JOURNAL OF APPLIED PHYSICS VOLUME 26, NUMBER 5 MAY, 1955

Interaction of Moving Charges with Wave Circuits

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(Received September 23, 1954)

When a charge moves along a nondispersive transmission line it induces forward and backward wave components in the line. For unaccelerated motion there is no radiation unless the charge moves with the speed of an unforced wave. For small accelerations, both radiation to the line and electromagnetic inertial effects caused by changes in the energy of the fields will be observed. The electromagnetic mass at velocities greater than the wave velocity is negative. Large accelerations can be handled by numerical computation. In the case of dispersive circuits, the charge radiates at frequencies for which the phase velocity of the circuit is equal to the velocity of the charge. This radiation is identified with Cherenkov radiation. Similar radiation takes place when a charge moves through a plasma.

INTRODUCTION

THE material presented here deals with the interaction of moving charges with an environment, called a circuit. The problem is attacked by methods commonly used in connection with microwave tubes, but the behavior disclosed, and the methods themselves, are related to phenomena and methods of classical electricity and magnetism.[1–3] This is apparent in connection with electromagnetic mass, radiation from an accelerated charge, and Cherenkov radiation.

In Part I of this paper, a very simple matter is considered: the motion of a charge near a nondispersive circuit. Matters dealt with include continuous transfer of energy from charge to circuit in the special case of synchronous velocity and electric forces on accelerated charges. Slow accelerations are dealt with by means of an electromagnetic mass, while rapid accelerations require another approach.

In Part II, uniform motion of a charge near a dispersive circuit is considered. In this case the feature of interest is the continuous transfer of energy to the circuit at any frequency for which the velocity of the charge is equal to the phase velocity of a mode of the circuit. This is a form of Cherenkov radiation, and Cherenkov radiation into a dielectric-filled wave guide is compared with Cherenkov radiation into an infinite dielectric medium.

The writer originally intended to include in this paper the theory of Cherenkov-like radiation to a smoothed-out electron gas (or plasma) by an electron moving through it. The treatment of this case follows from that of Part II; one needs merely to compute an impedance looking out into the gas. Because this matter has been treated in a somewhat different way by Pines and Bohm,[4] and because it is somewhat foreign to the body of the paper, this work has not been included. The result will be given, however. The power flow P from a charge q to a medium with plasma frequency ω_p is, in mks units,

$$P=\frac{K_0(\omega_p a/v)q^2\omega_p}{8\pi\epsilon a K_1(\omega_p a/v)}.$$

Here K_0 and K_1 are modified Bessel functions, and a is a lower-limiting or meaningful radius dependent on the spacing between the charges in the gas.

PART I

A MOVING CHARGE INTERACTING WITH A NONDISPERSIVE TRANSMISSION LINE

In this section we deal with the motion of a charge near a uniform, lossless, nondispersive transmission line. In dealing with a lossless, nondispersive line we can use very elementary mathematical methods. The procedure enables us to proceed by a series of small, clear steps and to keep our thoughts straight in doing so.

1. Differential Equation Approach

We will consider a uniform transmission line having a series inductance L and a shunt capacitance C per meter, as shown in Fig. 1. It will be assumed that a line charge density

$$q(z-vt) \text{ coulombs/meter}$$

moves parallel to and very close to the line all at the same velocity v, so that all the lines of force from portions of the charge terminate on the line at the same z-position at which they originate. Then the charge $q(z-vt)$ can be regarded as a charge induced on the line.

It will be assumed that all wave and particle velocities involved are small compared with the velocity of light.

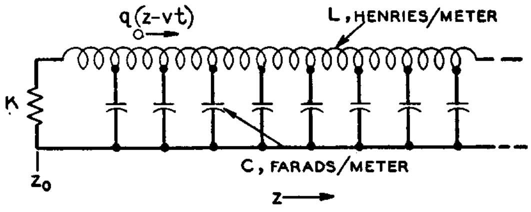

FIG. 1. A charge $q(z-vt)$ moves close to a distributed circuit which can carry an electromagnetic wave.

