$

$$\mathbf{H}^{(2)} = \frac{\partial}{\partial r}(H_\theta^{(1)}v_r)\mathbf{i}_\theta - \frac{1}{r}\frac{\partial}{\partial r}(H_\theta^{(1)}v_r)\mathbf{i}_z. \tag{54}$$

$\mathbf{H}$ and $\mathbf{H}^{(1)}$ are orthogonal, while $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ are collinear. Therefore, $d(V)$ is again equal to one, with the surfaces tangent to $\mathbf{H}^{(1)}$ as constant flux surfaces. The $\mathbf{H}^{(1)}$-lines in this example are helices with pitch independent of r, as are both the $\mathbf{H}$ and $\mathbf{H}^{(1)}$-lines in example (b). The constant flux surfaces in the two cases therefore differ only by a reflection across the x, y plane. These surfaces include circular cylinders with axis along the z axis, and right helicoids about the z axis, with pitch equal to the pitch of an $\mathbf{H}^{(1)}$-line.

Consider the cycle C lying in the $z = 0$ plane which is described initially by the equation

$$(r-2)^2 = 4\theta^2(1-\theta^2). \tag{55}$$

This cycle is a symmetrical figure eight, with one loop on each side of the line $\theta = 0$. Since the $H_z^{(\alpha)}$'s are functions of r only, the flux of $H_z^{(\alpha)}$ through one loop of the figure eight cancels the flux through the other loop. The $\mathbf{H}$-flux through C therefore has the constant value zero. The surface Σ generated by the $\mathbf{H}^{(1)}$-lines through C (i.e., the constant flux surface containing C) has the topology of a cylinder with a figure eight cross-section, and C is not contractable in Σ. Thus, the sufficient condition given above for a finite cycle to have constant flux is not necessary.

A similar argument applies to the cycle C' lying in the $\theta = 0$ plane which is described initially by the equation

$$(r-2)^2 = 4z^2(1-z^2). \tag{56}$$

C and C' both intersect themselves. As a similar example which does not intersect itself, consider the cycle C'', described initially by the parametric equations

$$\theta = \sin u, \tag{57}$$

$$r = 2 + \sin 2u, \tag{58}$$

$$z = \cos u. \tag{59}$$

The projections of C'' on the planes $z = 0$ and $\theta = 0$ by lines of constant r, θ and constant r, z, respectively, are C and C', so that C'' must also have constant flux.

In Section III we considered a plane wave travelling through a uniform external magnetic field $\mathbf{H}_0$ with the propagation vector $\mathbf{K}$ not perpendicular to $\mathbf{H}_0$. For such a wave the total magnetic field is given by

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_1, \tag{60}$$

where $\mathbf{H}_1$ is the small oscillating part of $\mathbf{H}$. To first order in the amplitude, we have

$$\mathbf{v}_p = (\mathbf{E}\times\mathbf{H})/H^2 \cong (\mathbf{E}\times\mathbf{H}_0)/H_0^2, \tag{61}$$

and, using Eqs. (45) and (61),

$$\begin{aligned}\mathbf{H}^{(1)} &\cong -\nabla\times[\mathbf{H}_0(\mathbf{E}\cdot\mathbf{H}_0)/H_0^2] \\ &\cong -i(\mathbf{K}\times\mathbf{H}_0)(\mathbf{E}\cdot\mathbf{H}_0)/H_0^2.\end{aligned} \tag{62}$$

Suppose that $\mathbf{E}$ and $\mathbf{H}_0$ are not perpendicular, and that $\mathbf{K}$ and $\mathbf{H}_0$ are not collinear. Then $\mathbf{H}^{(1)}$ will not vanish, and $\mathbf{v}_p$ will not be flux-preserving, i.e., $\mathbf{v}_p$ will differ from the flux-preserving velocity $\mathbf{v}_F$ given by Eq. (18). Since $\mathbf{H}^{(1)}\times\mathbf{v}_p$ is of second order in the amplitude,

$$\mathbf{H}^{(2)} \cong \dot{\mathbf{H}}^{(1)} \cong -i\omega\mathbf{H}^{(1)}. \tag{63}$$

Thus, $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ are again collinear $[d(V) = 1]$. They are both perpendicular to the plane of $\mathbf{H}_0$ and $\mathbf{K}$, and the class of constant flux surfaces consists of the cylindrical surfaces with generators perpendicular to $\mathbf{H}_0$ and $\mathbf{K}$. This result can also be derived from the fact that $\mathbf{v}_p$ and $\mathbf{v}_F$ differ only in their components perpendicular to the plane of $\mathbf{H}_0$ and $\mathbf{K}$.

We shall now give examples of somewhat more complicated fields for which $d(V)$ is two or three.

(g) The magnetic field is uniform and parallel to the y axis, and the electric field is produced by a point dipole in the x direction.

$$\mathbf{H} = \mathbf{i}_y \tag{64}$$

$$\mathbf{E} = \left(-\frac{1}{r^3} + \frac{3x^2}{r^5}\right)\mathbf{i}_x + \frac{3xy}{r^5}\,\mathbf{i}_y + \frac{3xz}{r^5}\,\mathbf{i}_z\,, \tag{65}$$

$$\mathbf{v}_p = -\frac{3xz}{r^5}\,\mathbf{i}_x + \left(-\frac{1}{r^3} + \frac{3x^2}{r^5}\right)\mathbf{i}_z\,, \tag{66}$$

$$\mathbf{H}^{(1)} = -\frac{15xyz}{r^7}\,\mathbf{i}_x + \left(-\frac{3y}{r^5} + \frac{15x^2y}{r^7}\right)\mathbf{i}_z\,, \tag{67}$$

$$\mathbf{H}^{(2)} = \left(\frac{6xy}{r^{10}} - \frac{60xyz^2}{r^{12}}\right)\mathbf{i}_x + \left(-\frac{6yz}{r^{10}} + \frac{60x^2yz}{r^{12}}\right)\mathbf{i}_z\,. \tag{68}$$

$\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ are not collinear, so that $d(V)$ is either two or three. $\mathbf{H}$ has a time-independent family of orthogonal surfaces, i.e., the planes perpendicular to the y axis, and these must be constant flux surfaces. This eliminates $d(V) = 3$, leaving $d(V) = 2$ as the only possibility. Therefore, for any $\alpha \geqq 1$, $\mathbf{H}^{(\alpha)}$ is perpendicular to the y axis, and the family of orthogonal surfaces coincides with the family of constant flux surfaces.

(h) The electric and magnetic fields are given by:

$$\mathbf{H} = \cos z\,\mathbf{i}_x + \sin z\,\mathbf{i}_y\,, \tag{69}$$

$$\mathbf{E} = \mathbf{i}_z\,. \tag{70}$$

This magnetic field is produced by a current

$$\mathbf{J} = \frac{1}{4\pi}\,\nabla \times \mathbf{H} = -\frac{1}{4\pi}\cos z\,\mathbf{i}_x - \frac{1}{4\pi}\sin z\,\mathbf{i}_y. \tag{71}$$

Since $\mathbf{H}$ is parallel to its own curl, there are no orthogonal surfaces.

$$\mathbf{v}_p = \sin z\;\mathbf{i}_z\,, \tag{72}$$

$$\mathbf{H}^{(1)} = (1 - 2\sin^2 z)\mathbf{i}_x + 2\sin z\cos z\,\mathbf{i}_y\,, \tag{73}$$

$$\mathbf{H}^{(2)} = \cos z(1 - 6\sin^2 z)\mathbf{i}_x + \sin z(4 - 6\sin^2 z)\mathbf{i}_y\,. \tag{74}$$

$\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ are not collinear. On the other hand, $\mathbf{H}^{(2)}$ and $\mathbf{v}_p$ are functions of z only, so that $\mathbf{H}^{(3)} = \nabla \times (\mathbf{H}^{(2)} \times \mathbf{v})$ must be in the x, y plane, i.e., in the plane

of $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$. Thus, $d(V) = 2$, and the constant flux surfaces are the planes parallel to the x, y plane.

