Electric field outside a parallel plate capacitor

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The problem of determining the electrostatic potential and field outside a parallel plate capacitor is reduced, using symmetry, to a standard boundary value problem in the half space $z \geqslant 0$. In the limit that the gap d between plates approaches zero, the potential outside the plates is given as an integral over the surface of one plate. This integral is evaluated for several special cases. The magnitude of the field just outside and near the center of a two-dimensional strip capacitor of width W is shown to agree with finite difference calculations when $W/d > 4$. The shapes of field lines outside a strip capacitor are determined, and circular lines are shown to occur near the edges. The determination of the electric field just outside and near the center of a parallel plate capacitor complements the recently published result for the magnetic field just outside and near the center of a long solenoid [J. A. Farley and R. H. Price, Am. J. Phys. **69**, 751–754 (2001)].
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I. INTRODUCTION

The question of the magnitude of the magnetic field outside a long (ideal) solenoid was recently addressed in this journal.[1] It was shown that the magnitude of the field just outside the solenoid and near its center is given by

$$B_{\text{out}} = \left(\frac{2A}{\pi L^2}\right) B_{\text{in}}, \tag{1}$$

where B_{in} is the uniform field inside and near the center, L is the length of the solenoid, and A is its cross-sectional area. This interesting result is independent of the shape of the cross section and holds for $L \gg \sqrt{A}$. The magnetic dipole field holds at large distances. Thus, using Eq. (1), the field is known in three regions: inside and near the center, just outside and near the center, and outside at large distances from the center, $r \gg L$.

A long solenoid produces a region of uniform magnetic field inside and near its center. To produce a region of uniform electric field, a parallel plate capacitor would be used with plate dimensions large compared to the gap d between the plates. The question I address in this paper is "Can the electric field outside the capacitor plates be determined." I will show that the field outside rectangular plates of dimensions $L \times W$ may be determined throughout a plane of symmetry perpendicular to the plates for $d \ll L, W$. This determination of the field includes regions both near and far from the plates as well as a region near the edges of an infinitely thin plate where the field becomes infinite. The field just outside and near the center of these plates is

$$E_{\text{out}} = \frac{2V\sqrt{L^2+W^2}}{\pi L W} = \left(\frac{2d\sqrt{L^2+W^2}}{\pi L W}\right) E_{\text{in}}, \tag{2}$$

where $E_{\text{in}} = V/d$ is the uniform field inside the plates and V is the potential difference between the plates.

A magnetic field must exist outside a solenoid because magnetic field lines form closed loops. An electric field must exist outside parallel capacitor plates for an equally fundamental reason: electrostatic field lines do *not* form closed loops $(\nabla \times \mathbf{E} = 0)$.

Equation (2) holds just outside and near the center of rectangular plates in the limit that d approaches zero. That such a field must exist in this region follows from a general argument based on the fact that $\mathbf{E}$ has zero curl and thus zero line integral around any closed path. For example, taking the path along the z axis of Fig. 1, I obtain

$$\int_{-d/2}^{0} E_z(0,0,z)\,dz + \int_0^{\infty} E_z(0,0,z)\,dz = 0. \tag{3}$$

The path is closed on an arc of radius r on which the $1/r^3$ dipole field holds. As $r \to \infty$, the contribution from the arc vanishes. By using symmetry with respect to the surface of zero potential, I obtain Eq. (3). To cancel the first contribution from the field inside, there must be a field outside and a corresponding surface charge density on the outer surfaces of each plate, as shown in Fig. 1. For example, consider circular plates of radius R. A uniform field is produced by a charge density distributed uniformly on a plane. It is thus natural to assume a field that is uniform just outside and near the center of the plates. This field decreases as z increases $(z \geqslant 0)$. The length scale for this decrease must be just R because I assume $d \ll R$. Making an order of magnitude estimate, I obtain from Eq. (3),

$$-E_{\text{in}}\left(\frac{d}{2}\right) + E_{\text{out}} R \approx 0, \tag{4}$$

which gives

$$E_{\text{out}} \approx \frac{V}{2R} = \left(\frac{d}{2R}\right) E_{\text{in}}. \tag{5}$$

This order of magnitude estimate turns out to be exact, in the region indicated, as will be shown in Sec. III.

