On the External Magnetic Field of a Closed-Loop Core

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An infinite solenoid, or a toroidal coil, has no external magnetic field for dc excitation but has a nonvanishing vector potential field. The development of the steady-state condition from the transient response to step excitation is discussed. Under ac excitation, an external magnetic field must be present, but in practical cases its magnitude is negligible.

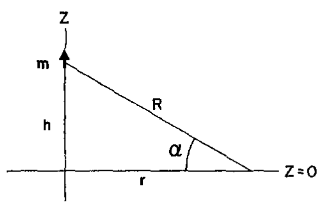

FIG. 1. Coordinate system for dipole source.

INTRODUCTION

A geometrically ideal transformer (toroidal core) has no external magnetic field under direct-current excitation, but with alternating excitation, a voltage is induced in closed paths linking the core. If a secondary turn is spaced a distance r from the core, any voltage induced in it is due to a change in exciting current at the earlier time, $t-r/c$.

Consider a step-function exciting current. Since

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\int \dot{\mathbf{B}}\cdot d\sigma \quad (1)$$

vanishes for $t<r/c$, there is correspondingly no change in the flux linking the turn before $t=r/c$. This implies that an increase $\Delta\phi$ in the core flux is accompanied by an external flux $-\Delta\phi$, which continues to link the turn until $t=r/c$. This suggests that the external flux propagates away from the core with the speed c.

It will be shown that the field of a "turned-on" magnetic dipole is, in fact, established by a propagation process but that in the case of an infinite solenoid, or a toroid, the time delay is related to the destructive interference of the flux by contributions arriving from various parts of the core.

MAGNETIC DIPOLE

Consider an oscillating magnetic dipole $m\exp(-j\omega t)$ oriented along the Z axis, and located at $z=h$ (Fig. 1). In the plane $z=0$, the vector potential is

$$A_\phi = (\mu m/4\pi R^2)\cos\alpha[1-(j\omega R/c)] \times\exp[-j\omega(t-R/c)]. \quad (2)$$

The cylindrical coordinate components of the flux density are

$$B_z = -\frac{\mu m}{4\pi R^3}\Big((1-3\sin^2\alpha)-(1-3\sin^2\alpha)\times\frac{j\omega R}{c}-\frac{\omega^2R^2}{c^2}\cos^2\alpha\Big)\exp\left[-j\omega\left(t-\frac{R}{c}\right)\right], \quad (3)$$

$$B_r = -\frac{\mu m}{4\pi R^3}\sin\alpha\cos\alpha\left(3-3\frac{j\omega R}{c}-\frac{\omega^2R^2}{c^2}\right)\times\exp\left[-j\omega\left(t-\frac{R}{c}\right)\right]. \quad (4)$$

The flux density normal to the cone of constant angle α is

$$B_\theta = \frac{\mu m}{4\pi R^3}\cos\alpha\left(1-\frac{j\omega R}{c}-\frac{\omega^2R^2}{c^2}\right)\times\exp\left[-j\omega\left(t-\frac{R}{c}\right)\right]. \quad (5)$$

If m is a unit step, $m\delta^{(-1)}(t)$, (5) yields

$$\hat{B}_\theta = \frac{\mu m}{4\pi R^3}\cos\alpha\left[\delta^{(-1)}\left(t-\frac{R}{c}\right)+\frac{R}{c}\delta\left(t-\frac{R}{c}\right)+\frac{R^2}{c^2}\delta^{(1)}\left(t-\frac{R}{c}\right)\right]. \quad (6)$$

The flux through the cone element dR is

$$d\phi = \tfrac{1}{2}(\mu m\cos^2\alpha)(dR/R^2)\times[\delta^{(-1)}+(R/c)\delta+(R^2/c^2)\delta^{(1)}] \quad (7)$$

so that the flux *behind* the wave front (excluding the singularity at the origin) is

$$\phi = \int_a^{ct}\frac{d\phi}{dR}dR = \tfrac{1}{2}(\mu m\cos^2\alpha)[a^{-1}-(ct)^{-1}] \quad (8)$$

and the flux *in* the wave front (contributed by the δ function) is

$$\phi_f = +\tfrac{1}{2}(\mu m\cos^2\alpha)(ct)^{-1}. \quad (9)$$

Thus as the wave front expands, it "deposits" flux; the total flux is constant. The process is comparable to the action of a paintbrush.

