External Field of an Ideal Toroid

CHESTER H. PAGE
Institute for Basic Standards
*National Bureau of Standards**
Electricity Division
Washington, D. C. 20234
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The external magnetic field of an ideal toroid vanishes for dc excitation but not for ac excitation. The external static field has a nonvanishing vector potential; the emf in a transformer winding is due to the time variation of this vector potential. An expression for the external magnetic field is derived, and it is shown that the external flux density is a negligible fraction of the flux density in the core.

INTRODUCTION

The external magnetic field of an ideal toroid vanishes for $\omega=0$; it is often incorrectly assumed to vanish for $\omega\neq 0$.

Consider a loose-spaced secondary turn, and the integral form of the field equation

$$\oint \mathbf{E}\cdot d\boldsymbol{\lambda}=\int\nabla\times\mathbf{E}\cdot d\boldsymbol{\sigma}=-\int\dot{\mathbf{B}}\cdot d\boldsymbol{\sigma}$$

or

$$\oint \mathbf{A}\cdot d\boldsymbol{\lambda}=\int\mathbf{B}\cdot d\boldsymbol{\sigma}.$$

A change of flux $\Delta\phi$ in the core *cannot* induce a voltage in the turn before $t=r/c$, where r is distance from core to turn. This implies $\int\mathbf{B}\cdot d\boldsymbol{\sigma}=0$ for $t<r/c$, so there must be an external flux, $-\Delta\phi$.

$\nabla\cdot\mathbf{B}\equiv 0$ suggests that a step in the exciting current generates flux as circles of infinitesimal diameter surrounding the current, and that these expand to their steady-state pattern. Flux arriving from different source points produces a complicated superposition, resulting in a high order of cancellation outside the toroid. The details of this phenomenon can be readily exhibited for the field of an infinitely long solenoid, with step-function current, but will not be discussed in this paper.

FORMULATION OF THE PROBLEM

Consider a toroid generated by an arbitrary cross section and excited by a surface current of angular frequency ω. The surface current is taken to be such that the field is of the *magnetic type* (transverse electric), i.e., zero surface charge and zero scalar potential. The differential equations for the vector potential are

$$\nabla\times\nabla\times\mathbf{A}=\omega^2\mu\epsilon\mathbf{A}, \tag{1}$$

$$\nabla\cdot\mathbf{A}=0, \tag{2}$$

with the boundary conditions as follows:

Outgoing wave at infinity. (B1)

The discontinuity in the tangential magnetic field is the surface current density:

$$[\mathbf{n}\times\mathbf{H}]\equiv\mathbf{n}\times(\mathbf{H}_e-\mathbf{H}_i)=\mathbf{K}, \qquad \text{(B2)}$$

$$[\mathbf{n}\times\mathbf{A}]=0, \qquad \text{(B3)}$$

$$[\epsilon\mathbf{n}\cdot\mathbf{A}]=0. \qquad \text{(B4)}$$

Condition (B3) follows from $\phi=0$ and (B4) from the lack of surface charge.

In cylindrical coordinates:

$$\nabla\times\mathbf{A}=-\mathbf{r}_1\frac{\partial A_\theta}{\partial z}+\boldsymbol{\theta}_1\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)+\mathbf{z}_1 r^{-1}\frac{\partial}{\partial r}(rA_\theta), \tag{3}$$

$$\nabla\times\nabla\times A=-\mathbf{r}_1\frac{\partial}{\partial z}\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)+\boldsymbol{\theta}_1\left[-\frac{\partial^2 A_\theta}{\partial z^2}-\frac{\partial}{\partial r}\left(r^{-1}\frac{\partial}{\partial r}(rA_\theta)\right)\right]+\mathbf{z}_1 r^{-1}\frac{\partial}{\partial r}r\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right), \tag{4}$$

$$A_\theta=0, \qquad \partial\mathbf{A}/\partial\theta=0. \tag{5}$$

Then

$$B_\theta=(\partial A_r/\partial z)-(\partial A_z/\partial r)$$

and Eq. (1) gives

$$\partial B_\theta/\partial z = -k^2\mu_r\epsilon_r A_r, \tag{6a}$$

$$r^{-1}(\partial/\partial r)(rB_\theta) = k^2\mu_r\epsilon_r A_z, \tag{6b}$$

where

$$k^2 \equiv \omega^2\mu_0\epsilon_0 = \omega^2/c_0^2,$$

$$\nabla\cdot\mathbf{A} = (k^2\mu_r\epsilon_r)^{-1}r^{-1} \times\left[-\frac{\partial}{\partial r}\left(r\frac{\partial B_\theta}{\partial z}\right)+\frac{\partial}{\partial z}\frac{\partial}{\partial r}(rB_\theta)\right]\equiv 0. \tag{6c}$$