(i) To destroy the coplanarity of the $\mathbf{H}^{(\alpha)}$'s in example (h), we introduce a uniform space charge to make $\mathbf{E}$ a function of x.

$$\mathbf{H} = \cos z\,\mathbf{i}_x + \sin z\,\mathbf{i}_y\,, \tag{75}$$

$$\mathbf{E} = x\,\mathbf{i}_x\,, \tag{76}$$

$$\mathbf{v}_p = x\sin z\,\mathbf{i}_z\,, \tag{77}$$

$$\mathbf{H}^{(1)} = x(1 - 2\sin^2 z)\mathbf{i}_x + 2x\sin z\cos z\,\mathbf{i}_y - \sin z\cos z\,\mathbf{i}_z\,, \tag{78}$$

$$\mathbf{H}^{(2)} = x^2\cos z(1 - 6\sin^2 z)\mathbf{i}_x + x^2\sin z(4 - 6\sin^2 z)\mathbf{i}_y + x\sin z(4\sin^2 z - 2)\mathbf{i}_z\,, \tag{79}$$

$$\mathbf{H}^{(3)} = x^3(1 - 20\sin^2 z + 24\sin^4 z)\mathbf{i}_x + x^3\sin z\cos z(8 - 24\sin^2 z)\mathbf{i}_y + x^2\sin z\cos z(-3 + 18\sin^2 z)\mathbf{i}_z\,. \tag{80}$$

$\mathbf{H}^{(1)}$, $\mathbf{H}^{(2)}$, and $\mathbf{H}^{(3)}$ are linearly independent, so that $d(V) = 3$, and there are no constant flux surfaces.

VI. LINE- AND SURFACE-PRESERVATION

A velocity $\mathbf{v}$ is said to be line-preserving if any line l, moving with velocity $\mathbf{v}$, which coincides initially with a line of force, continues to coincide with some line of force during the course of its motion; i.e., if l is initially tangent to $\mathbf{H}$, it will remain tangent to $\mathbf{H}$. This condition may also be expressed as follows. If l is a line of force at time t, draw infinitesimal vectors $\mathbf{v}\,dt$ at every point of l. Then l', the locus of the endpoints of these vectors, is a line of force at time $t + dt$. The line-preserving property of a velocity $\mathbf{v}$ will also be referred to by stating that $\mathbf{v}$ transforms lines of force into lines of force.

We shall now prove that $\mathbf{v}$ is line-preserving if and only if $\mathbf{H}$ and $\mathbf{H}^{(1)}$ are collinear. To do this, we consider space and time together as a four-dimensional continuum, and introduce the four-vectors $\tilde{\mathrm{H}} = (\mathbf{H}, 0)$ and $\tilde{\mathrm{v}} = (\mathbf{v}, 1)$. If, in Fig. 2, $\tilde{\mathrm{A}}$ is interpreted as $-\tilde{\mathrm{v}}$, and $\tilde{\mathrm{B}}$ as $\tilde{\mathrm{H}}$, the segments (2, 3) and (4, 5) will be $\mathbf{H}$-lines at times t and $t + \epsilon$, respectively. According to Eq. (11), the four-vector $\tilde{\Delta} = \tilde{\mathrm{x}}_5 - \tilde{\mathrm{x}}_1$ is given by

$$\tilde{\Delta} = -\epsilon\delta[\tilde{\mathrm{H}}, \tilde{\mathrm{v}}], \tag{81}$$

provided that the four segments are chosen to satisfy Eqs. (8). If the index μ runs from 1 to 4,

$$\Delta_4 = -\,\epsilon\delta\left(v_\mu\,\frac{\partial H_4}{\partial x_\mu} - H_\mu\,\frac{\partial v_4}{\partial x_\mu}\right) = 0, \tag{82}$$

so that points 1 and 5, and hence 1 and 4, have the same time coordinate. Points 1 and 4 therefore lie on the line into which the **H**-line (2, 3) is transformed at time $t + \epsilon$ by the velocity **v**. This line will itself be a line of force if and only if it also contains the point 5, in which case the spatial part of $\tilde{\Delta}$ will be collinear with **H**. Thus, a necessary and sufficient condition for line-preservation is that **H** and the spatial part of $[\tilde{\mathrm{H}}, \tilde{\mathrm{v}}]$ should be collinear[10].

If $i = 1, 2$, or 3,

$$[\tilde{\mathrm{H}}, \tilde{\mathrm{v}}]_i = v_\mu \frac{\partial H_i}{\partial x_\mu} - H_\mu \frac{\partial v_i}{\partial x_\mu} = [\mathbf{H}, \mathbf{v}]_i + \dot{H}_i, \tag{83}$$

and the spatial part of $[\tilde{\mathrm{H}}, \tilde{\mathrm{v}}]$ is given by

$$\begin{aligned}[\mathbf{H}, \mathbf{v}] + \dot{\mathbf{H}} &= (\mathbf{v}\cdot\nabla)\mathbf{H} - (\mathbf{H}\cdot\nabla)\mathbf{v} + \dot{\mathbf{H}} \\ &= \nabla \times (\mathbf{H} \times \mathbf{v}) + \mathbf{v}(\nabla\cdot\mathbf{H}) - \mathbf{H}(\nabla\cdot\mathbf{v}) + \dot{\mathbf{H}} \\ &= \mathbf{H}^{(1)} - \mathbf{H}(\nabla\cdot\mathbf{v}),\end{aligned} \tag{84}$$

which lies along **H** if and only if **H** and $\mathbf{H}^{(1)}$ are collinear.

If **H** and $\mathbf{H}^{(1)}$ are collinear, the collinearity theorem for $\alpha = 0$ implies that for any β, $\mathbf{H}^{(\beta)}$ is collinear with **H**, and therefore with $\mathbf{H}^{(1)}$. Thus, the dimension $d(V)$ of the derivative space is either zero or one for any line-preserving velocity. If $d(V) = 1$, the constant flux surfaces are the surfaces tangent to **H**, i.e., the tubes of force, and the infinitesimal cycles with constant flux are those for which the flux vanishes.

The necessary and sufficient condition for flux-preservation is that $\mathbf{H}^{(1)}$ should vanish. It is therefore clear that any flux-preserving velocity must also be line-preserving, although a line-preserving velocity may not be flux-preserving.

The condition for line-preservation can be written in the form:

$$\begin{aligned}\mathbf{H} \times \mathbf{H}^{(1)} &= \mathbf{H} \times [\dot{\mathbf{H}} + \nabla \times (\mathbf{H} \times \mathbf{v})] \\ &= -\mathbf{H} \times [\nabla \times (\mathbf{E} + \mathbf{v} \times \mathbf{H})] = 0.\end{aligned} \tag{85}$$

For the particle drift velocity $\mathbf{v}_p$, this reduces to

$$-\mathbf{H} \times [\nabla \times \{\mathbf{H}(\mathbf{E}\cdot\mathbf{H})/H^2\}] = 0. \tag{86}$$

In particular, this result shows that $\mathbf{v}_p$ is line-preserving if (but not only if) **E** and **H** are perpendicular, in accordance with the general result established earlier.

For a plane wave in a uniform magnetic field $\mathbf{H}_0$, it is easy to verify that there is one and only one line-preserving velocity proportional to $\exp i(\mathbf{K}\cdot\mathbf{x} - \omega t)$,

[10] Since $\tilde{\mathrm{H}}$, $\tilde{\Delta}$, and $[\tilde{\mathrm{H}}, \tilde{\mathrm{v}}]$ all have vanishing time components, this is equivalent to saying that $\tilde{\mathrm{H}}$ and $[\tilde{\mathrm{H}}, \tilde{\mathrm{v}}]$ should themselves be collinear.

provided that $\mathbf{H}_0\cdot\mathbf{K} \neq 0$. This is of course identical with the flux-preserving velocity $\mathbf{v}_F$ given by Eq. (18). Thus, the particle drift velocity $\mathbf{v}_p$ will be line-preserving if and only if $\mathbf{v}_p = \mathbf{v}_F$.