II. ELECTRIC FIELD OUTSIDE A PARALLEL PLATE CAPACITOR

The boundary value problem in Fig. 1 can be simplified by using symmetry about the zero potential surface as shown in Fig. 2(a). Rectangular plates of length L and width W are assumed. The $x=0$ plane is shown. One plate at potential $+V/2$ is a distance $d/2$ above an infinite grounded conduct-

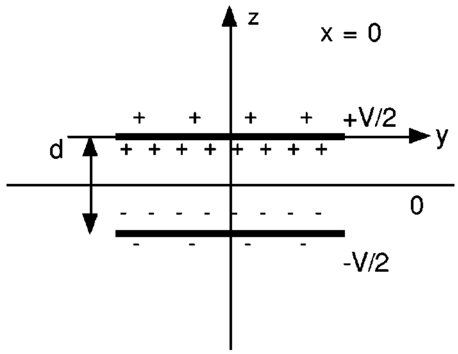

Fig. 1. Two identical parallel plates a distance d apart are seen in cross section in the $x=0$ plane. This plane is perpendicular to the plates and passes through their centers. The plates are assigned potentials $\pm V/2$ so the surface of zero potential is halfway between the plates at $z=-d/2$ (the coordinate origin is taken at the surface of the upper plate).

ing plane at $z=-d/2$. This problem is equivalent by symmetry to the original problem. The final step is to use the limit that d go to zero to obtain the situation shown in Fig. 2(b). The plate is lowered into the plane replacing the previously grounded section by a section at $+V/2$. The field obtained in this boundary value problem should give a good approximation to the field outside the plates in Fig. 2(a) when $d \ll L, W$. By using Green's theorem,[2] the potential can then be expressed as an integral over the surface at $+V/2$. The integral can be evaluated to give the field on the z axis of circular plates and in the $x=0$ plane of the parallel plates just described. At large distances, a dipole field is obtained with the dipole moment magnitude qd, where q is the magnitude of the charge on the inner surfaces of the plates.

The formula for the potential obtained from Green's theorem is[2]

$$\Phi(x,y,z) = -\frac{1}{4\pi}\int\int \Phi(x',y',0)\frac{\partial G}{\partial n'}\,dx'\,dy'. \tag{6}$$

G must vanish at $z=0$. An image charge at $(x,y,-z)$ gives the solution for G:

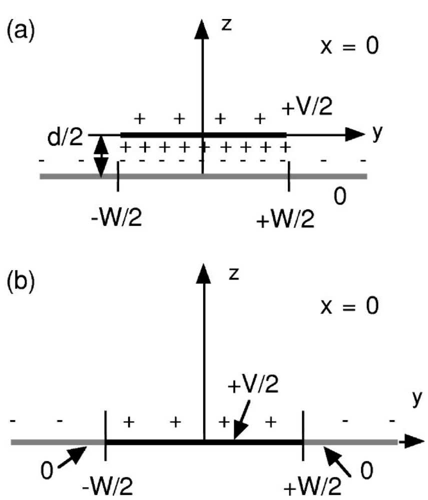

Fig. 2. (a) A boundary value problem equivalent by symmetry to that in Fig. 1. An infinite grounded conducting plane is at $z=-d/2$. Rectangular plates of width W are assumed. The length of a plate, L, can be finite or infinite. (b) The plate at is lowered into the $z=0$ plane to obtain a boundary value problem for $z \geq 0$ that approximates the problem of (a) for $d \ll L$, W.

$$G = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \frac{(-1)}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}. \tag{7}$$

The (outward) normal derivative is

$$\frac{\partial G}{\partial n'} = \left[-\frac{\partial G}{\partial z'}\right]_{z'=0} = \frac{-2z}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}. \tag{8}$$

Finally, the potential for $z \geq 0$ is

$$\Phi(x,y,z) = \frac{V}{4\pi}z\int\int \frac{dy'\,dx'}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}. \tag{9}$$

The limits on the integrals depend on the choice of plates. Both circular and rectangular plates are considered below.