It is convenient to let $rB_\theta \equiv \psi$:

$$\partial\psi/\partial z = -k^2\mu_r\epsilon_r rA_r, \tag{7a}$$

$$\partial\psi/\partial r = k^2\mu_r\epsilon_r rA_z, \tag{7b}$$

yielding

$$\frac{\partial^2\psi}{\partial r^2} - r^{-1}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} + k^2\mu_r\epsilon_r\psi = 0, \tag{8}$$

with the general solution

$$\psi = \int e^{\beta z} r\mathcal{C}_1(\lambda_r r) f(\beta)\, d\beta, \tag{9}$$

where

$$\lambda_r^2 \equiv \beta^2 + k^2\mu_r\epsilon_r \tag{10}$$

and $\mathcal{C}_1$ is the general cylinder function of first order.

The boundary conditions on A can be related to ψ. Equations (7) yield

$$\nabla\psi = \mu_r\epsilon_r k^2 r\mathbf{A}\times(\mathbf{r}_1\times\mathbf{z}_1) \tag{11}$$

so that the tangential and normal derivatives of ψ are

$$\partial\psi/\partial\tau = \mu_r\epsilon_r k^2 rA_n, \tag{12a}$$

$$\partial\psi/\partial n = \mu_r\epsilon_r k^2 rA_t. \tag{12b}$$

Boundary conditions (B3) and (B4) yield

$$(\mu_r\epsilon_r)^{-1}(\partial\psi/\partial n)]_{\text{int}} = (\partial\psi/\partial n)]_{\text{ext}}, \tag{13a}$$

$$\mu_r^{-1}(\partial\psi/\partial\tau)]_{\text{int}} = (\partial\psi/\partial\tau)]_{\text{ext}}, \tag{13b}$$

where μ_r and ϵ_r are for the core, the external medium being air. From (13b), the discontinuity $[\psi/\mu]$ is uniform on the surface, so that $rK = [\psi/\mu]$ is uniform on the surface, in agreement with the assumed lack of surface charge. This allows subdividing the core into elementary cores, each linked by the current I, and computing the total field by integration over the cross section of the core.

An approximate solution ψ for the case of a material core $(\mu_r,\ \epsilon_r)$ can be derived from the solution for the air core case, ϕ. Externally, take

$$\psi_e = \mu_r\phi_e \tag{14}$$

and deduce the internal solution at the surface, from

$$[\psi/\mu] = [\phi] \tag{15}$$

or

$$\psi]_e - (\psi/\mu_r)]_i = \phi]_e - \phi]_i,$$

where] indicates a value at the surface.

$$\psi]_i = \mu_r\{(\mu_r-1)\phi]_e + \phi]_i\} = \mu_r^2\phi]_e - \mu_r[\phi], \tag{16a}$$

$$(\partial\psi/\partial n)]_i = \mu_r\epsilon_r(\partial\psi/\partial n)]_e = \mu_r^2\epsilon_r(\partial\phi/\partial n)]_e = \mu_r^2\epsilon_r(\partial\phi/\partial n)]_i. \tag{16b}$$

The internal solution ψ_i can be found (if desired) from these internal boundary values of ψ and $\partial\psi/\partial n$.

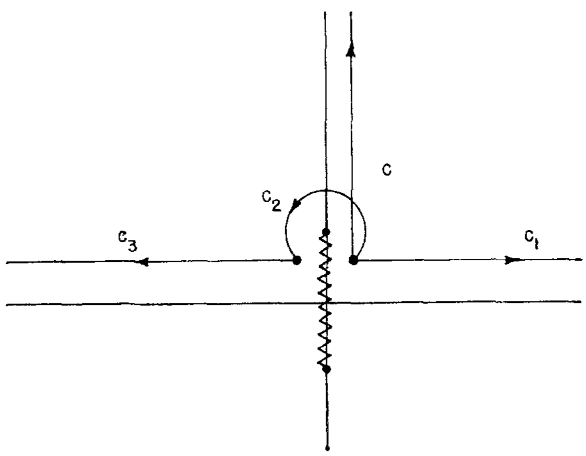

Fig. 1. Integration contours for Eq. (24).