In the case of example (f) of Section IV, we showed by direct calculation that **H** and $\mathbf{H}^{(1)}$ are perpendicular rather than collinear. It follows that lines of force will not move into lines of force, as we may verify by the following simple argument: Since $\mathbf{v}_p$ is directed radially inward, a line of force l will, in the course of its motion, change into a helix l' with the same pitch as l but smaller radius. But we have already seen that the pitch of a line of force is proportional to the square of its radius, so that l' cannot itself be a line of force.

In example (b), **H** and $\mathbf{H}^{(1)}$ are collinear, so that $\mathbf{v}_p$ is line-preserving. This result also follows from the fact that in this case the pitch of a line of force is independent of its radius.

We shall now introduce the concept of surface-preservation, which will be useful in Section VII. A velocity **v** is said to be surface-preserving if there exists a family of surfaces, moving with velocity **v**, which remain tangent to **H** in the course of their motion. Surfaces having this property will be called invariant surfaces. They will, of course, be tubes of force if they have the topology of a cylinder.

If **v** is line-preserving, every surface tangent to **H** is an invariant surface. Therefore, any line-preserving velocity must also be surface-preserving.

A necessary and sufficient condition for surface-preservation is that $\mathbf{H}^{(2)}$ and the commutator product $[\mathbf{H}, \mathbf{H}^{(1)}]$ should both be linear combinations of **H** and $\mathbf{H}^{(1)}$. To prove necessity, we again consider space and time together as a four-dimensional manifold, and introduce the four-vectors $\tilde{\mathrm{H}} = (\mathbf{H}, 0)$ and $\tilde{\mathrm{v}} = (\mathbf{v}, 1)$. Assume that there exists a family of invariant surfaces, let **A** be any divergence-free vector field which is tangent to the invariant surfaces, and let $\tilde{\mathrm{A}} = (\mathbf{A}, 0)$.

As in the derivation of the line-preservation condition, it is easy to prove that the commutator product $[\tilde{\mathrm{A}}, \tilde{\mathrm{v}}]$ must have a vanishing time component and a spatial component tangent to the invariant surfaces; otherwise the invariant surfaces could not remain tangent to **A** in the course of their motion. We can then deduce that $\dot{\mathbf{A}} + [\mathbf{A}, \mathbf{v}]$, and hence $\mathbf{A}^{(1)}$, must be tangent to the invariant surfaces. Taking **A** equal to **H**, we see that $\mathbf{H}^{(1)}$ is tangent to the invariant surfaces. Similarly, taking **A** equal to $\mathbf{H}^{(1)}$, $\mathbf{H}^{(2)}$ must be tangent to the invariant surfaces. The vectors **H**, $\mathbf{H}^{(1)}$, and $\mathbf{H}^{(2)}$ are therefore coplanar. Furthermore, $\mathbf{H}^{(2)}$ is a linear combination of **H** and $\mathbf{H}^{(1)}$. This follows from the collinearity theorem if **H** and $\mathbf{H}^{(1)}$ are collinear, and from the coplanarity of **H**, $\mathbf{H}^{(1)}$, and $\mathbf{H}^{(2)}$ if **H** and $\mathbf{H}^{(1)}$ are not collinear. Finally, $[\mathbf{H}, \mathbf{H}^{(1)}]$ is also a linear combination of **H** and $\mathbf{H}^{(1)}$, since the invariant surfaces are tangent to both **H** and $\mathbf{H}^{(1)}$.

To prove sufficiency, we observe that if $[\mathbf{H}, \mathbf{H}^{(1)}]$ is a linear combination of **H** and $\mathbf{H}^{(1)}$, there will exist a family of surfaces tangent to both **H** and $\mathbf{H}^{(1)}$. If, in addition, $\mathbf{H}^{(2)}$ is a linear combination of $\mathbf{H}$ and $\mathbf{H}^{(1)}$, it is easy to show that a surface, moving with velocity $\mathbf{v}$, which coincides initially with one of the surfaces tangent to both $\mathbf{H}$ and $\mathbf{H}^{(1)}$ will, in the course of its motion, continue to coincide with one of these surfaces.

There are several possibilities to be distinguished. First, $\mathbf{v}$ may be flux-preserving, in which case $\mathbf{H}^{(1)}$ will vanish $[d(V) = 0]$. Second, $\mathbf{v}$ may be line-preserving but not flux-preserving, in which case $\mathbf{H}^{(1)}$ will not vanish, but will be collinear with $\mathbf{H}[d(V) = 1]$. In either case, $\mathbf{v}$ will also be surface-preserving, and every surface tangent to $\mathbf{H}$ will be an invariant surface. Third, we may have $\mathbf{H}^{(1)}$ not collinear with $\mathbf{H}$, but $\mathbf{H}^{(2)}$ collinear with $\mathbf{H}^{(1)}$. The derivative space again has dimension one in this case, and every surface tangent to $\mathbf{H}^{(1)}$ is a constant flux surface. The velocity $\mathbf{v}$ will be surface-preserving if and only if $[\mathbf{H}, \mathbf{H}^{(1)}]$ is a linear combination of $\mathbf{H}$ and $\mathbf{H}^{(1)}$. In that case, the integral surfaces of $\mathbf{H}$ and $\mathbf{H}^{(1)}$ will be the invariant surfaces, and they will also constitute a sub-class of the constant flux surfaces. Fourth, we may have $\mathbf{H}^{(1)}$ not collinear with $\mathbf{H}$, and $\mathbf{H}^{(2)}$ not collinear with $\mathbf{H}^{(1)}$, but $\mathbf{H}$, $\mathbf{H}^{(1)}$, and $\mathbf{H}^{(2)}$ coplanar. We then have $d(V) = 2$, and the condition for surface-preservation is the same as the condition for the existence of constant flux surfaces, i.e., that $[\mathbf{H}, \mathbf{H}^{(1)}]$ should[11] lie in the plane of $\mathbf{H}$, $\mathbf{H}^{(1)}$, and $\mathbf{H}^{(2)}$. If this condition is satisfied, the integral surfaces of $\mathbf{H}$ and $\mathbf{H}^{(1)}$ constitute both the constant flux and the invariant surfaces. Finally, $\mathbf{H}$, $\mathbf{H}^{(1)}$, and $\mathbf{H}^{(2)}$ may be linearly independent. The dimension of the derivative space is either two or three, depending on whether or not $\mathbf{H}^{(3)}$ is coplanar with $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$, but $\mathbf{v}$ will not be surface-preserving, so that there will be no family of invariant surfaces. It should be pointed out that in all these cases, an invariant surface must also be a constant flux surface, since it is tangent to all the $\mathbf{H}^{(\alpha)}$'s. The constant value of the flux through an infinitesimal cycle lying in an invariant surface is, of course, zero.

As an example of a surface-preserving velocity, we consider the particle drift velocity in a plane wave with a uniform external magnetic field $\mathbf{H}_0$. As we have already seen, $\mathbf{v}_p$ need not be line-preserving in such a situation. Choosing a coordinate system in which $\mathbf{H}_0$ is along the z axis and $\mathbf{K}$ is in the x, z plane, $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ are both parallel to the y axis. Also,

$$[\mathbf{H}, \mathbf{H}^{(1)}] \cong -(\mathbf{H}_0 \cdot \nabla)\mathbf{H}^{(1)} \cong -i(\mathbf{H}_0 \cdot \mathbf{K})\mathbf{H}^{(1)}, \tag{87}$$

so that the conditions for surface-preservation are fulfilled. An invariant surface is generated by a line of force translated along the y axis. It therefore oscillates about a plane perpendicular to the x axis, the displacement from this plane having the form of a wave travelling in the z direction.

