ANNALS OF PHYSICS: **28**, 225–319 (1964)

A Covariant Formulation of Classical Electrodynamics for Charges of Finite Extension

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A covariant formulation of classical electrodynamics for charges of finite extension is developed. The nonelectromagnetic forces necessary for stability are taken into account implicitly by requiring that the charge distribution retain a given shape throughout the course of its motion; a general prescription is given for constructing charge-current densities appropriate to such rigid charge distributions. A detailed treatment of the relativistic kinematics of rotating rigid bodies is presented and application is then made to obtain, from an action principle, the equations of motion for a spherically symmetric charge distribution interacting with an applied field; the associated conservation theorems relating to the linear and angular momentum are discussed. Finally, approximations to the equations of motion are obtained by making perturbation expansions in powers of the size of the charge distribution.

I. INTRODUCTION

It is well known that the concept of a stable charge held together solely by the action of electromagnetic forces is inconsistent with classical Maxwellian electrodynamics. Within the classical framework the instability problem can be treated in one of two ways. The first approach involves the explicit introduction of other fields to provide the component of attraction needed to balance out the Coulomb repulsion between elements of the charge. The second approach is phenomenological, being based on an assumption similar to that underlying ordinary rigid body mechanics: one simply assumes that the charge is rigidly held together in some prescribed shape by nonelectromagnetic forces whose precise nature is left unspecified. Although this problem has had a long history,[1] a complete relativistic formulation with the second approach does not seem to appear in the literature.

The purpose of this article, then, is to present a Lorentz-covariant formula-

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† Supported in part by the U. S. Atomic Energy Commission.

‡ Supported in part by the Company Independent Research Program of TRW Space Technology Laboratories, Inc.

[1] See, for example, ref. *1*.

tion of classical electrodynamics for rigid charges of finite extension. This formulation is characterized by the following restrictions and assumptions: (1) It is assumed that the classical Maxwell equations are applicable under all circumstances. (2) Quantum effects are not considered—the formulation is strictly classical. (3) The development is confined within the framework of the special theory of relativity; in this connection it is unlikely that the formulation given here can be extended to the general theory, at least not without a drastic revision.

Sections II and III deal with the construction of covariant charge-current densities for rigid charge distributions. Section IV presents a detailed discussion of the relativistic kinematics of a rotating rigid body. The derivation of equations of motion from an action principle is given in Section V and the associated conservation laws are considered in Section VI. Finally, Section VII deals with perturbation expansions in powers of the size of the charge distribution.

II. CONSTRUCTION OF COVARIANT CHARGE-CURRENT DENSITIES FOR SPHERICALLY SYMMETRIC DISTRIBUTIONS NEGLECTING EFFECTS OF SPATIAL ROTATION

In this section we are primarily interested in illustrating the main ideas involved and consequently the discussion presented here is not as elegant nor as general as it could be. In particular, we shall restrict ourselves to the case of a spherically symmetric charge distribution and neglect the effects of spatial rotation. A general prescription for obtaining covariant charge-current densities applicable to any shape and including the effects of spatial rotation is given in the next section.

The charge-current density appropriate to a spherically symmetric charge distribution will be formulated in terms of the motion of the center of the charge distribution according to the following prescription: *In the sequence of inertial frames in which the charge center is instantaneously at rest, each frame in the sequence being related to the laboratory frame by a Lorentz transformation,*[2] *the size and shape of the charge distribution remain the same throughout the entire history of the motion of the charge.* It now remains to translate this prescription into a mathematical expression which is manifestly covariant.

Suppose that, in some given (inertial) laboratory frame S, the center of the charge distribution follows a path given by[3]

$$x_i = z_i(t) \tag{2.1}$$

[2] We use the term "Lorentz transformation" in its narrowest sense to denote a rotation in 4-space involving the time axis and one space axis.

[3] Roman indices run 1 to 3, Greek indices run 1 to 4; we will use the summation convention throughout.