Equation (9) predicts a dipole field at large distances corresponding to a dipole moment $\mathbf{p}=(0,0,p_z)$. If we expand the integrand for large r ($r^2=x^2+y^2+z^2$), and keep only the leading term, we find

$$\Phi(x,y,z) \to \frac{VA}{4\pi}\frac{z}{r^3}, \tag{10}$$

where A is the area of a plate. This result may be compared to the potential of a point dipole at the origin

$$\Phi = k\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right). \tag{11}$$

The constant k is used to include both Gaussian and SI units. Thus,

$$k = \begin{cases} 1 & \text{(Gaussian)}, \\ 1/(4\pi\epsilon_0) & \text{(SI)}. \end{cases} \tag{12}$$

The comparison shows that $p_x = p_y = 0$ and

$$p_z = \frac{VA}{4\pi k}. \tag{13}$$

For later reference, I give the field of such a dipole on the positive z axis ($r=z$),

$$E_z = \frac{2p_z}{r^3}k, \tag{14}$$

and on the y axis ($r=|y|$)

$$E_z = -\frac{p_z}{r^3}k. \tag{15}$$

The dipole moment can be related to the charges on the plates. Using

$$E_{\text{in}} = \frac{V}{d} = 4\pi\sigma_{\text{in}}k, \tag{16}$$

where σ_{in} is the magnitude of the charge per unit area on the inside surfaces of a plate, the dipole moment can be shown to become

$$p_z=\frac{A}{4\pi k}(4\pi k\sigma_{\text{in}}d)=qd, \quad (17)$$

where $q=A\sigma_{\text{in}}$ is the magnitude of the charge on the inside surface of a plate.

III. CIRCULAR PLATES

The case of circular plates (disks) of radius R can be found as a problem in Jackson's text.[3] Only the results will be summarized here. The potential on the z axis is given by [let $x=y=0$ in Eq. (9), and evaluate the integral using cylindrical coordinates]

$$\Phi(0,0,z)=\frac{V}{2}\left[1-\frac{z}{\sqrt{z^2+R^2}}\right]. \quad (18)$$

The field on the positive z axis is then

$$E_z(0,0,z)=\frac{V}{2}\frac{R^2}{(R^2+z^2)^{3/2}}. \quad (19)$$

At $z=0$ the field is

$$E_z(0,0,0)=\frac{V}{2R}. \quad (20)$$

This result shows that the order of magnitude estimate made in the first section, Eq. (5), is exact. For $z \gg R$, the field on the axis takes the form of Eq. (14).

The solution for $z \geqslant 0$ can be extended to $z \leqslant 0$. If the integral giving the potential is evaluated assuming $z \leqslant 0$, the result is [Eq. (9) is odd in z]

$$\Phi(0,0,z)=\frac{V}{2}\left[-1-\frac{z}{\sqrt{z^2+R^2}}\right]. \quad (21)$$

Equations (18) and (21) show the discontinuity that exists in the potential at $z=0$ corresponding to circular plates at $+V/2$ and $-V/2$ with a separation approaching zero.

The additional charge on the outside surfaces of each plate increases the capacitance above the standard result. The fringing field makes important contributions, however, and a more detailed analysis[4,5] is needed to determine those contributions.

IV. RECTANGULAR PLATES: $L \gg W$

A two-dimensional version of the rectangular plate problem is obtained in the limit as L becomes infinite. Because this limit is simpler to evaluate and is of interest in its own right, it will be considered first.

To find the field on the z axis, I let $x=y=0$ in the integral of Eq. (9) and evaluate the integral over x' obtaining

$$\Phi(0,0,z)=\frac{V}{4\pi}zL\int_{-W/2}^{+W/2}\frac{dy'}{(y'^2+z^2)\sqrt{y'^2+z^2+L^2/4}}. \quad (22)$$

I then take the limit of infinite L and obtain

$$\Phi(0,0,z)=\frac{V}{2\pi}z\int_{-W/2}^{+W/2}\frac{dy'}{(y'^2+z^2)}=\frac{V}{\pi}\arctan\left(\frac{W}{2z}\right). \quad (23)$$