Another case in which surface-preservation holds without line-preservation is

[11] An equivalent form of this condition is that $[\mathbf{H}^{(1)}, \mathbf{H}^{(2)}]$ should lie in the plane of $\mathbf{H}$, $\mathbf{H}^{(1)}$, and $\mathbf{H}^{(2)}$.

example (f) of Section IV, where $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ are collinear. The invariant surfaces are circular cylinders about the z axis, of which the radii are decreasing functions of time. These surfaces constitute only a small sub-class of the complete class of constant flux surfaces.

In example (h) of Section V, $\mathbf{H}$, $\mathbf{H}^{(1)}$, and $\mathbf{H}^{(2)}$ are all parallel to the x, y plane, but no two of them are collinear. Therefore, we again have surface-preservation without line-preservation, with the planes parallel to the x, y plane as invariant surfaces. These planes move parallel to the z axis with velocity $\sin z$. In contrast to example (f), the invariant surfaces are the only constant flux surfaces, since $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ are not collinear.

Finally, in examples (g) and (i) of Section V, $\mathbf{H}$, $\mathbf{H}^{(1)}$, and $\mathbf{H}^{(2)}$ are linearly independent, so that in these cases there is no family of invariant surfaces.

VII. PARTICLE ORBITS IN AN ELECTROMAGNETIC FIELD

One of the reasons for investigating the motion of magnetic fields is the possibility that in certain cases information concerning the motion of a field can be used to make inferences concerning the particle orbits in that field. Since the field motion is related only to the transverse component of the particle motion, we should expect interesting results mainly in the case where the transverse motion is separable from the longitudinal motion, i.e., when the transverse motion can be integrated without any knowledge of the longitudinal motion.

We shall assume throughout this section that it is permissible to replace the particle motion by the motion of the guiding center, and that the transverse velocity is given by $\mathbf{v}_p = (\mathbf{E} \times \mathbf{H})/H^2$, without any correction for drifts due to inhomogeneities in the magnetic field. This last assumption will hold if the magnetic moment of the particle in its Larmor circle is sufficiently small. Thus, if $\mathbf{V}$ is the velocity of a particle, we may write

$$\mathbf{V} = \frac{\mathbf{E} \times \mathbf{H}}{H^2} + \mathbf{v}_{\parallel} = \mathbf{v}_p + \mathbf{v}_{\parallel}. \tag{88}$$

To say that the transverse motion is separable means that there exists a curvilinear coordinate system (ξ, η, ζ), where ξ, η, and ζ are functions of $\mathbf{r}$ and t, such that the equations of motion for ξ and η do not involve ζ or $\mathbf{v}_{\parallel}$. Such a coordinate system will have the following property: If two particles start off on the same line of constant ξ and η, but with different values of ζ or $d\zeta/dt$, they will at all times be joinable by a line of constant ξ and η. Thus, by integrating the equations of motion for ξ and η, we can find out which member of a certain family of lines a particle is on at any time, even if we know nothing about its ζ-motion.

The equation of motion for ξ is

$$\frac{d\xi}{dt} = \dot{\xi} + \mathbf{V}\cdot\nabla\xi = \dot{\xi} + \mathbf{v}_p\cdot\nabla\xi + \mathbf{v}_{\parallel}\cdot\nabla\xi. \tag{89}$$

The third term on the right hand side must clearly vanish, if the transverse motion is to be separable. Thus,

$$\mathbf{v}_{\|}\cdot\nabla\xi = \mathbf{v}_{\|}\cdot\nabla\eta = 0. \tag{90}$$

Equations (90) imply that ξ and η must be constant along $\mathbf{v}_{\|}$-lines, i.e., along lines of force.

The right-hand side of Eq. (89) is a function of $\mathbf{r}$ and t, given by the transformation equations between the coordinate systems (x, y, z) and (ξ, η, ζ). For the transverse motion to be separable, this function should be expressible as a function of ξ, η, and t. Hence,

$$\frac{\partial}{\partial\zeta}(\dot{\xi} + \mathbf{v}_p\cdot\nabla\xi) = 0. \tag{91}$$

Since the derivative with respect to ζ is taken while holding ξ, η, and t constant, Eq. (91) implies that

$$\mathbf{H}\cdot\nabla[\dot{\xi} + \mathbf{v}_p\cdot\nabla\xi] = 0, \tag{92}$$

which may be rewritten in the form

$$\begin{aligned}
\frac{\partial}{\partial t}(\mathbf{H}\cdot\nabla\xi) - \dot{\mathbf{H}}\cdot\nabla\xi + H_i\frac{\partial}{\partial x_i}\left(v_j\frac{\partial\xi}{\partial x_j}\right)
&= -\dot{\mathbf{H}}\cdot\nabla\xi + H_i v_j\frac{\partial^2\xi}{\partial x_i\partial x_j} + H_i\frac{\partial v_j}{\partial x_i}\frac{\partial\xi}{\partial x_j}\\
&= -\dot{\mathbf{H}}\cdot\nabla\xi + v_j\left[\frac{\partial}{\partial x_j}\left(H_i\frac{\partial\xi}{\partial x_i}\right) - \frac{\partial H_i}{\partial x_j}\frac{\partial\xi}{\partial x_i}\right] + H_i\frac{\partial v_j}{\partial x_i}\frac{\partial\xi}{\partial x_j}\\
&= -\dot{\mathbf{H}}\cdot\nabla\xi + H_i\frac{\partial v_j}{\partial x_i}\frac{\partial\xi}{\partial x_j} - v_j\frac{\partial H_i}{\partial x_j}\frac{\partial\xi}{\partial x_i}\\
&= -\dot{\mathbf{H}}\cdot\nabla\xi + H_j\frac{\partial v_i}{\partial x_j}\frac{\partial\xi}{\partial x_i} - v_j\frac{\partial H_i}{\partial x_j}\frac{\partial\xi}{\partial x_i}\\
&= -(\dot{\mathbf{H}} + [\mathbf{H}, \mathbf{v}_p])\cdot\nabla\xi\\
&= -[\dot{\mathbf{H}} + \nabla\times(\mathbf{H}\times\mathbf{v}_p) + \mathbf{v}_p(\nabla\cdot\mathbf{H}) - \mathbf{H}(\nabla\cdot\mathbf{v}_p)]\cdot\nabla\xi\\
&= -\mathbf{H}^{(1)}\cdot\nabla\xi = 0,
\end{aligned} \tag{93}$$

where repeated use has been made of the fact that $\mathbf{H}\cdot\nabla\xi = H_i\partial\xi/\partial x_i = 0$. A similar relation may be derived for η. Thus, ξ and η are both constant along the $\mathbf{H}^{(1)}$-lines. Since they are also constant along the $\mathbf{H}$-lines, $\mathbf{H}$ and $\mathbf{H}^{(1)}$ must be collinear. But this is simply the condition for line-preservation, so that separability of the transverse motion implies that the particle drift velocity is line-

preserving. Conversely, if $\mathbf{v}_p$ is line-preserving, it is easy to see that the lines of force may be used as lines of constant ξ and η for a coordinate system of the type described above. Thus, line-preservation for the particle drift velocity is a necessary and sufficient condition for separability of the transverse motion.