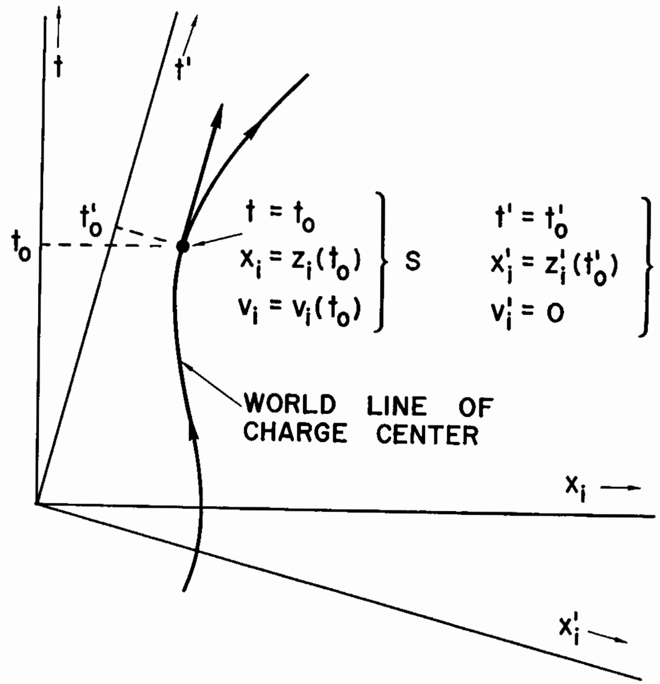

FIG. 1. World line of charge center

The only restriction on this path is that the magnitude of the associated velocity remain less than c. With reference[4] to Fig. 1, at $t = t_0$ the charge center is at position $z_i(t_0)$ in S and has velocity

$$v_i(t_0) = \left.\frac{dz_i(t)}{dt}\right|_{t=t_0}$$

Let S' be an inertial frame in which the charge center is instantaneously at rest at the corresponding point $t' = t_0'$, $x_i' = z_i'(t_0')$, with S' related to S by a Lorentz transformation associated with the velocity $v_i(t_0)$:

$$x_i' = x_i + \left[\frac{\gamma(t_0) - 1}{v_j(t_0)v_j(t_0)}\right] v_i(t_0)v_k(t_0)x_k - \gamma(t_0)v_i(t_0)t \tag{2.2}$$

$$t' = \gamma(t_0)\left[t - \frac{1}{c^2} v_k(t_0)x_k\right] \tag{2.3}$$

where

[4] The ideas expressed in the first several figures were first given by Fermi (*2*); the author is indebted to Prof. Rohrlich for bringing this to his attention.

$$\gamma(t_0) = \left[1 - \frac{1}{c^2} v_j(t_0) v_j(t_0)\right]^{-1/2}$$

Assume that for the special case in which the charge center is at rest at the origin in S, the charge-current density is given by

$$\left.\begin{aligned} j_\mu &= [0, ic\rho] \\ \rho &= ef(x_i x_i) = ef(r^2) \end{aligned}\right\} \tag{2.4}$$

Then in the case being considered here, in accordance with the prescription given previously, the charge-current density in S' on the hyperplane $t' = t_0'$ is given by

$$\left.\begin{aligned} j_\mu' &= [0, ic\rho'] \\ \rho' &= ef([x_i' - z_i'(t_0')][x_i' - z_i'(t_0')]) \\ &= ef([x_i' - z_i'(t')][x_i' - z_i'(t')])\,|_{t'=t_0'} \end{aligned}\right\} \tag{2.5}$$

where, from (2.2) and (2.3)

$$z_i'(t_0') = z_i(t_0) + \left[\frac{\gamma(t_0) - 1}{v_j(t_0)v_j(t_0)}\right] v_i(t_0)v_k(t_0)z_k(t_0) - \gamma(t_0)v_i(t_0)t_0 \tag{2.6}$$

$$t_0' = \gamma(t_0)\left[t_0 - \frac{1}{c^2} v_k(t_0) z_k(t_0)\right] \tag{2.7}$$