AIR CORE SOLUTION

Consider an elementary toroid located at $z=s$, $r=\rho$. Solution behavior at $r=0$ and $r=\infty$ requires the outgoing Hankel function $H_1^{(1)}(\lambda r)$ for $r>\rho$ and the Bessel function $J_1(\lambda r)$ for $r<\rho$. The solution is

$$d\phi=\frac{k^2\mu_0 I dsd\rho}{4\pi}r\int_0^{i\infty}\cosh\beta(z-s)\begin{Bmatrix}H_1^{(1)}(\lambda r)J_1(\lambda\rho)\\ J_1(\lambda r)H_1^{(1)}(\lambda\rho)\end{Bmatrix}d\beta, \tag{17}$$

where the upper line in the braces applies for $r\geq\rho$, the lower line for $r\leq\rho$, and $\lambda^2\equiv\beta^2+k^2$. The vector potential components are

$$dA_r=-\frac{\mu_0 I dsd\rho}{4\pi}\int\beta\sinh\beta(z-s)\begin{Bmatrix}H_1^{(1)}(\lambda r)J_1(\lambda\rho)\\ J_1(\lambda r)H_1^{(1)}(\lambda\rho)\end{Bmatrix}d\beta, \tag{18}$$

$$dA_z=\frac{\mu_0 I dsd\rho}{4\pi}\int\lambda\cosh\beta(z-s)\begin{Bmatrix}H_0^{(1)}(\lambda r)J_1(\lambda\rho)\\ J_0(\lambda r)H_1^{(1)}(\lambda\rho)\end{Bmatrix}d\beta, \tag{19}$$

so that dA_z has a discontinuity across $r=\rho$:

$$[dA_z]=\frac{i\mu_0 I dsd\rho}{2\pi^2\rho}\int_0^{i\infty}\cosh\beta(z-s)\,d\beta$$

since

$$J_1(z)H_0^{(1)}(z)-J_0(z)H_1^{(1)}(z)\equiv 2i/\pi z.$$

Integration of (19) over a core of finite cross section, however, yields a continuous A_z.

The solution (17) is singular at $z=s$, $r=\rho$. Its behavior at this point can be determined from the limit of the line integral $\oint d\mathbf{A}\cdot d\mathbf{l}$ on a contour surrounding the singular point:

$$\begin{aligned}d\phi&=\rho dB_\theta=\rho(d\Phi/dsd\rho)=\rho\oint(d\mathbf{A}/dsd\rho)\cdot d\mathbf{l}\\
&=\frac{\mu_0 I\rho}{4\pi}\left\{\int_{s-\delta}^{s+\delta}\left.\frac{dA_z}{dsd\rho}\right)_{r=\rho-\epsilon}dz+\int_{\rho-\epsilon}^{\rho+\epsilon}\left.\frac{dA_r}{dsd\rho}\right)_{z=s+\delta}dr+\int_{s+\delta}^{s-\delta}\left.\frac{dA_z}{dsd\rho}\right)_{r=\rho+\epsilon}dz+\int_{\rho+\epsilon}^{\rho-\epsilon}\left.\frac{dA_r}{dsd\rho}\right)_{z=s-\delta}dr\right\}\\
&=\frac{\mu_0 I\rho}{2\pi}\int_0^{i\infty}\sinh\beta\delta\left(\frac{\lambda}{\beta}\{J_0[\lambda(\rho-\epsilon)]H_1^{(1)}(\lambda\rho)-H_0^{(1)}[\lambda(\rho+\epsilon)]J_1(\lambda\rho)\}\right.\\
&\quad\left.+\frac{\beta}{\lambda}\{H_0^{(1)}[\lambda(\rho+\epsilon)]J_1(\lambda\rho)-H_0^{(1)}(\lambda\rho)J_1(\lambda\rho)-J_0[\lambda(\rho-\epsilon)]H_1^{(1)}(\lambda\rho)+J_0(\lambda\rho)H_1^{(1)}(\lambda\rho)\}\right)d\beta.\end{aligned} \tag{20}$$