If the particle drift velocity is not only line-preserving but also flux-preserving, we will be able to derive further information concerning the particle orbits. Consider the prism bounded by the surfaces $\xi = a$, $\xi = a + d\xi$, $\eta = b$, and $\eta = b + d\eta$. The surface Σ of this prism is clearly a tube of force, and we designate the flux through Σ by $d\Phi = \alpha\, d\xi\, d\eta$, where α is a function of ξ, η, and t, but not of ζ. Since the cross-sectional area of Σ is given by $|\nabla\xi\times\nabla\eta|^{-1}\, d\xi\, d\eta$, we may write

$$\begin{aligned}
\alpha &= \frac{H}{|\nabla\xi\times\nabla\eta|}\\
&= \frac{H^2}{\mathbf{H}\cdot(\nabla\xi\times\nabla\eta)},
\end{aligned} \tag{94}$$

using the fact that $\nabla\xi$ and $\nabla\eta$ are both perpendicular to $\mathbf{H}$, and choosing the coordinate system (ξ, η, ζ) so as to satisfy the following conditions: (1) $\mathbf{H}\cdot\nabla\zeta > 0$. (2) The vectors $\nabla\xi$, $\nabla\eta$, and $\nabla\zeta$ form a right-handed triad. Since α is independent of ζ, the right-hand side of Eq. (94) may be evaluated at any point of a given line of force.

If $\mathbf{v}_p$ is flux-preserving, we have

$$\frac{d}{dt}(\alpha\, d\xi\, d\eta) = 0. \tag{95}$$

Let (ξ_t, η_t) and (ξ_0, η_0) represent the values of (ξ, η) of a particle at times t and t_0, respectively, where t is regarded as a variable and t_0 as a constant. Equation (95) then yields

$$\alpha(\xi_t, \eta_t, t)\, d\xi_t\, d\eta_t = \alpha(\xi_0, \eta_0, t_0)\, d\xi_0\, d\eta_0\,. \tag{96}$$

The coordinates ξ_t and η_t are functions of ξ_0, η_0, and t, so that Eq. (96) may be rewritten in the form

$$\alpha(\xi_t, \eta_t, t)J\begin{pmatrix}\xi_t\ \eta_t\\ \xi_0\ \eta_0\end{pmatrix} = \alpha(\xi_0, \eta_0, t_0), \tag{97}$$

where J is the Jacobian determinant of (ξ_t, η_t) with respect to (ξ_0, η_0), holding t constant. The left-hand side of Eq. (97) is a constant of the motion. It is, however, unusual in that it depends not only upon the particle's positional coordinates (ξ_t, η_t), but also on their derivatives with respect to their initial values.

Consider the equations of motion determining the values of ξ and η for an individual particle in a coordinate system of the type described above:

$$\frac{d\xi}{dt} = \dot{\xi} + \mathbf{v}_p \cdot \nabla \xi = f(\xi, \eta, t), \tag{98}$$

$$\frac{d\eta}{dt} = \dot{\eta} + \mathbf{v}_p \cdot \nabla \eta = g(\xi, \eta, t). \tag{99}$$

Since ξ, η, and $\mathbf{v}_p$ are known functions of $\mathbf{r}$ and t, the expressions f and g are known functions of ξ, η, and t. The transverse motion will be completely known when Eqs. (98) and (99) have been solved to yield ξ_t and η_t as functions of ξ_0, η_0, and t. The solution of these coupled differential equations may sometimes be facilitated by the use of Eq. (97), in the case where $\mathbf{v}_p$ is flux-preserving. For example, suppose that there exists a one-dimensional family of surfaces Σ tangent to $\mathbf{H}$ and moving with velocity $\mathbf{v}_p$ for which the integrated motion is known. We can then choose our coordinate system in such a way that, at any instant, the surfaces Σ are surfaces of constant η, with the value of η assigned to a given surface independent of time. With this choice of coordinate system, we have

$$\frac{d\eta}{dt} = 0, \tag{100}$$

and

$$\eta = \eta_0 . \tag{101}$$

The coupled equations of motion then reduce to the single differential equation

$$\frac{d\xi}{dt} = f(\xi, \eta_0, t). \tag{102}$$

Of course, if there were two independent families of surfaces tangent to $\mathbf{H}$ for which the motion were known, the problem would be completely solved, since we could choose the coordinate system so as to make both ξ and η constant.

If there is a value $\xi_0 = a$ for which Eq. (102) is easily solvable, Eq. (97) may be used to obtain a solution for any value of ξ_0. Thus, we suppose that $\xi_t(a, \eta_0, t)$ is a known function of η_0 and t. Using the relations

$$\frac{\partial \eta_t}{\partial \xi_0} = 0, \tag{103}$$

and

$$\frac{\partial \eta_t}{\partial \eta_0} = 1, \tag{104}$$

Eq. (97) reduces to

$$\alpha(\xi_t, \eta_0, t) \frac{\partial \xi_t}{\partial \xi_0} = \alpha(\xi_0, \eta_0, t_0), \tag{105}$$

which may be integrated to yield

$$\int_{\xi_t(a,\eta_0,t)}^{\xi_t} \alpha(\xi, \eta_0, t)\, d\xi = \int_a^{\xi_0} \alpha(\xi, \eta_0, t_0)\, d\xi. \tag{106}$$

Since the integrands in Eq. (106) are known functions of ξ, Eq. (106) gives the value of ξ_t as a function of ξ_0, η_0, and t.

To illustrate this procedure we will look at a simple example. Our results for this example could very easily be obtained by less abstract methods, but it will serve to illustrate the concepts we are using. We consider the electromagnetic field

$$\mathbf{H} = s(z, r) \cos \omega t\, \mathbf{i}_r + p(z, r) \cos \omega t\, \mathbf{i}_z, \tag{107}$$

$$\mathbf{E} = q(z, r) \sin \omega t\, \mathbf{i}_\theta, \tag{108}$$

where s, p, and q satisfy the relations

$$s(z, r) = -s(-z, r), \tag{109}$$

$$p(z, r) = p(-z, r), \tag{110}$$

$$q(z, r) = q(-z, r). \tag{111}$$

From Maxwell's equations, we obtain the further relations

$$\frac{1}{r}\frac{\partial}{\partial r}(rs) + \frac{\partial p}{\partial z} = 0, \tag{112}$$

$$s = -\frac{1}{\omega}\frac{\partial q}{\partial z}, \tag{113}$$

and

$$p = \frac{1}{\omega r}\frac{\partial}{\partial r}(rq). \tag{114}$$

Since $\mathbf{E} \cdot \mathbf{H}$ vanishes, the particle drift velocity is flux-preserving. Its θ component vanishes, so that the half-planes with edge along the z axis are stationary. We have, therefore, a one-dimensional family of surfaces tangent to $\mathbf{H}$ for which the motion is known. Choosing $\eta = \theta$, we have $\eta_t = \eta_0$.

Let l represent the line of force passing through a particle. For the coordinate ξ of the particle, we choose the distance from the z axis to the point of intersection of l with the plane $z = 0$. Clearly, $\dot{\xi}$ vanishes, and we have for $z = 0$:

$$\mathbf{v}_p = \frac{E_\theta}{H_z}\,\mathbf{i}_r = \frac{q(o, r)\sin\omega t}{p(o, r)\cos\omega t}\,\mathbf{i}_r, \tag{115}$$

$$\xi = r, \tag{116}$$

and

$$\nabla\xi = \mathbf{i}_r\,. \tag{117}$$

Equation (98), the equation of motion for ξ, is therefore

$$\frac{d\xi}{dt} = \frac{q(o, \xi)\sin\omega t}{p(o, \xi)\cos\omega t}. \tag{118}$$

This may be integrated directly to yield

$$\frac{1}{\omega}\log\frac{\cos\omega t_0}{\cos\omega t} = \int_{\xi_0}^{\xi_t}\frac{p(o, \xi)\,d\xi}{q(o, \xi)}. \tag{119}$$

Using Eq. (114), the right-hand side of Eq. (119) reduces to

$$\frac{1}{\omega}\int_{\xi_0}^{\xi_t}\frac{d\xi}{\xi q(o, \xi)}\frac{\partial}{\partial\xi}[\xi q(o, \xi)] = \frac{1}{\omega}\log\frac{\xi_t q(o, \xi_t)}{\xi_0 q(o, \xi_0)}, \tag{120}$$

so that

$$\xi_t q(0, \xi_t)\cos\omega t = \xi_0 q(0, \xi_0)\cos\omega t_0\,, \tag{121}$$

which gives ξ_t as a function of ξ_0. With ξ_t and η_t both known as functions of ξ_0 and η_0, we are now able to tell which line of force a particle will be on at time t, if we know which line it was on at time t_0.