As z approaches zero on the positive z axis, the potential approaches $+V/2$. The field on the z axis is

$$E_z(0,0,z)=\frac{VW}{2\pi}\frac{1}{(z^2+W^2/4)}, \quad (24)$$

and the uniform field just outside each plate has the magnitude

$$E_z(0,0,0)=\frac{2V}{\pi W}. \quad (25)$$

Note that Eq. (2) reduces to Eq. (25) for $L \gg W$. The behavior at large z is that of a two-dimensional dipole, which is

$$E_z=\frac{2\xi}{z^2}k, \quad (26)$$

where ξ is the dipole moment per unit length along the x (long) axis, which is

$$\xi=\frac{WV}{4\pi k}=\frac{(4\pi k\sigma_{\text{in}}d)W}{4\pi k}=\lambda_{\text{in}}d, \quad (27)$$

where $\lambda_{\text{in}}=\sigma_{\text{in}}W$ is the magnitude of the charge per unit length on the inside surface of each conducting strip.

The field can also be determined throughout the $x=0$ plane. From Eq. (9), the integral over x' gives

$$\frac{2L}{[(y-y')^2+z^2]\sqrt{L^2+4(y-y')^2+4z^2}}. \quad (28)$$

In the limit of infinite L, the potential becomes

$$\Phi(y,z)=\frac{Vz}{2\pi}\int_{-W/2}^{+W/2}\frac{dy'}{(y-y')^2+z^2}, \quad (29)$$

so that

$$\Phi(y,z)=\frac{V}{2\pi}\left[\arctan\left(\frac{W/2+y}{z}\right)+\arctan\left(\frac{W/2-y}{z}\right)\right]. \quad (30)$$

A potential of this form is postulated in a problem in Stratton.[6] Equation (30) clearly reduces to Eq. (23) when $y=0$, and it gives either $+V/2$ or zero when z approaches zero for $z \geqslant 0$. The components of $\mathbf{E}$ are then

$$E_z(y,z)=\frac{V}{2\pi}\left\{\frac{(W/2+y)}{[z^2+(W/2+y)^2]}+\frac{(W/2-y)}{[z^2+(W/2-y)^2]}\right\}, \quad (31)$$

and

$$E_y(y,z)=-\frac{Vz}{2\pi}\left\{\frac{1}{[z^2+(W/2+y)^2]}-\frac{1}{[z^2+(W/2-y)^2]}\right\}. \quad (32)$$

On the z axis $(y=0)$, the z component reduces to Eq. (24) and the y component is zero, as required. The y component also vanishes for $z=0$. As another check, the divergence and curl of $\mathbf{E}$ are both zero.

The equations for the potential and field can be extended to $z \leqslant 0$ to describe parallel conducting strips at potentials $+V/2$ and $-V/2$ with a separation approaching zero. The potential, Eq. (30), is odd in z. It has limiting values of $+V/2$ from above or $-V/2$ from below for $|y|<W/2$. But, for $|y|>W/2$, both limits give zero as required.

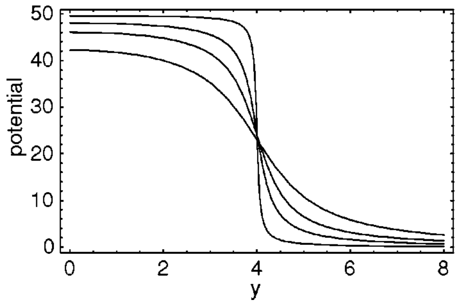

Fig. 3. The potential, Eq. (30), plotted as a function of y in cm for $z=1.00$, 0.50, 0.25, and 0.05 cm using $V=100$ statvolts and $W=8$ cm with L infinite.

The behavior of the field near the edges of the plates is illustrated in Figs. 3 and 4. Figure 3 shows the potential as a function of y for $z=1.00$, 0.50, 0.25, and 0.05 cm for $V=100$ statvolts and $W=8$ cm. The "square-step" discontinuity at $y=W/2=4$ cm is clearly seen to develop as the plate is approached. In Fig. 4, the z component of the field is plotted for same parameter values as Fig. 3. The functional form for $z\rightarrow 0$ is

$$E_z(y,0)=\frac{V}{2\pi}\left(\frac{1}{W/2+y}+\frac{1}{W/2-y}\right). \tag{33}$$

The field becomes infinite at the ends because infinitely thin plates are assumed. A detailed examination of the fringing field for infinite L and $W\gg d$ can be found in Cross.[7]