For $\epsilon\to 0$,

$$d\phi\to\frac{\mu_0 I\rho}{2\pi}\int_0^{i\infty}(\sinh\beta\delta)\frac{\lambda}{\beta}[J_0(\lambda_0)H_1^{(1)}(\lambda\rho)-H_0^{(1)}(\lambda\rho)J_1(\lambda\rho)]d\beta$$

$$=-\frac{i\mu_0 I}{\pi^2}\int_0^{i\infty}\frac{\sinh\beta\delta}{\beta}d\beta=\frac{\mu_0 I}{\pi^2}\int_0^{\infty}\frac{\sin x\delta}{x}dx=\frac{\mu_0 I}{2\pi}. \tag{21}$$

The proposed elementary solution (17) can therefore be written

$$d\phi=\frac{\mu_0 I}{2\pi}\left(\tfrac{1}{2}k^2\int_0^{i\infty} r\cosh\beta(z-s)\begin{Bmatrix}H_1^{(1)}(\lambda r)J_1(\lambda\rho)\\ J_1(\lambda r)H_1^{(1)}(\lambda\rho)\end{Bmatrix}d\beta+\delta(z-s,\,r-\rho)\right)ds\,d\rho, \tag{22}$$

where the integral over β is inapplicable at the singular point.

For the complete air core, integration yields

$$\phi_e=\frac{\mu_0 Ik}{4\pi}\int_0^{i\infty} r\iint_{\text{core}} ds\,d\rho\cosh\beta(z-s)\begin{Bmatrix}H_1^{(1)}(\lambda r)J_1(\lambda\rho)\\ J_1(\lambda r)H_1^{(1)}(\lambda\rho)\end{Bmatrix}d\beta, \tag{23}$$

$$\phi_i=\phi_e+(\mu_0 I/2\pi).$$

For $k^2=0$, $\phi_e=0$, $\phi_i=\mu_0 I/2\pi$.

The external solution for a material core and the surface value of the internal solution follow from (14) and (16a):

$$\psi_e=\mu_r\phi_e,$$

$$\psi]_i=\mu_r^2\phi]_e+(\mu_r\mu_0 I/2\pi)=\mu_r\{\psi]_e+(\mu_0 I/2\pi)\}.$$

The high order of cancellation of external flux can be demonstrated by finding upper bounds to the ratio $\psi]_e/\psi]_i$.

Let the ratio

$$\frac{\psi]_e}{\mu_0 I/2\pi}\equiv R\mu_r.$$

We shall first show that $\mu_r R\ll 1$ for toroids whose dimensions are small relative to the wavelength, so that $\psi_i\doteq\mu_r\mu_0 I/2\pi$ and $\psi]_e/\psi]_i\doteq R$:

$$R=\tfrac{1}{2}k^2 r\iint ds\,d\rho\int_0^{i\infty}\cosh\beta(z-s)\begin{Bmatrix}H_1^{(1)}(\lambda r)J_1(\lambda\rho)\\ J_1(\lambda r)H_1^{(1)}(\lambda\rho)\end{Bmatrix}d\beta. \tag{24}$$

Now λ has branch points at $\beta=\pm ik$, We take the contour of integration as C in Fig. 1.

The Hankel functions of the first kind behave at infinity over the upper half-plane; the hyperbolic cosine prevents deforming the contour to, say, C_1. But

$$\cosh\beta(z-s)=\tfrac{1}{2}\{\exp[\beta(z-s)]+\exp[-\beta(z-s)]\}.$$

For $z>s$, the first term allows deformation of C to C_2+C_3; the second term permits C_1. For $z<s$, the opposite deformations are allowed. Thus

$$\int_C\cosh\beta(z-s)HJ\,d\beta=\int_{C_1}\tfrac{1}{2}\{\exp[-\beta(z-s)]\}HJ\,d\beta+\left(\int_{C_2}+\int_{C_3}\right)\tfrac{1}{2}\{\exp[\beta(z-s)]\}HJ\,d\beta.$$