An alternative method of solution is to use Eq. (97), the flux-preservation condition. From Eq. (94), evaluated at $z = 0$,

$$\alpha(\xi, \eta, t) = \xi p(0, \xi)\cos\omega t. \tag{122}$$

Equation (106) then becomes

$$\cos\omega t\int_0^{\xi_t}\xi p(o, \xi)\,d\xi = \cos\omega t_0\int_0^{\xi_0}\xi p(o, \xi)\,d\xi, \tag{123}$$

using the fact that $\xi_t = 0$ when $\xi_0 = 0$. In view of Eq. (114), the left-hand side of Eq. (124) is equal to

$$\frac{1}{\omega}\cos\omega t\int_0^{\xi_t}\frac{\partial}{\partial\xi}[\xi q(o, \xi)]\,d\xi = \frac{1}{\omega}\cos\omega t\,\xi_t q(o, \xi_t), \tag{124}$$

with a similar relation for the right-hand side. Equation (121) follows immediately.

In situations where $\mathbf{v}_p$ is not line-preserving, the transverse motion will not be completely separable, as we have proved above. However, it may still be possible to separate one component of the transverse motion. In that case, it will be possible to tell which member of a certain family of surfaces a particle is on at any time, but it will not be possible to specify its location further without examining the longitudinal motion.

Suppose that one component of the transverse motion is separable. Then there exists a coordinate system (ξ, η, ζ) in which the equation of motion for ξ does not involve $\mathbf{v}_\parallel$, η, or ζ. As before, we write Eq. (89), observe that the third term on the right-hand side must vanish, and conclude that ξ must be constant along **H**-lines. Also, we must have

$$\frac{\partial}{\partial\eta}(\dot{\xi} + \mathbf{v}_p\cdot\nabla\xi) = \frac{\partial}{\partial\zeta}(\dot{\xi} + \mathbf{v}_p\cdot\nabla\xi) = 0. \tag{125}$$

Hence,

$$\begin{aligned}\mathbf{H}\cdot\nabla(\dot{\xi} + \mathbf{v}_p\cdot\nabla\xi) &= (\mathbf{H}\cdot\nabla\xi)\frac{\partial}{\partial\xi}(\dot{\xi} + \mathbf{v}_p\cdot\nabla\xi)\\ &\quad + (\mathbf{H}\cdot\nabla\eta)\frac{\partial}{\partial\eta}(\dot{\xi} + \mathbf{v}_p\cdot\nabla\xi) + (\mathbf{H}\cdot\nabla\zeta)\frac{\partial}{\partial\zeta}(\dot{\xi} + \mathbf{v}_p\cdot\nabla\xi) = 0.\end{aligned} \tag{126}$$

As before, we may conclude that

$$\begin{aligned}(\dot{\mathbf{H}} + [\mathbf{H}, \mathbf{v}_p])\cdot\nabla\xi &= [\mathbf{H}^{(1)} - \mathbf{H}(\nabla\cdot\mathbf{v}_p)]\cdot\nabla\xi\\ &= \mathbf{H}^{(1)}\cdot\nabla\xi = 0,\end{aligned} \tag{127}$$

so that ξ is constant along $\mathbf{H}^{(1)}$-lines. Thus, the surfaces of constant ξ are tangent to both **H** and $\mathbf{H}^{(1)}$, and $[\mathbf{H}, \mathbf{H}^{(1)}]$ must be a linear combination of **H** and $\mathbf{H}^{(1)}$.

If, in the above argument, we substitute $\mathbf{H}^{(1)}$ for **H** and $\mathbf{H}^{(2)}$ for $\mathbf{H}^{(1)}$, we may conclude that ξ is constant along $\mathbf{H}^{(2)}$-lines, i.e., that the surfaces of constant ξ are tangent to $\mathbf{H}^{(2)}$. As in the derivation of the condition for surface-preservation, it follows that $\mathbf{H}^{(2)}$ is a linear combination of **H** and $\mathbf{H}^{(1)}$. Thus, it is possible to separate one component of the transverse motion only if $\mathbf{H}^{(2)}$ and $[\mathbf{H}, \mathbf{H}^{(1)}]$ are both linear combinations of **H** and $\mathbf{H}^{(1)}$, i.e., only if $\mathbf{v}_p$ is surface-preserving. Conversely, if $\mathbf{v}_p$ is surface-preserving, we can use the invariant surfaces as surfaces of constant ξ in a coordinate system in which the ξ-motion is separable from the longitudinal motion and from the other component of the transverse motion.

When $\mathbf{v}_p$ is flux-preserving, we can derive a constant of the (ξ, η)-motion, given by Eq. (97). Similarly, in the case where only the ξ-motion is separable, we can derive a constant of the motion if the derivative space has dimension one, i.e.. if every surface tangent to $\mathbf{H}^{(1)}$ is a constant flux surface. This situation arises whenever the following conditions hold: (1) $\mathbf{H}^{(2)}$ is collinear with $\mathbf{H}^{(1)}$, but not with **H**. (2) [**H**, $\mathbf{H}^{(1)}$] is coplanar with **H** and $\mathbf{H}^{(1)}$. Under these circumstances, we can pick a constant flux surface π which intersects all the surfaces of constant ξ, designated by $\Sigma(\xi)$. Let $C(\xi)$ be the intersection of $\Sigma(\xi)$ with π, and let $\beta(\xi, t)\, d\xi$ be the magnetic flux between the surfaces $\Sigma(\xi)$ and $\Sigma(\xi + d\xi)$. Then $\beta(\xi, t)\, d\xi$ is also the flux between $C(\xi)$ and $C(\xi + d\xi)$. Since these curves lie in the constant flux surface π, we have

$$\beta(\xi_t, t)\, d\xi_t = \beta(\xi_0, t_0)\, d\xi_0, \tag{128}$$

or

$$\beta(\xi_t, t) \frac{\partial \xi_t}{\partial \xi_0} = \beta(\xi_0, t_0), \tag{129}$$

which is the constant of the motion referred to above.

If ξ_t is known as a function of t for the single value $\xi_0 = a$, Eq. (129) may be integrated to yield

$$\int_{\xi_t(a)}^{\xi_t} \beta(\xi, t)\, d\xi = \int_a^{\xi_0} \beta(\xi, t_0)\, d\xi. \tag{130}$$

Since the integrands are known functions of ξ, the coordinate ξ_t is now completely determined as a function of ξ_0 and t. We are therefore able to tell which invariant surface a particle is on at any time, if we know which invariant surface it was on initially.

To illustrate, we again consider the example given in connection with complete separability of the transverse motion, adding an electrostatic field $\mathbf{E}_s$ which has no θ component, and which is not perpendicular to **H**. This extra field will add a θ component to $\mathbf{v}_p$, and $\mathbf{v}_p$ will, in general, no longer be flux-preserving, or even line-preserving. Clearly, the vectors $\mathbf{H}^{(1)} = -\nabla \times [\mathbf{H}(\mathbf{E}_s \cdot \mathbf{H})/H^2]$, $\dot{\mathbf{H}}^{(1)}$, $\nabla \times (\mathbf{H}^{(1)} \times \mathbf{v}_p)$, and $\mathbf{H}^{(2)}$ will all be in the θ direction. Therefore, $\mathbf{v}_p$ is surface-preserving, with the surfaces generated by rotating lines of force about the z axis as invariant surfaces. Since $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ are collinear, the derivative space has dimension one, and any surface of revolution about the z axis is a constant flux surface.