[1] G. A. Schott, *Electromagnetic Radiation* (The Cambridge University Press, New York, 1912).
[2] W. Heitler, *Quantum Theory of Radiation* (Clarendon Press, Oxford, England, 1944), second edition, Sec. I.
[3] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), pp. 261–265.
[4] David Pines and David Bohm, Phys. Rev. **85**, 338–353 (1952).

Thus, if V is the voltage across the shunt capacitance, to a good approximation the field E in the z-direction at the inductance and at the charge is

$$E=-\partial V/\partial z. \quad (1.1)$$

The transmission line equations are then

$$\frac{\partial V}{\partial z}=-L\frac{\partial I}{\partial t}, \quad (1.2)$$

$$\frac{\partial I}{\partial z}=-C\frac{\partial V}{\partial t}+\frac{\partial}{\partial t}q(z-vt). \quad (1.3)$$

By differentiating and combining one obtains

$$\frac{\partial^2 V}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}=-L\frac{\partial^2}{\partial t^2}q(z-vt), \quad (1.4)$$

$$c=\frac{1}{(LC)^{\frac{1}{2}}}. \quad (1.5)$$

Here, c is the velocity of an unforced wave on the line.

If we assume the velocity v to be constant, we easily see that Eq. (1.4) has a solution made up of two parts. One is a special solution

$$V=-\frac{Lv^2}{1-(v/c)^2}q(z-vt). \quad (1.6)$$

It is convenient to rewrite this in terms of the characteristic impedance of the line, K

$$K=(L/C)^{\frac{1}{2}} \quad (1.7)$$

so that

$$V=-\frac{(v/c)}{1-(v/c)^2}Kvq(z-vt). \quad (1.8)$$

Here, $vq(z-vt)$ is the convection current of the electron stream. Thus, the voltage is given by the product of a current and an impedance times a dimensionless factor, which is as it should be.

To give a complete description of all possible excitations on the line we may add two unforced waves traveling to the left and right, solutions which make the left-hand side of Eq. (1.4) equal to zero. These are

$$V=f(z+ct), \quad (1.9)$$

$$V=g(z-ct), \quad (1.10)$$

where $f(z+ct)$ and $g(z-ct)$ may be arbitrary functions of the variable.

2. Summing Up Currents Induced in the Line

It seems of some interest and value to approach the problem in a slightly different manner. In this case, in deducing the fields produced by the moving charge, we will consider the charge as causing a current $J(z_1,t_1)$ amperes/meter to flow into the line, as shown in Fig. 2. Half of this current will flow to the right with a velocity c and half will flow to the left with the same velocity. We will then evaluate the current at a position z at a time t.

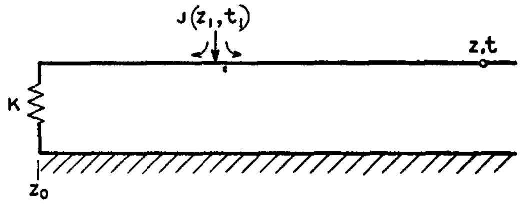

FIG. 2. A current density $J(z_1,t_1)$ flows into a transmission line and excites waves traveling to the left and to the right.

In order to make the problem reasonable physically, we will assume that the line extends indefinitely in the $+z$-direction, and is terminated in its characteristic impedance K at $z=z_0$.

We will call the part of the current at z at the time t due to waves traveling in the $+z$-direction I_+. This current may be expressed as

$$I_+=\tfrac{1}{2}\int_{z_0}^{z} J[z_1,t_1(z_1)]dz_1. \quad (2.1)$$

That is, it is a summing up of current which entered the line earlier and traveled to the right with the velocity c to reach point z at a time t. Thus

$$t_1(z_1)=t-\frac{z-z_1}{c}. \quad (2.2)$$

Now, if $q(z-vt)$ is the induced charge per meter, the induced current per meter, $J(z,t)$ is

$$J(z,t)=\frac{\partial}{\partial t}q(z-vt) = -vq'(z-vt), \quad (2.3)$$

where $q'(x)$ is the derivative of the function with respect to its argument.