Now

$$H_1^{(1)}(\lambda r)=J_1(\lambda r)+iY_1(\lambda r),$$

$$H_1^{(1)}(\lambda re^{i\pi})=J_1(\lambda r)-iY_1(\lambda r)=H_1^{(2)}(\lambda r),$$

$$J_1(\lambda re^{i\pi})=-J_1(\lambda r), \tag{25}$$

so that the integral on C_3 can be expressed as an integral on C_1 and combined with the other integral on C_1. The result is:

$$R=\frac{k^2r}{2}\iint dsd\rho\int_0^\infty \exp(-\beta\mid z-s\mid)J_1(\lambda r)J_1(\lambda\rho)d\beta$$
$$+\tfrac{1}{2}(k^2r)\iint dsd\rho\int_{C_2}\tfrac{1}{2}[\exp(\beta\mid z-s\mid)]\begin{Bmatrix}H_1^{(1)}(\lambda r)J_1(\lambda\rho)\\ J_1(\lambda r)H_1^{(1)}(\lambda\rho)\end{Bmatrix}d\beta. \quad (26)$$

The first term in the expansion of R for small k arises from setting $k=0$ in the integrals (which shrinks the path C_2), resulting in

$$R\doteq R_0\equiv\tfrac{1}{2}(k^2r)\iint dsd\rho\int_0^\infty \exp(-\lambda\mid z-s\mid)J_1(\lambda r)J_1(\lambda\rho)d\lambda. \quad (27)$$

To find upper bounds on R_0, we use the following:

$$\left|\int_{\rho 1}^{\rho 2}J_1(\lambda\rho)d\rho\right|=\left|\frac{J_0(\lambda\rho_1)-J_0(\lambda\rho_2)}{\lambda}\right|<\frac{2}{\lambda}\qquad\text{(useful for large }\lambda\text{)}, \quad (28a)$$

$$\mid J_1(\lambda\rho)\mid\leq 1/\sqrt{2}\quad\text{and}\quad\mid J_1(\lambda r)\mid\leq 1/\sqrt{2}, \quad (28b)$$

$$\left|e^{-\lambda z}\int_{s1}^{z}e^{\lambda s}ds+e^{\lambda z}\int_{z}^{s2}e^{-\lambda s}ds\right|=\left|\frac{2-\exp[-\lambda(z-s_1)]-\exp[-\lambda(s_2-z)]}{\lambda}\right|<\frac{2}{\lambda}, \quad (29a)$$

$$\exp(-\lambda\mid z-s\mid)\leq 1. \quad (29b)$$

Combining these gives

$$\mid\iint\exp(-\lambda\mid z-s\mid)J_1(\lambda\rho)dsd\rho\mid$$
$$\leq 4/\lambda^2\qquad\text{(useful for large }\lambda\text{)}$$
$$\leq A/\sqrt{2}, \quad (30)$$

where $A=\iint dsd\rho$ is the cross section area of the core.

Using these bounds for two ranges of λ,

$$\mid R_0\mid\leq\tfrac{1}{2}(k^2r)\left\{\int_0^L\frac{A}{\sqrt{2}}\frac{1}{\sqrt{2}}d\lambda+\int_L^\infty\frac{4}{\lambda^2}\frac{1}{\sqrt{2}}d\lambda\right\}$$
$$=\tfrac{1}{2}(k^2r)\left\{\tfrac{1}{2}(AL)+\frac{2\sqrt{2}}{L}\right\}. \quad (31)$$

This minimizes for $L^2=4\sqrt{2}/A$, giving

$$\mid R_0\mid\leq 2^{-3/4}k^2A^{1/2}r. \quad (32)$$

In highly precise electrical measurements, as in the realization of the standards of capacitance and resistance from their theoretical definitions, much use is made of accurate ratio transformers operating at an angular frequency of 10^4 rad/s. A typical core has a square cross section, with inner and outer diameters of 10 and 15 cm, respectively. The ratio R_0 given by Eq. (32) is 1.2×10^{-12}, which is negligible compared to the leakage flux from other sources (discrete wire winding, inhomogeneities in the core, etc.).

The factor k^2 that appears in Eq. (23) for external flux density, but not in (18), and (19) for the vector potential, shows that the external electric field (hence the emf induced in a secondary winding), is essentially due to the sinusoidal variation of the vector potential of the static (dc) case. The correction to this approximation is proportional to ω^2 and is, in fact, the negligible term given by Eq. (32).

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