Let us choose the plane $z = 0$ as a constant flux surface intersecting all the invariant surfaces. Then, if we define ξ and η as before,

$$\beta(\xi, t) = 2\pi\xi p(0, \xi) \cos \omega t. \tag{131}$$

Also, $\xi_t = 0$ if $\xi_0 = 0$, so that Eq. (130) reduces to

$$2\pi \cos \omega t \int_0^{\xi_t} \xi p(0, \xi)\, d\xi = 2\pi \cos \omega t_0 \int_0^{\xi_0} \xi p(0, \xi)\, d\xi, \tag{132}$$

which is identical with Eq. (123). Thus, addition of $\mathbf{E}_s$ does not affect the ξ-motion, which can be computed just as if $\mathbf{v}_p$ were flux-preserving. Of course, this fact could have been predicted beforehand, since the addition of $\mathbf{E}_s$ does not change the r and z components of $\mathbf{v}_p$. It does, however, change the θ component, so that η_t is no longer independent of t. In fact, it is no longer possible even to compute η_t independently of the longitudinal motion.

VIII. RELATIVISTIC INVARIANCE

The concept of a line of force is intrinsically nonrelativistic. The points of a line of force must all have the same time coordinate, but because of the relativity of simultaneity, this condition cannot be satisfied by the same line in every coordinate system. However, a moving line of force can be defined covariantly under certain conditions. Suppose there exists a doubly infinite family of two-dimensional surfaces Σ in space-time which intersect every space-like hyperplane π along lines of force in π, i.e., lines of force in the Lorentz frame in which the points of π all have the same time coordinate. Under these circumstances, the surfaces Σ, which we shall call covariant magnetic surfaces (c.m.s.), will appear in any coordinate system as surfaces traced out by moving lines of force.

A family of c.m.s. will exist if and only if $\mathbf{E} \cdot \mathbf{H}$ vanishes and, in that case, only if the lines of force are assigned the particle drift velocity $\mathbf{v}_p$ or some velocity differing from $\mathbf{v}_p$ only by a component along **H**. The scalar $\mathbf{E} \cdot \mathbf{H}$ is of course a relativistic invariant. To prove the "if" part of this statement, we look for vectors l_μ satisfying

$$F_{\mu\nu} l_\nu = 0, \tag{133}$$

where $F_{\mu\nu}$ is the relativistic electromagnetic field tensor. Such vectors will exist, since

$$\det (F_{\mu\nu}) = -(\mathbf{E} \cdot \mathbf{H})^2 = 0. \tag{134}$$

It is easy to verify that every minor determinant of $F_{\mu\nu}$ also contains $\mathbf{E} \cdot \mathbf{H}$, so that if $\mathbf{E} \cdot \mathbf{H}$ vanishes, the rank of $F_{\mu\nu}$ is at most two, and there are at least two vectors satisfying Eq. (133). Also, we can verify by examining the second order minor determinants that the rank of $F_{\mu\nu}$ cannot be less than two unless **E** and **H** both vanish identically, a possibility which we shall not consider. Therefore, $F_{\mu\nu}$ has rank two, and Eq. (133) has exactly two linearly independent solutions, l_μ^1 and l_μ^2.

The c.m.s. will be the integral surfaces of l_μ^1 and l_μ^2. To show that such surfaces exist, we write

$$F_{\mu\nu}[l^1, l^2]_\nu = F_{\mu\nu} l_\rho^2 \frac{\partial l_\nu^1}{\partial x_\rho} - F_{\mu\nu} l_\rho^1 \frac{\partial l_\nu^2}{\partial x_\rho}. \tag{135}$$

Differentiating Eq. (133),

$$F_{\mu\nu} \frac{\partial l_\nu}{\partial x_\rho} + \frac{\partial F_{\mu\nu}}{\partial x_\rho} l_\nu = 0, \tag{136}$$

which may be combined with Eq. (135) to yield

$$F_{\mu\nu}[l^1, l^2]_\nu = \frac{\partial F_{\mu\nu}}{\partial x_\rho}(l_\rho^1 l_\nu^2 - l_\rho^2 l_\nu^1) = \frac{1}{2}\frac{\partial F_{\mu\nu}}{\partial x_\rho}(l_\rho^1 l_\nu^2 - l_\rho^2 l_\nu^1) + \frac{1}{2}\frac{\partial F_{\mu\rho}}{\partial x_\nu}(l_\nu^1 l_\rho^2 - l_\nu^2 l_\rho^1), \tag{137}$$

where the indices ρ and ν have been interchanged in the second term of the right-hand side. Using the antisymmetric character of $F_{\mu\nu}$, and Maxwell's equation in covariant form:

$$\frac{\partial F_{\mu\nu}}{\partial x_\rho} + \frac{\partial F_{\rho\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\rho}}{\partial x_\mu} = 0, \tag{138}$$

we find that

$$\begin{aligned} F_{\mu\nu}[l^1, l^2]_\nu &= \frac{1}{2}\left(\frac{\partial F_{\mu\nu}}{\partial x_\rho} + \frac{\partial F_{\rho\mu}}{\partial x_\nu}\right)(l_\rho^1 l_\nu^2 - l_\rho^2 l_\nu^1) = -\frac{1}{2}\frac{\partial F_{\nu\rho}}{\partial x_\mu}(l_\rho^1 l_\nu^2 - l_\rho^2 l_\nu^1) \\ &= \frac{1}{2}F_{\nu\rho}\left(\frac{\partial l_\rho^1}{\partial x_\mu} l_\nu^2 - \frac{\partial l_\rho^2}{\partial x_\mu} l_\nu^1\right) = -\frac{1}{2}\frac{\partial l_\rho^1}{\partial x_\mu}(F_{\rho\nu} l_\nu^2) + \frac{1}{2}\frac{\partial l_\rho^2}{\partial x_\mu}(F_{\rho\nu} l_\nu^1) = 0. \end{aligned} \tag{139}$$

Thus, the commutator product of l_μ^1 and l_μ^2 also satisfies Eq. (133) and is therefore a linear combination of l_μ^1 and l_μ^2. This proves that a family of integral surfaces exists, and we must now show that these are also c.m.s.

Let π be a space-like hyperplane (s.l.h.), and choose a frame of reference in which every point of π has the same time coordinate. The magnetic field tensor in π is given by $H_{ij} = F_{ij}$, where the indices i, j run from 1 to 3. The magnetic field vector in π is

$$H_i = \tfrac{1}{2}\epsilon_{ijk} H_{jk}, \tag{140}$$

where ϵ_{ijk} is the unit alternating tensor. A vector dx_i is along a line of force if and only if $\mathbf{H} \times \mathbf{dx}$ vanishes, i.e., if and only if

$$\begin{aligned} \epsilon_{ijk} H_j\, dx_k &= \tfrac{1}{2}\epsilon_{ijk}\epsilon_{jlm} H_{lm}\, dx_k \\ &= \tfrac{1}{2}(\delta_{kl}\delta_{im} - \delta_{km}\delta_{il}) H_{lm}\, dx_k \\ &= H_{ik}\, dx_k = 0, \end{aligned} \tag{141}$$

using the antisymmetric character of H_{ik}.

Let Σ be one of the integral surfaces mentioned above, and let dx_μ be a four-vector in the line of intersection of Σ with π. Then $dx_4 = 0$, since dx_μ is in π, and

$$H_{ik}\, dx_k = F_{ik}\, dx_k = F_{i\mu}\, dx_\mu = 0, \tag{142}$$

since dx_μ is in Σ. Hence, Σ is a covariant magnetic surface.