The equation for the electric field lines (outside the plates) in the yz plane is the function $z(y)$ given by

$$\frac{dz}{dy}=\frac{E_z}{E_y}=\frac{z^2-y^2+W^2/4}{2yz}. \tag{34}$$

The field lines near the edges of the plates in Fig. 2(b) are circles centered on each edge. If we shift the origin to the right edge, $z'=z$, $y'=(y-W/2)$, and assume large W, the equation for field lines in the new coordinates is $dz'/dy'=-(y'/z')$. This result gives circular lines centered on the edge.[8] The equation for the field lines outside the plates can be solved for arbitrary W. In the new coordinates,

$$\frac{dz'}{dy'}=\frac{z'^2-y'^2-y'W}{2z'y'+z'W}, \tag{35}$$

and its solution for $z'>0$ is

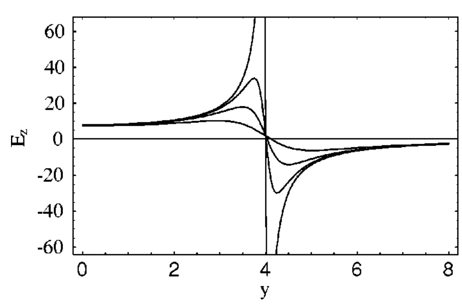

Fig. 4. The z component of the field, Eq. (31), plotted as a function of y in cm for $z=1.00$, 0.50, 0.25, and 0.05 cm using $V=100$ statvolts and $W=8$ cm with L infinite.

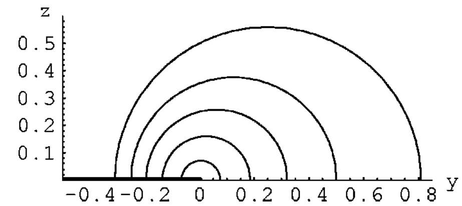

Fig. 5. Field lines of a strip capacitor in one quadrant of the yz plane for $W=1$ and $C=1.02$, 1.10, 1.25, 1.50, and 2.00 cm. The z axis goes through the center of a plate.

$$z'(y')=\frac{1}{2}\sqrt{W+2y'}\sqrt{C-\frac{(W^2+2Wy'+4y'^2)}{(W+2y')}}. \tag{36}$$

The allowed range of y' for lines in the right half of the yz plane is $y'_-<y'<y'_+$, where $(C>W)$

$$y'_\pm=\left(\frac{C-W}{4}\right)\left[1\pm\sqrt{1+\frac{4W}{(C-W)}}\right]. \tag{37}$$

Figure 5 shows lines plotted using $W=1$ and $C=1.02$, 1.10, 1.25, 1.50, and 2.00 (the z axis goes through the centers of the plates). Note that these lines become circular as the edge is approached. To show the relation to circular lines, the solution may be rewritten as

$$z'^2+y'^2=\frac{(C-W)}{4}[W+2y']. \tag{38}$$

For $W=0$, Eq. (38) gives the field lines of a two-dimensional (linear) dipole.

V. RECTANGULAR PLATES: ARBITRARY *L*, *W*

The field on the z axis for arbitrary L and W is considered first. The integral in Eq. (22) is evaluated using MATHEMATICA giving

$$\Phi(0,0,z)=\frac{V}{\pi}\arctan\left(\frac{WL}{2z\sqrt{L^2+W^2+4z^2}}\right). \tag{39}$$

The potential approaches $V/2$, as required, as z approaches zero for $z\geqslant 0$. The field on the z axis is then

$$E_z(0,0,z)=\frac{2LWV(L^2+W^2+8z^2)}{\pi(L^2+4z^2)(W^2+4z^2)\sqrt{L^2+W^2+4z^2}}. \tag{40}$$

Equation (2) then follows, letting $z=0$ in Eq. (40). Equations (39) and (40) reduce to Eqs. (23) and (24) for $L\rightarrow\infty$ as required. By expanding about infinite z, I obtain

$$E_z\rightarrow\frac{LWV}{2\pi}\frac{1}{z^3}, \tag{41}$$

which gives the dipole form, Eq. (14), with moment qd, as derived previously. Equation (40) and the dipole field are plotted in Fig. 6 for $V=100$ statvolts, $L=4$ cm, and $W=8$ cm.