EXTENDED SOURCE

Let us next consider a uniform linear distribution of magnetic dipole moment along the entire z axis. Equations (2)–(5), with m replaced by ρdh, give the contributions from the element dh. Noting that

$$R = r\sec\alpha, \quad (10)$$

$$dh = r\sec^2\alpha d\alpha, \quad (10a)$$

we have

$$A_\theta = \frac{\mu\rho\exp(-j\omega t)}{2\pi r}\int_0^{\pi/2}\cos\alpha\left(1-\frac{j\omega r}{c}\sec\alpha\right)\times\exp\left(\frac{j\omega r\sec\alpha}{c}\right)d\alpha, \quad (11)$$

$$B_z = -\frac{\mu\rho\exp(-j\omega t)}{2\pi r^2}\int_0^{\pi/2}\cos\alpha\Big((1-3\sin^2\alpha)-(1-3\sin^2\alpha)\frac{j\omega r}{c}\sec\alpha-\frac{\omega^2r^2}{c^2}\Big)\times\exp\left(\frac{j\omega r\sec\alpha}{c}\right)d\alpha, \quad (12)$$

$$B_r = 0. \quad (13)$$

In (12), the factor $(1-3\sin^2\alpha)$ produces a sign reversal at $h=r/\sqrt{2}$. For $\omega=0$, the integral becomes

$$\int_0^{\pi/2}(1-3\sin^2\alpha)d(\sin\alpha) = \sin\alpha\cos^2\alpha\Big]_0^{\pi/2} = 0, \quad (14)$$

showing that the dc field from the region $h<r/\sqrt{2}$ is cancelled by that from $h>r/\sqrt{2}$. For $\omega\neq 0$, the first term under the integral sign may be integrated by parts, yielding

$$\int\exp\left(\frac{j\omega r\sec\alpha}{c}\right)d(\sin\alpha\cos^2\alpha) = -\frac{j\omega r}{c}\int\exp\left(\frac{j\omega r\sec\alpha}{c}\right)\sin^2\alpha d\alpha, \quad (15)$$

showing that the summation of the elementary quasistatic field (magnitude independent of ω) produces an induction field term (magnitude proportional to ω).

For the second term of (12), we have

$$-\frac{j\omega r}{c}\int (1-3\sin^2\alpha)\exp\left(\frac{j\omega r\sec\alpha}{c}\right)d\alpha$$

$$=-\frac{j\omega r}{c}\int (\cos^2\alpha-2\sin^2\alpha)\times\exp\left(\frac{j\omega r\sec\alpha}{c}\right)d\alpha$$

$$=-\frac{j\omega r}{c}\int \exp\left(\frac{j\omega r\sec\alpha}{c}\right)\times[\cos\alpha d(\sin\alpha)-2\sin^2\alpha d\alpha], \quad (16)$$

and integration of the first term of (16) by parts gives

$$=-(j\omega r/c)\int \exp(j\omega r\sec\alpha/c)[\sin^2\alpha-(j\omega r/c)\times\sin\alpha\cos\alpha\sec\alpha\tan\alpha-2\sin^2\alpha]d\alpha$$

$$=(j\omega r/c)\int \exp(j\omega r\sec\alpha/c)\sin^2\alpha d\alpha-(\omega^2r^2/c^2)\int\exp(j\omega r\sec\alpha/c)\sin^2\alpha\sec\alpha d\alpha.$$

The first term of this result is an induction field term which cancels that given by (15). The remaining term is a radiation field (magnitude proportional to ω^2) which combines with the third term of (12):

$$B_z=\frac{+\mu\rho\exp(-j\omega t)\omega^2}{2\pi c^2}\int_0^{\pi/2}\exp\left(\frac{j\omega r\sec\alpha}{c}\right)\times(\cos\alpha+\sin^2\alpha\sec\alpha)d\alpha$$

$$=\frac{\mu\rho\exp(-j\omega t)\omega^2}{2\pi c^2}\int_0^{\pi/2}\exp\left(\frac{j\omega r\sec\alpha}{c}\right)\sec\alpha d\alpha.$$

$$=\frac{\mu\rho\exp(-j\omega t)\omega^2}{2\pi c^2}\int_1^{\infty}\frac{\exp[(j\omega r/c)x]dx}{(x^2-1)^{1/2}}. \quad (17)$$

This last integral is a well-known representation of a Hankel function, yielding

$$B_z=[j\omega^2\mu\rho\exp(-j\omega t)/4c^2]H_0^{(1)}(\omega r/c). \quad (18)$$

Integrating (11) gives

$$A_\theta=(j\mu\rho\omega/4c)\exp(-j\omega t)H_1^{(1)}(\omega r/c), \quad (19)$$

which, for $\omega\rightarrow 0$, becomes

$$A_{\theta 0}=\mu\rho/2\pi r. \quad (20)$$

The contributions from the elements of the infinite line source thus result in cancellation of the quasistatic and induction flux fields, but *not* of the quasistatic vector potential field.