By using Eqs. (2.3) and (2.2) in connection with Eq. (2.3) we obtain

$$I_+=-\frac{v}{2}\int_{z_0}^{z} q'[(1-v/c)z_1-v(t-z/c)]dz_1. \quad (2.4)$$

This of course integrates directly to give

$$I_+=-\frac{v}{2(1-v/c)}\Big|_{z_0}^{z} q[(1-v/c)z_1-v(t-z/c)],$$

$$I_+=-\frac{v}{2(1-v/c)}\{q(z-vt) -q[(v/c)(z-ct)+z_0(1-v/c)]\}. \quad (2.5)$$

We can obtain the part of the current at z_1 at a time t_1 due to current flowing to the left by a similar procedure

$$I_-=\tfrac{1}{2}\int_z^\infty J[z_1,t_1(z_1)]dz, \quad (2.6)$$

$$I_-=\tfrac{1}{2}\int_z^\infty q'[z_1(1+v/c)-c(t+z/c)]dz_1, \quad (2.7)$$

$$I_-=-\frac{vq(z-vt)}{2(1+v/c)}. \quad (2.8a)$$

Let us now examine Eqs. (2.5) and (2.8a). We see that I_+ consists of two parts. The first term in the brackets represents a current traveling along with the velocity v of the charge. The second term represents a current traveling along with the velocity of propagation c of the transmission system.

Suppose that we assume $v\neq c$, and consider the field in the vicinity of the charge far to the right of $z=z_0$, a long time after the charge has passed that point. If $q(z-vt)$ is a narrow function of z, that is, a pulse, the part of I_+ which travels with the velocity c will then be far away from the charge, and in the vicinity of the charge we will have

$$I_+=-\frac{v}{2(1-v/c)}q(z-vt) \quad (2.9)$$

and

$$I_-=-\frac{v}{2(1+v/c)}q(z-vt). \quad (2.8b)$$

The total current I will be the sum of I_+ and I_-, or

$$I=I_++I_-,$$

$$I=-\frac{1}{1-(v/c)^2}vq(z-vt). \quad (2.10)$$

We will note that I_+ and I_- can be considered as current traveling to the left and right as unforced waves with the velocity c. This is clear from Eqs. (2.1) and (2.6) together with Eq. (2.2). The charge in its motion continually builds up the current toward one edge of the charge and cancels it out toward the other edge, so that the *pattern* of current moves with the velocity v. Nonetheless the actual current flow at any instant can be considered as an excitation of the unforced waves of the transmission line, and if the charge suddenly vanished these waves would travel off to the left and to the right with their original spatial distribution but with a velocity c.

Thus, we can obtain the voltages V_+ and V_- corresponding to the currents (2.9) and (2.8b) by means of the usual transmission-line relations

$$V_+=KI_+, \quad (2.11)$$

$$V_-=-KI_-. \quad (2.12)$$

Here, Eq. (2.12) takes the form which it does because we have counted I_- as positive when it flows in the $+z$-direction.

Accordingly, we have

$$V_+=-\frac{1}{2(1-v/c)}Kvq(z-vt), \quad (2.13)$$

$$V_-=\frac{1}{2(1+v/c)}Kvq(z-vt), \quad (2.14)$$

$$V=V_++V_-,$$

$$V=-\frac{(v/c)}{1-(v/c)^2}Kvq(z-vt). \quad (2.15)$$

We see that this agrees with Eq. (1.8), the special solution of the differential equation.

The advance has been in breaking V and I up into + and − components, each simpler in form than their sum, and each obeying the usual transmission-line relations (2.11) and (2.12).