As in Sec. IV, the field can also be determined throughout the $x=0$ plane. From Eq. (9), the integral over x' gives

$$\frac{2L}{[(y-y')^2+z^2]\sqrt{L^2+4(y-y')^2+4z^2}}. \tag{42}$$

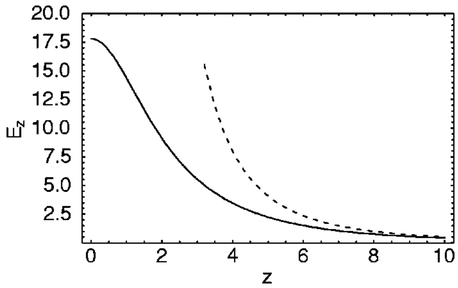

Fig. 6. The field on the z axis, Eq. (40) (solid curve), and the dipole field (dashed curve), plotted as a function of z in cm for $L=4$ cm, $W=8$ cm, and $V=100$ statvolts.

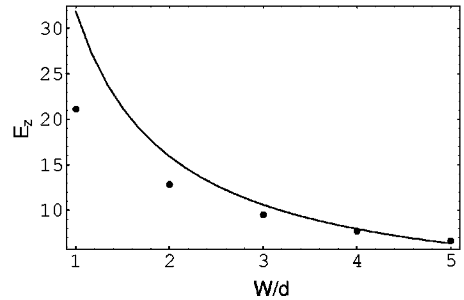

Fig. 7. The field, Eq. (25), just outside and near the center of two parallel conducting strips obtained by assuming $d \ll W$ compared to that field computed numerically for five ratios of W/d (points).

Using MATHEMATICA as before, the potential is determined to be

$$\Phi(0,y,z)=\frac{V}{2\pi}\left\{\arctan\left[\frac{L(W-2y)}{2zD_-}\right]+\arctan\left[\frac{L(W+2y)}{2zD_+}\right]\right\}, \tag{43}$$

where

$$D_\pm=\sqrt{L^2+(W\pm 2y)^2+4z^2}. \tag{44}$$

Equation (43) reduces to Eq. (39) when $y=0$ so the field on the z axis agrees with Eq. (40). The field on the y axis is

$$E_z(0,y,0)=\frac{V}{\pi L}\left[\frac{\sqrt{L^2+(W-2y)^2}}{(W-2y)}+\frac{\sqrt{L^2+(W+2y)^2}}{(W+2y)}\right]. \tag{45}$$

Equation (45) reduces to Eq. (33) in the limit of large L. For large y,

$$E_z(0,y,0)\to -\frac{LWV}{4\pi}\frac{1}{y^3}, \tag{46}$$

which has the form of Eq. (15) with the same dipole moment as obtained before.

VI. NUMERICAL EXAMPLE

I have found the uniform field, Eq. (2), just outside and near the center of rectangular parallel plates in the limit that $d \ll L, W$. For the two-dimensional case, $L \gg W$, and Eq. (2) reduces to Eq. (25). The question of when Eq. (25) actually becomes an accurate approximation is found by using a two-dimensional finite difference code[9,10] to calculate the potential and field for finite gaps, and letting the ratio W/d increase and comparing with Eq. (25). The comparison is shown in Fig. 7. The plot shows good agreement for $W/d > 4$.

The code is based on the standard five point finite difference formula for the Laplacian on a square grid of spacing h. The formula has an error whose leading term is $O(h^2)$. However, additional error is produced by the discontinuity at the edge of a plate. Also, in order to compare with the theory, the region covered by the grid needs to be relatively large. In practice, a compromise is sought between the ideal of both a large grid and small h. The square region actually used corresponds to Fig. 2(a) with closure added on three sides with the plate centered at $y=0$. A coarse 40×40 grid with $h=1$ cm was used to setup the problem. The plate at $V/2=50$ statvolts was located at $z=1$ cm, corresponding to $d=2$ cm and an interior field of 50 statvolts/cm. Two finer grids were used in the multigrid code[11] to obtain the potential on a fine grid with $h=1/4$. Approximately 26 000 grid points were used on the finest grid. Five runs were made using $W=2$, 4, 6, 8, and 10 cm giving ratios $W/d=1$, 2, 3, 4, and 5. As shown in Fig. 7, the calculated points begin to overlap Eq. (25) for $W/d>4$.