We next transform (2.5) back to S to obtain $j_\mu = [j_i, ic\rho]$. Applying the transformation corresponding to (2.2) and (2.3) for the 4-vector j_μ,

$$j_i = j_i' + \left[\frac{\gamma(t_0) - 1}{v_j(t_0)v_j(t_0)}\right] v_i(t_0)v_k(t_0)j_k' + \gamma(t_0)v_i(t_0)\rho' \tag{2.8}$$

$$\rho = \gamma(t_0)\left[\rho' + \frac{1}{c^2} v_k(t_0) j_k'\right] \tag{2.9}$$

we obtain

$$j_\mu = \{e[\gamma(t_0)v_i(t_0), ic\gamma(t_0)]f([x_j' - z_j'(t')][x_j' - z_j'(t')])\}\,|_{t'=t_0'} \tag{2.10}$$

From Eqs. (2.3) and (2.7), the relation $t' = t_0'$ is equivalent to

$$\gamma(t_0)\left[t - \frac{1}{c^2} v_k(t_0) x_k\right] = \gamma(t_0)\left[t_0 - \frac{1}{c^2} v_k(t_0) z_k(t_0)\right] \tag{2.11}$$

this is, t_0 is a solution of

$$t_0 = t - \frac{1}{c^2} v_k(t_0)[x_k - z_k(t_0)] \tag{2.12}$$

Using (2.2), (2.6), and (2.12), we find

$$\begin{aligned} &f([x_j' - z_j(t')][x_j' - z_j'(t')])\,|_{t'=t_0'} \\ &\quad = f([x_j - z_j(t_0)][x_j - z_j(t_0)] - c^2[t - t_0]^2)\,|_{t_0 = t - (1/c^2)v_k(t_0)[x_k - z_k(t_0)]} \end{aligned} \tag{2.13}$$

Finally, substituting (2.13) into (2.10) gives, for the charge-current density in S,

$$\begin{aligned} j_\mu &= [j_i, ic\rho] \\ &= \{e[\gamma(t_0)v_i(t_0), ic\gamma(t_0)] \\ &\quad \cdot f([x_j - z_j(t_0)][x_j - z_j(t_0)] - c^2[t - t_0]^2)\}\,|_{t_0 = t - (1/c^2)v_k(t_0)[x_k - z_k(t_0)]} \end{aligned} \tag{2.14}$$

Before proceeding, we will elaborate briefly on the preceding result. If we assume for the moment that the form factor $f(r^2)$ vanishes for $r > b$, say, and that the acceleration of the charge center is small, then the history of the charge distribution in S will appear as shown in Fig. 2. For a given value of t_0, Eq. (2.12) determines a hyperplane which is perpendicular to the world line of the charge center. For lack of better terminology, we refer to this hyperplane as a "t_0-hyperplane." As t_0 ranges from $-\infty$ to $+\infty$, a band of world lines, centered about that of the charge center, is swept out. On a given t_0-hyperplane, the expression for the charge-current density has the simple form given by (2.5).