CLOSED-LOOP CORE

Consider a closed-loop core (Fig. 2) excited by a surface current distribution which, for $\omega=0$, has a zero external magnetic field. The situation is analogous to that of the infinite solenoid. For sinusoidal excitation of the same surface current pattern, we expect the external field to be primarily the electric field associated with the alternation of the quasistatic vector potential field. A weak magnetic field (radiation field proportional to ω^2) is also expected. It is of interest to find the relative magnitudes of the external and internal flux densities.

We can evaluate the average flux density along C_1, the boundary of the window, in terms of the average flux density in the core. The conditions assumed on the surface-current excitation imply that the external field is of the magnetic type (transverse electric), with

$$\mathbf{E}=-\dot{\mathbf{A}}.$$

Then

$$\oint_{C_1}\mathbf{H}\cdot d\mathbf{l}=\int_W \nabla\times\mathbf{H}\cdot\mathbf{n}da$$

$$=\int_W \mathbf{J}\cdot\mathbf{n}da+\int_W\dot{\mathbf{D}}\cdot\mathbf{n}da$$

$$=0+\omega^2\epsilon\int_W \mathbf{A}\cdot\mathbf{n}da, \quad (21)$$

so that

$$\oint_{C_1}\mathbf{B}\cdot d\mathbf{l}=k^2\int_W\mathbf{A}\cdot\mathbf{n}da, \quad (22)$$

where

$$k^2=\omega^2\mu\epsilon=\omega^2/c^2=4\pi^2/\lambda^2 \quad (23)$$

with λ the wavelength in the external medium.

At the point Q on W, $\mathbf{A}$ can be estimated from

$$\oint_{C_2}\mathbf{A}\cdot d\mathbf{l}=\int\mathbf{B}\cdot d\boldsymbol{\sigma}=\Phi$$
$$=\bar{A}P_2, \quad (24)$$

where P_2 is the length of C_2. We make the reasonable assumption that the average A in the window does not exceed the average A on the contour C_3 on the core surface:

$$\int_W\mathbf{A}\cdot\mathbf{n}da\leq S_W\bar{A}_3\leq S_W\Phi/P_3=S_WS_C\bar{B}_C/P_C, \quad (25)$$

where S_W is the area of the window, S_C is a cross-sectional area of the core, $P_3=P_C$ is the perimeter of the core section, and $\bar{B}_C$ is the average flux density in the section.

Introducing the window perimeter P_W we have

$$\bar{B}_1P_W\leq k^2(S_WS_C\bar{B}_C/P_C) \quad (26)$$

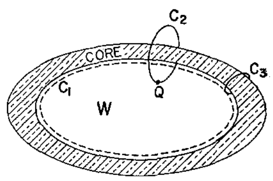

FIG. 2. Closed-loop core.

or

$$\bar{B}_1/\bar{B}_C\leq k^2(S_W/P_W)(S_C/P_C). \quad (27)$$

Now for a plane figure of greatest diameter L, we have

$$S/PL\leq\tfrac{1}{4}, \quad (28)$$

so (27) can be transformed to

$$\bar{B}_1/\bar{B}_C\leq(k^2/16)L_WL_C=(\pi^2/4)(L_WL_C/\lambda^2), \quad (29)$$

with L_W the greatest diameter of the window and L_C the greatest diameter of the core section.

A typical practical case is illustrated by

$$\omega=10^4\text{ rad/s},\qquad L_W\sim10\text{ cm},\qquad L_C\sim10\text{ cm},$$

which yields

$$\bar{B}_1/\bar{B}_C\leq10^{-11}.$$

Thus even in a precision transformer with a desired accuracy of a few parts in 10^9, the unavoidable leakage flux associated with the existence of the field is negligible.

CONCLUSIONS

An infinite solenoid or a closed-loop "toroidal" coil can exhibit an external field in which the magnetic flux density vanishes for $\omega=0$ and is negligible at low frequency, but in which the vector potential is not small. This external vector potential field of negligible curl is responsible for transformer operation.