Before passing on, it is perhaps worth examining the expressions

$$I=-\frac{1}{1-(v/c)^2}vq(z-vt), \quad (2.10)$$

$$V=-\frac{(v/c)}{1-(v/c)^2}Kvq(z-vt). \quad (2.15)$$

As v approaches 0 the current I approaches the convection current of the moving charge. The voltage V, however, approaches zero.

For $v<c$ the voltage is proportional to the charge by a negative factor; that is, a positive charge produces a negative voltage (as in the case of a negative capacitance). In fact, at a given velocity the voltage has the same spatial distribution as the charge, and the relation is just as if the circuit consisted of a capacitance

$$-\frac{1-(v/c)^2}{(v/c)Kv}$$

farads per meter.

As v approaches c, that is, as the charge approaches synchronism with the unforced wave, I and V approach infinity. This is because I_+ and V_+ approach infinity, for I_- and V_- remain finite.

3. Charge Injected with Velocity of Unforced Water

The fact that the I and V of the special solution, as given by Eqs. (2.10) and (2.15), approach infinity as v approaches c does not mean that a charge shot along the line with the velocity c will produce an infinite field. In obtaining the expression (2.9) for I_+ it was assumed that a wave traveling with the velocity v became separated from a wave traveling with the velocity c and this will not occur if $v=c$.

Let us investigate what does happen when $v=c$. In doing this, we will expand the second term in Eq. (2.5) in powers of $(1-v/c)$

$$\begin{aligned} &q[(v/c)(z-ct)+z_0(1-v/c)] \\ &\quad=q[(z-vt)-(z-z_0)(1-v/c)] \\ &\quad=q(z-vt)-(z-z_0)(1-v/c)q'(z-vt)+\cdots. \end{aligned} \tag{3.1}$$

Thus, for $v=c$, I_+ becomes

$$I_+=-\frac{v(z-z_0)}{2}q'(z-vt). \tag{3.2}$$

In other words, I_+ increases in amplitude as the distance from the beginning of the line, which is also the point of injection of the charge. Energy is continually transferred from the charge to the circuit, and the charge must experience a retarding force. We note that for a nondispersive circuit, an unaccelerated charge experiences a retarding force only when its velocity is equal to that for a wave on the circuit.

We note from Eq. (2.8) that there is no such peculiarity of behavior for the backward current, because the denominator contains the factor $(1+v/c)$ rather than $(1-v/c)$.

4. Changes in Velocity

Let us now consider cases in which $v\neq c$ and in which Eqs. (2.9), (2.8), (2.13), and (2.14) apply. Let us concern ourselves first with I_+

$$I_+=-\frac{vq(z-vt)}{2(1-v/c)}. \tag{2.9}$$

Now, suppose that at the time t_a when the charge is at z_a we suddenly change the velocity of the charge to v_a;

$$v_a=v+\delta v. \tag{4.1}$$

At the moment of change the current and field in the line cannot suddenly change. This will be true if the current of the new forward (+) waves [and that of the new backward (−) waves as well] does not change. The new I_+ must be made up of a component traveling with a velocity v_a [as in Eq. (2.9)] and one with a velocity c. The combination which is equal to Eq. (2.9) at $t=t_a$ and for which the part with velocity v_a obeys relation (2.9) is

$$\begin{aligned} I_+=&-\frac{v_aq[z-v_at+(v_a-v)t_a]}{2(1-v_a/c)} \\ &+\frac{v_aq[z-ct+(c-v)t_a]}{2(1-v_a/c)}-\frac{vq[z-ct+(c-v)t_a]}{2(1-v/c)}. \end{aligned} \tag{4.2}$$

We see that at $t=t_a$ this gives the same value as the previous expression for I_+.