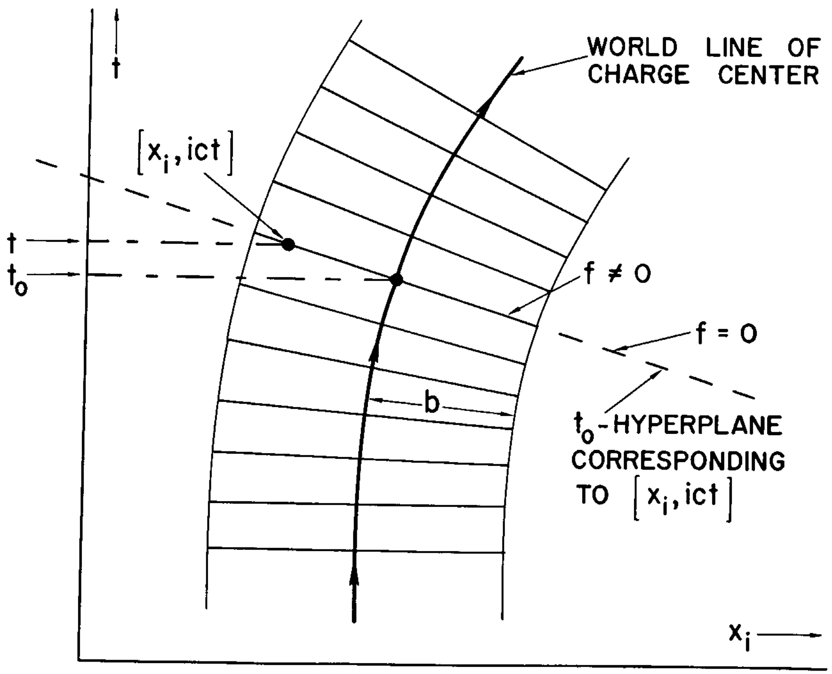

FIG. 2. World history of a spherically symmetric, rigid charge distribution

Using relations of the form of (D.82) now leads to Eqs. (7.42) and (7.43) for $\Sigma_{\mu\nu}^{(+)}$ and $\Sigma_{\mu\nu}^{(-)}$.

APPENDIX E

In this appendix we shall calculate the energy-momentum radiated by a charge distribution characterized by the charge-current density given by (4.36). The calculation will be carried out to second order in the parameter λ.

First of all we require the appropriate second order expression for the potential $A_\mu^{(\mathrm{self})}(x)$. Using Eqs. (D.15) and (7.16) we obtain

$$A_\mu^{(\mathrm{self})}(x) = 2e\int_{-\infty}^{\infty} ds' \int_{-\infty}^{\infty} d^4\xi'\theta(v_\rho'(\lambda\xi_\rho' + z_\rho' - x_\rho))\delta((x - z' - \lambda\xi')^2) \cdot \left[v_\mu' - \frac{1}{c}(\eta_{\mu\nu}' + \omega_{\mu\nu}')\xi_\nu'\right] f(\xi'^2)\delta(v_\alpha'\xi_\alpha') \tag{E.1}$$

where the expansion parameter λ has been introduced in a manner appropriate for obtaining an expansion valid at points x_μ outside the charge distribution. Expanding the integrand of (E.1) we obtain

$$\begin{aligned} A_\mu^{(\mathrm{self})}(x) = 2e\int_{-\infty}^{\infty} ds'\theta(v_\rho'(z_\rho' - x_\rho)) \int_{-\infty}^{\infty} d^4\xi' f(\xi'^2)\delta(v_\alpha'\xi_\alpha') \\ \cdot\left\{v_\mu' - \frac{1}{c}(\eta_{\mu\nu}' + \omega_{\mu\nu}')\xi_\nu'\right\} \\ \cdot\{\delta((x - z')^2) - 2\lambda\xi_\sigma'(x_\sigma - z_\sigma')\delta'((x - z')^2) + \lambda^2\xi_\sigma'\xi_\beta' \\ \cdot[\delta_{\sigma\beta}\delta'((x - z')^2) + 2(x_\sigma - z_\sigma')(x_\beta - z_\beta')\delta''((x - z')^2)] + O(\lambda^3)\} \end{aligned} \tag{E.2}$$

where we have used the relation

$$\theta(v_\rho'(\lambda\xi_\rho' + z_\rho' - x_\rho)) = \theta(v_\rho'(z_\rho' - x_\rho)) \tag{E.3}$$

which is valid if no differential pair creation takes place. The terms in (E.2) which are odd in λ are also odd in ξ_μ' and these integrate to zero. Using (D.5) then leads to