We now assume the existence of a family of c.m.s., and prove that $\mathbf{E}\cdot\mathbf{H}$

vanishes. Let Σ be a c.m.s., and let π be an s.l.h., both passing through an arbitrarily chosen point P, and let dx_μ be a vector along the intersection of Σ with π. As before, we pick a coordinate system in which every point of π has the same time coordinate. Then $dx_4 = 0$, and since Σ is a c.m.s., dx_i is along a line of force in π. Therefore,

$$F_{\mu\nu}\, dx_\nu = F_{\mu j}\, dx_j = \begin{cases} H_{ij}\, dx_j = 0, & \text{for } \mu = i \\ iE_j\, dx_j, & \text{for } \mu = 4, \end{cases} \tag{143}$$

and the vector $F_{\mu\nu}\, dx_\nu$ is perpendicular to π. But this is true for any π belonging to the doubly infinite family of s.l.h. intersecting Σ along dx_μ. Hence $F_{\mu\nu}\, dx_\nu = 0$, which is possible only if $\det(F_{\mu\nu}) = -(\mathbf{E}\cdot\mathbf{H})^2$ vanishes.

Any vector dx_μ in a c.m.s. Σ can be interpreted as the proper velocity of a point of the magnetic field. The three-dimensional velocity in a particular frame of reference is then given by $v_i = i\, dx_i/dx_4$. Using the fact that $\mathbf{E}\cdot\mathbf{H} = 0$, Eq. (133) may readily be solved for dx_μ, and hence for v_i. This velocity turns out to be the particle drift velocity $(\mathbf{E} \times \mathbf{H})/H^2$ plus an arbitrary component along $\mathbf{H}$. It will be less than or greater than the velocity of light according as the invariant $F_{\mu\nu}F_{\mu\nu} = 2(H^2 - E^2)$ is positive or negative. In the latter case, the c.m.s. will be space-like surfaces, and the lines of force will appear to move backwards in time in some frames of reference.

The condition $\mathbf{E}\cdot\mathbf{H} = 0$ for the existence of a family of c.m.s. is the same as the condition given in Section III for the existence of a field velocity satisfying statement VC4′. The velocity involved is also the same, i.e., the particle drift velocity, if we restrict ourselves to velocities perpendicular to the magnetic field.

We may define a covariant magnetic hypersurface (c.m.h.) as a hypersurface in space-time which intersects every s.l.h. π on a tube of force in π. It would seem offhand that the existence of a singly infinite family of c.m.h. is a weaker condition than the existence of a doubly infinite family of c.m.s. However, it can be shown that the existence of a family of c.m.h. implies that $\det(F_{\mu\nu})$ vanishes, and hence that a family of c.m.s. also exists.

APPENDIX

The condition for the existence of integral surfaces, which was stated without complete proof in Section II, is given by the following theorem, which we shall now prove: Two linearly independent vector fields $\tilde{A}$ and $\tilde{B}$ in a space R_n of dimension n will generate a family of integral surfaces if and only if the commutator product $[\tilde{A}, \tilde{B}]$ is a linear combination of $\tilde{A}$ and $\tilde{B}$.

To carry out the proof, we choose an $(n - 1)$-dimensional subspace R_{n-1} nowhere tangent to $\tilde{A}$ and everywhere tangent to $\tilde{B}$. For any $\tilde{B}$-line b in R_{n-1}, let $\Sigma(b)$ be the surface generated by the $\tilde{A}$-lines through b. (See Fig. 3.)

We assume first that $[\tilde{A}, \tilde{B}]$ is a linear combination of $\tilde{A}$ and $\tilde{B}$, and prove that the surfaces $\Sigma(b)$ are integral surfaces of $\tilde{A}$ and $\tilde{B}$. Clearly, they are tangent to

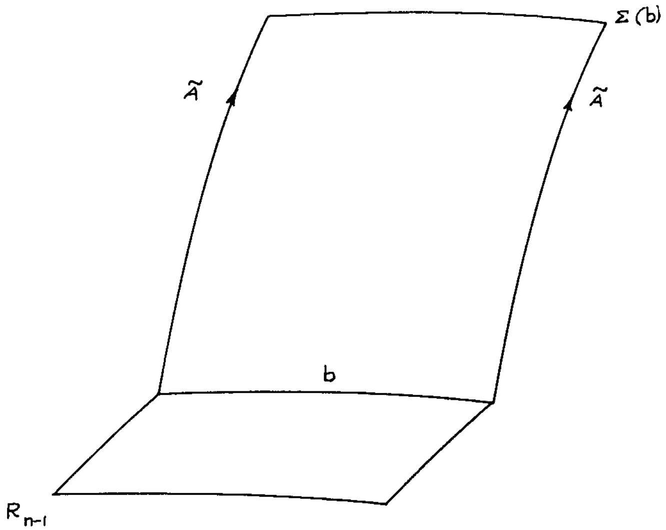

FIG. 3

Ã, since they are generated by Ã-lines. To show that they are also tangent to B̃, we introduce a curvilinear coordinate system $\{x_1 \cdots x_n\}$ such that the $(n - 2)$ coordinates $x_3 \cdots x_n$ are constant on each of the surfaces $\Sigma(b)$.

The standard way of defining the components A_i of a vector Ã in such a coordinate system is to introduce local base vectors $\tilde{e}_i$ at each point satisfying

$$\tilde{e}_i \cdot \tilde{\nabla}\phi = \partial\phi/\partial x_i, \tag{A1}$$

where ϕ is an arbitrary scalar function of position. These base vectors have the property that in the displacement defined by $\tilde{e}_i$, x_i changes by unity while all the other coordinates remain constant. Of course, the $\tilde{e}_i$ need not be orthogonal or of unit length. In fact, any statement concerning their orthogonality or length would be meaningless, since we have not introduced a metric tensor for R_n. [A metric is not needed for the dot product in Eq. (A1), since $\tilde{e}_i$ and $\tilde{\nabla}\phi$ are contravariant and covariant vectors, respectively.] Every contravariant vector Ã may now be written in the form $A_i\tilde{e}_i$. The components of [Ã, B̃], as defined in this way, are given by the right-hand side of Eq. (2).

We now compute the change in B̃ resulting from an infinitesimal displacement $\tilde{A}\delta$ along an Ã-line.

$$\begin{aligned} dB_i &= A_j \frac{\partial B_i}{\partial x_j}\delta = \left(A_j \frac{\partial B_i}{\partial x_j} - B_j \frac{\partial A_i}{\partial x_j} + B_j \frac{\partial A_i}{\partial x_j}\right)\delta \\ &= \left(-[\tilde{A}, \tilde{B}]_i + B_j \frac{\partial A_i}{\partial x_j}\right)\delta. \end{aligned} \tag{A2}$$

Since [Ã, B̃] is a linear combination of Ã and B̃, this reduces to

$$dB_i = \left(fA_i + gB_i + B_j \frac{\partial A_i}{\partial x_j}\right)\delta, \tag{A3}$$

where f and g are scalar functions of position. If $\alpha > 2$, A_α will vanish, so that Eq. (A3) becomes

$$dB_\alpha = gB_\alpha\delta. \tag{A4}$$

Since B_α vanishes on R_{n-1}, Eq. (A4) implies that B_α must vanish everywhere. The vector field B̃ is therefore tangent to $\Sigma(b)$, so that $\Sigma(b)$ is an integral surface of Ã and B̃.

To prove the converse, we assume that there exists an integral surface through the B̃-line b. This integral surface must clearly be $\Sigma(b)$, since the latter is generated by the Ã-lines through b. Equation (A2) may be derived as before, and yields

$$dB_\alpha = \left(-[\tilde{A}, \tilde{B}]_\alpha + B_j \frac{\partial A_\alpha}{\partial x_j}\right)\delta. \tag{A5}$$

Since A_α and B_α both vanish for $\alpha > 2$, $[\tilde{A}, \tilde{B}]_\alpha$ must also vanish, so that [Ã, B̃] is tangent to $\Sigma(b)$, i.e., a linear combination of Ã and B̃.

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