VII. SUMMARY AND PROBLEMS FOR STUDENTS

A charged parallel plate capacitor has a charge on the outer faces of its plates and an increasing charge density as the edges of a plate are approached. The existence of charge on the outer faces is required by the condition that the line integral of the electrostatic field around any closed path must vanish. By using symmetry and the condition $d \ll L, W$ for rectangular plates of length L and width W, the boundary value problem of Fig. 1 can be reduced to that of Fig. 2(b), which is a standard textbook problem.[3] The solution for the potential outside the plates is then given by Eq. (9), which has been evaluated in a plane of symmetry perpendicular to the plates. A uniform field is obtained just outside and near the center of each plate, Eq. (2). For $L \gg W$, the potential and field are found on the z axis, Eq. (23) and Eq. (24), respectively. In addition, the potential and field components are determined throughout the $x=0$ plane, Eqs. (30)–(32). For arbitrary L and W, I have calculated the potential and field on the z axis, Eqs. (39) and (40). Finally, the potential was found throughout the $x=0$ plane, Eq. (43). I also determined the potential and field on the z axis of circular plates (see Ref. 3). Dipole fields follow as limiting cases, and the dipole moments are determined by charge on the inside surfaces of the plates. As the edges of the infinitely thin plates are approached, field components increase without limit. Finite difference calculations in two dimensions show agreement between the calculated and predicted field just outside and near the center of a strip capacitor of width W for $W/d>4$. The equation giving the shapes of field lines outside a strip capacitor is determined, and circular lines are shown to occur near the edges.

Problem 1. Consider the parallel plate capacitor in Fig. 1 with the z axis perpendicular to the plates, as shown. Take the origin to be at the upper surface of the positive plate. (a) Show that the vanishing of the line integral of **E** around any closed path leads to the requirement that

$$\int_{-d/2}^{\infty} E_z(0,0,z)\,dz=0. \tag{47}$$

Hint: Use symmetry, the fact that the plates look like a point dipole at large distances r, and that the dipole field decreases as $1/r^3$. (b) Explain why Eq. (47) requires each plate to have charge of the same sign on its outside surface, as shown in Fig. 1. (c) Assume circular plates of radius R and a uniform field E_{out} just outside and near the center of each plate and make a rough estimate of E_{out} using Eq. (47). Assume that an initially uniform field decreases over a length scale R.

Problem 2. (a) Evaluate the electrostatic potential in two dimensions, Eq. (30), in the following limiting cases: (1) $z \to 0^+$ for $|y| < W/2$, (2) $z \to 0^-$ for $|y| < W/2$, (3) $z \to 0^+$ for $|y| > W/2$, and (4) $z \to 0^-$ for $|y| > W/2$. Show that these limits are consistent with two semi-infinite conducting strips of width W with equal and opposite charges that are separated by a vanishingly small gap (a two-dimensional strip capacitor). (b) Calculate the field components and show explicitly that the divergence and curl of $\mathbf{E}$ are both zero (or, equivalently, that the Laplacian of the potential is zero). (c) Plot or sketch the potential and the z component of the field near the surface of the plate (z small, $0 < y < W$, where $y = 0$ is the center of a strip). Discuss the behavior at $y = W/2$. (d) Expand the field on the z axis for $z \gg W$ and determine the dipole moment per unit length of a strip. Show that it is equal to $\lambda_{\text{in}} d$, where $\lambda_{\text{in}} = \sigma_{\text{in}} W$ is the magnitude of the charge per unit length on the inside surface of each strip.

Problem 3. (a) Determine the electric field on the axis of a parallel plate capacitor with circular plates of radius R using Fig. 2(b) (see, for example, Ref. 3). (b) Show that the field on the axis takes the form of Eq. (14) for $z \gg R$, and determine the dipole moment p_z in terms of R and other parameters. (c) Show that p_z can be put in the form qd, where q is the magnitude of the charge on the inside surface of each plate and d is their separation.

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