$$\begin{aligned} A_\mu^{(\mathrm{self})}(x) = 2e\int_{-\infty}^{\infty} ds' v_\mu'\theta(v_\rho'(z_\rho' - x_\rho))\delta((x - z')^2) \\ + \lambda^2\cdot 2e\langle r^2\rangle \int_{-\infty}^{\infty} ds'\theta(v_\rho'(z_\rho' - x_\rho))\Big\{v_\mu'\delta'((x - z')^2) \\ + \tfrac{2}{3}v_\mu'[(x - z')^2 + (v_\sigma'(x_\sigma - z_\sigma'))^2]\delta''((x - z')^2) \\ + \frac{2}{3}\frac{1}{c}(\eta_{\mu\nu}' + \omega_{\mu\nu}')[(x_\nu - z_\nu') + v_\nu' v_\sigma'(x_\sigma - z_\sigma')]\delta'((x - z')^2)\Big\} + O(\lambda^4) \end{aligned} \tag{E.4}$$

Making use of the relation $x\delta''(x) = -2\delta'(x)$ and dropping the prime on the integration variable s', we may write (E.4) in the form

$$\begin{aligned} A_\mu^{(\mathrm{self})}(x) = 2e\int_{-\infty}^{\infty} ds v_\mu\theta(v_\rho(z_\rho - x_\rho))\delta((x - z)^2) \\ + \lambda^2\cdot 2e\langle r^2\rangle \int_{-\infty}^{\infty} ds\theta(v_\rho(z_\rho - x_\rho))\left\{\frac{1}{6}\frac{d}{ds}\left[v_\mu \frac{d}{ds}\delta((x - z)^2)\right]\right. \\ \left. + \frac{2}{3}\frac{1}{c}\left(\frac{1}{2}\eta_{\mu\nu} + \omega_{\mu\nu}\right)(x_\nu - z_\nu)\delta'((x - z)^2)\right\} + O(\lambda^4) \end{aligned} \tag{E.5}$$

The first term in the λ^2 integral on the right of (E.5) integrates to zero and we are left with

$$\begin{aligned} A_\mu^{(\mathrm{self})}(x) = 2e\int_{-\infty}^{\infty} ds v_\mu\theta(v_\rho(z_\rho - x_\rho))\delta((x - z)^2) \\ + 4\lambda^2 \int_{-\infty}^{\infty} ds[\mu_{\mu\nu}(s) + \mu_{\mu\nu}'(s)]\theta(v_\rho(z_\rho - x_\rho))\delta'((x - z)^2) + O(\lambda^4) \end{aligned} \tag{E.6}$$

where the magnetic and electric dipole moment tensors $\mu_{\mu\nu}$ and $\mu_{\mu\nu}'$ are defined by (7.32) and (7.33), respectively. The potential given by (E.6) is that which would result from a charge-current density appropriate to a point charge e and a point dipole moment $(\mu_{\mu\nu} + \mu_{\mu\nu}')$:

$$j_\mu(x) = ec\int_{-\infty}^{\infty} ds v_\mu\delta^4(x - z) + \lambda^2 c\frac{\partial}{\partial x_\nu}\int_{-\infty}^{\infty} ds[\mu_{\mu\nu}(s) + \mu_{\mu\nu}'(s)]\delta^4(x - z) + O(\lambda^4) \tag{E.7}$$

We shall denote the combined dipole moment by $m_{\mu\nu}$,

$$m_{\mu\nu} = \mu_{\mu\nu} + \mu_{\mu\nu}' \tag{E.8}$$

and the value of s corresponding to the retarded time by s_0; the latter satisfies the relation

$$(x - z(s_0))^2 = 0 \tag{E.9}$$

Finally, upon making use of (E.9) in (E.6), we obtain the desired second order expression,

$$A_\mu^{(\mathrm{self})}(x) = \left\{-\frac{1}{[v_\rho(x_\rho - z_\rho)]}\left[ev_\mu + \lambda^2\frac{d}{ds}\left(\frac{m_{\mu\nu}(x_\nu - z_\nu)}{v_\sigma(x_\sigma - z_\sigma)}\right)\right]\right\}_{s=s_0} + O(\lambda^4) \tag{E.10}$$

In calculating the radiation, we find it easiest to work in a definite frame. The usual expression for the energy radiated per unit time into the solid angle

$d\Omega$ in the direction $\mathbf{n} = \mathbf{R}/R$, where $\mathbf{R} = \mathbf{r} - \mathbf{z}$, is given by

$$\frac{dW}{d\Omega\, dt} = \lim_{R\to\infty} \left\{ \left| \frac{1}{4\pi c} \right| R\mathbf{n} \times \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \right|^2 \right\}_{t=t_{\text{ret}}} \tag{E.11}$$

where t_{ret} is the retarded time associated with s_0. For $R \to \infty$, we have

$$x_\mu - z_\mu(s_0) \xrightarrow[R\to\infty]{} Rn_\mu \tag{E.12}$$

where

$$n_\mu = [\mathbf{n}, i] \tag{E.13}$$

It should be noted that n_μ does not transform like a 4-vector.

By virtue of the form of the right-hand side of (E.10) the partial derivative appearing on the right of (E.11) should be written in the form

$$\frac{\partial}{\partial t} = ic \frac{\partial}{\partial x_4} - \frac{c}{(v_\rho n_\rho)} \frac{d}{ds_0} \tag{E.14}$$

Operating on the right of (E.10) with $\partial/\partial x_4$ leads to terms of order $1/R^2$ or higher and we have effectively

$$\frac{\partial}{\partial t} \xrightarrow[R\to\infty]{} -\frac{c}{(v_\rho n_\rho)} \frac{d}{ds_0} \tag{E.15}$$

Defining $\alpha_\mu(s)$ by the relation

$$A_\mu^{(\text{self})}(x) \xrightarrow[R\to\infty]{} \frac{\alpha_\mu(s)}{R} \bigg|_{s=s_0} \tag{E.16}$$

we obtain

$$\frac{dW}{d\Omega\, dt} = \frac{c}{4\pi} \frac{\dot\alpha_\mu \dot\alpha_\mu}{(v_\rho n_\rho)^2} \bigg|_{s=s_0} \tag{E.17}$$

or

$$\frac{dW}{d\Omega\, ds_0} = -\frac{1}{4\pi} \frac{\dot\alpha_\mu \dot\alpha_\mu}{v_\rho n_\rho} \bigg|_{s=s_0} \tag{E.18}$$

where, on the left of (E.17), we have used

$$\frac{d}{dt} = -\frac{c}{(v_\rho n_\rho)} \frac{d}{ds_0} \tag{E.19}$$

Finally, dropping the subscript on s_0, we obtain for the energy radiated per path length

$$\frac{dW}{ds} = -\frac{1}{4\pi} \int d\Omega \frac{\dot\alpha_\mu \dot\alpha_\mu}{v_\rho n_\rho} \tag{E.20}$$

In our case, applying (E.16) to (E.10) gives

$$\alpha_\mu = -\frac{ev_\mu}{v_\rho n_\rho} + \lambda^2 \left[\frac{(\dot v_\sigma n_\sigma) m_{\mu\nu} n_\nu}{(v_\rho n_\rho)^3} - \frac{\dot m_{\mu\nu} n_\nu}{(v_\rho n_\rho)^2} \right] + O(\lambda^4) \tag{E.21}$$

When differentiating (E.21) n_μ is to be regarded as constant.

If G_μ is the 4-momentum radiated per path length then we must have

$$\begin{aligned} G_4 = \frac{i}{c}\frac{dW}{ds} &= \frac{ie^2}{4\pi c} (\dot v_\mu \dot v_\nu - \dot v^2 v_\mu v_\nu) \int d\Omega \frac{n_\mu n_\nu}{(v_\rho n_\rho)^5} \\ &\quad + \lambda^2 \cdot \frac{2ie}{4\pi c} (\dot v_\mu v_\sigma - \dot v_\sigma v_\mu) \int d\Omega \\ &\quad \cdot \left[\frac{\ddot m_{\sigma\nu}}{(v_\rho n_\rho)^5} - \frac{3(\dot v_\lambda n_\lambda)\dot m_{\sigma\nu}}{(v_\rho n_\rho)^6} + \frac{3(\dot v_\lambda n_\lambda)^2 m_{\sigma\nu}}{(v_\rho n_\rho)^7} - \frac{(\ddot v_\lambda n_\lambda) m_{\sigma\nu}}{(v_\rho n_\rho)^6} \right] n_\mu n_\nu \end{aligned} \tag{E.22}$$

where we have substituted (E.21) into (E.20) and dropped terms higher than second order. It may be noted that the term of order λ^2 is the interference term between the point charge contribution and the dipole contribution to $\dot\alpha_\mu$. All the integrals in (E.22) may be evaluated in a straightforward manner by starting with the integral

$$J = \frac{i}{4\pi} \int \frac{d\Omega}{(v_\rho n_\rho)^3} = -\frac{v_4}{(v_\rho v_\rho)^2} \tag{E.23}$$

and then differentiating with respect to v_μ an appropriate number of times; for example, we have

$$\begin{aligned} \frac{i}{4\pi} \int d\Omega \frac{n_\mu n_\nu}{(v_\rho n_\rho)^5} &= \frac{1}{12} \frac{\partial^2 J}{\partial v_\mu \partial v_\nu} \\ &= \frac{1}{3(v_\rho v_\rho)^3} [v_\mu \delta_{\nu 4} + v_\nu \delta_{\mu 4} + v_4 \delta_{\mu\nu}] - \frac{2v_\mu v_\nu v_4}{(v_\rho v_\rho)^4} \\ &= -\tfrac{1}{3}[v_\mu \delta_{\nu 4} + v_\nu \delta_{\mu 4} + v_4 \delta_{\mu\nu} + 6v_\mu v_\nu v_4] \end{aligned} \tag{E.24}$$

We thus obtain

$$\begin{aligned} G_4 = \frac{2}{3}\frac{e^2}{c} \dot v^2 v_4 + \lambda^2 \cdot \frac{2}{15}\frac{e}{c} [&5\ddot m_{4\nu} \dot v_\nu + 3\dot v^2 \dot m_{4\nu} v_\nu - 2\dot v^2 m_{4\nu} \dot v_\nu \\ &+ (\dot v_\sigma \ddot v_\sigma) m_{4\nu} v_\nu - 4(\dot v_\sigma m_{\sigma\nu} v_\nu)\ddot v_4 + (\ddot v_\sigma m_{\sigma\nu} v_\nu - 9\dot v_\sigma \dot m_{\sigma\nu} v_\nu)\dot v_4 \\ &+ 5(\ddot v_\sigma m_{\sigma\nu} \dot v_\nu + 3\dot v^2 \dot v_\sigma m_{\sigma\nu} v_\nu - 3\dot v_\sigma \ddot m_{\sigma\nu} v_\nu) v_4] \end{aligned} \tag{E.25}$$

Replacing the subscript 4 by the subscript μ in (E.25) then gives the expression for G_μ, the 4-momentum radiated per path length. The resulting expression for G_μ may be written in the form

$$
\begin{aligned}
G_\mu = & -\Big\{\frac{2}{3}\frac{e^2}{c}[\ddot{v}_\mu - \dot{v}^2 v_\mu] + \lambda^2 \cdot \frac{2}{3}\frac{e}{c}\Big[-2\frac{d^3 m_{\mu\nu}}{ds^3}v_\nu + 2\dot{v}^2\dot{m}_{\mu\nu}v_\nu \\
& + 5(\dot{v}_\sigma\ddot{v}_\sigma)m_{\mu\nu}v_\nu + \dot{v}_\mu\dot{v}_\sigma\dot{m}_{\sigma\nu}v_\nu - \left(m_{\mu\nu}\frac{d^3 v_\nu}{ds^3} + v_\mu v_\sigma m_{\sigma\nu}\frac{d^3 v_\nu}{ds^3}\right) \\
& - (\dot{m}_{\mu\nu}\ddot{v}_\nu + v_\mu v_\sigma \dot{m}_{\sigma\nu}\ddot{v}_\nu) - 3(\ddot{m}_{\mu\nu}\dot{v}_\nu + v_\mu v_\sigma \ddot{m}_{\sigma\nu}\dot{v}_\nu) \\
& + 3\dot{v}^2(m_{\mu\nu}\dot{v}_\nu + v_\mu v_\sigma m_{\sigma\nu}\dot{v}_\nu)\Big]\Big\} \\
& + \frac{d}{ds}\Big\{\frac{2}{3}\frac{e^2}{c}\dot{v}_\mu + \lambda^2\cdot\frac{2}{3}\frac{e}{c}\Big[-(m_{\mu\nu}\ddot{v}_\nu + v_\mu v_\sigma m_{\sigma\nu}\ddot{v}_\nu) - 2\ddot{m}_{\mu\nu}v_\nu \\
& + \frac{13}{5}\dot{v}^2 m_{\mu\nu}v_\nu - \frac{4}{5}\dot{v}_\mu\dot{v}_\sigma m_{\sigma\nu}v_\nu\Big]\Big\}
\end{aligned} \qquad \text{(E.26)}
$$

To obtain the reaction force, $f_\mu^{(\text{reac})}$, we postulate the relation

$$\frac{1}{c} f_\mu^{(\text{reac})} = -G_\mu + \frac{dH_\mu}{ds} \qquad \text{(E.27)}$$

where H_μ is to be determined by the requirement

$$v_\mu f_\mu^{(\text{reac})} = 0 \qquad \text{(E.28)}$$

Now the expression (E.26) has been written in such a way that contracting the first curly bracket on the right with v_μ will lead to a null result. Hence the curly bracket following the operator d/ds on the right of (E.26) represents one solution for H_μ. The contents of the first curly bracket on the right of (E.26) should then be $(1/c)f_\mu^{(\text{reac})}$; however, when we write this result for $f_\mu^{(\text{reac})}$ in the form

$$f_\mu^{(\text{reac})} = eF_{\mu\nu}^{(\text{rad})}v_\nu \qquad \text{(E.29)}$$

we find that $F_{\mu\nu}^{(\text{rad})}$ thus obtained does not agree with (7.35). The explanation for this discrepancy is as follows. If we define $\tilde{m}_{\mu\nu}$ by the relation

$$\tilde{m}_{\mu\nu} = m_{\mu\nu} - v_\mu m_{\nu\lambda}v_\lambda + v_\nu m_{\mu\lambda}v_\lambda \qquad \text{(E.30)}$$

then $\tilde{m}_{\mu\nu}$ satisfies

$$v_\mu \frac{d}{ds}[\dot{\tilde{m}}_{\mu\nu}\dot{v}_\nu] = 0 \qquad \text{(E.31)}$$

Hence a term $k[\dot{\tilde{m}}_{\mu\nu}\dot{v}_\nu] = k[\dot{\mu}_{\mu\nu}\dot{v}_\nu]$, with k arbitrary, can always be added to H_μ. In the present case we must choose $k = -2\lambda^2 e/c$ in order to obtain agreement with (7.35). This means, then, that a knowledge of G_μ together with the relations (E.27) and (E.28) does not suffice to determine $f_\mu^{(\text{reac})}$ uniquely.

ACKNOWLEDGMENT

The author wishes to thank Dr. H. C. Corben for many stimulating and interesting discussions.

RECEIVED: January 8, 1964

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