We see that Eq. (4.2) consists of two parts; a part traveling with the speed of the charge, and a part traveling with the velocity of propagation of the line, c. If we assume that δv in Eq. (4.2) is very small, we can rewrite these to give

$$\begin{aligned} I_+=&-\left[v+\frac{\delta v}{(1-v/c)}\right]\frac{q[z-(v+\delta v)t+\delta vt_a]}{2(1-v/c)} \\ &+\frac{\delta v}{2(1-v/c)^2}q[z-ct+(c-v)t_a]. \end{aligned} \tag{4.3}$$

Thus, the amplitude of the current accompanying the charge is changed, and a small part of the current proportional to the velocity change δv is shaken off or radiated and travels as an unforced wave with the velocity c. As time passes, the pulses represented by the two components will separate, forming two distinct and essentially nonoverlapping pulses.

It is now of interest to examine the energy carried by these pulses. In terms of the inductance L per unit length and the capacitance C per unit length, the total stored magnetic and electric energy per unit length W is

$$W=\tfrac{1}{2}(I^2L+V^2C). \tag{4.4}$$

This can be expressed in terms of I_+ and I_- as

$$W=W_++W_-=\tfrac{1}{2}[(I_++I_-)^2L+(I_+-I_-)^2K^2C]. \tag{4.5}$$

Here, W_+ is the stored energy per unit length, the forward wave and W_- is the stored energy per unit length of the backward wave. By using Eqs. (1.5) and (1.7) we obtain

$$W_+=\frac{K}{c}I_+^2;\quad W_-=\frac{K}{c}I_-^2. \tag{4.6}$$

To get the total energy associated with the forward and backward waves we should integrate W_+ and W_-, as given by Eq. (4.6) with respect to z at a particular time. The only term that varies with distance is the square of the charge, and we will call the integral of the square of the charge per unit length with respect to distance Q^2/L. Then the energy corresponding to the first part of Eq. (4.3), the pulse moving with the charge, which we will call W_{+M}, is

$$W_{+M}=\frac{(v/c)^2Kc}{4(1-v/c)^2}\frac{Q^2}{L}\left[1+\frac{2(\delta v/v)}{(1-v/c)}+\frac{(\delta v/v)^2}{(1-v/c)^2}\right]. \tag{4.7}$$

While the part corresponding to the second part of Eq. (4.3), which we will call W_R will be

$$W_{+R}=\frac{(v/c)^2KcQ^2(\delta v/v)^2}{4(1-v/c)^4L}. \tag{4.8}$$

The forward wave part of the electromagnetic energy carried along with the charge is W_{+M} while W_{+R} is the forward wave part of the electromagnetic energy detached from the charge or "radiated."

Suppose we reach the final velocity v_a in two discrete steps, carried out at times far enough separated so that the "radiated" pulses do not overlap. Then the total "radiated" energy will be halved, while to the first order the change in the energy carried along will be the same as before.

We can apply this argument only to sudden changes in velocity separated in time enough so that the "radiated" pulses do not overlap. However, it indicates very strongly that for low accelerations the electromagnetic energy carried along becomes independent of acceleration while the energy radiated becomes negligible.

To this approximation, for small accelerations we will neglect the radiated energy and take the energy carried along by the forward component

$$W_{+M}=\frac{(v/c)^2KcQ^2}{4(1-v/c)^2L}. \tag{4.9}$$

If we make further use of these methods we find the energy carried along by the backward component to be

$$W_{-M}=\frac{(v/c)^2KcQ^2}{4(1+v/c)^2L}. \tag{4.10}$$

Thus, W_M the total energy carried along is

$$W_M=W_{+M}+W_{-M},$$
$$W_M=\frac{(v/c)^2[1+(v/c)^2]KcQ^2}{2[1-(v/c)^2]^2}. \tag{4.11}$$

Now, suppose that we assume a law of force

$$f=dM/dt, \tag{4.12}$$

where f is the force and M is momentum. We know that

$$dW/dt=fv=\frac{dM}{dt}v, \tag{4.13}$$

$$M=\int dW/v. \tag{4.14}$$

Or, integrating by parts

$$M=(W/v)+\int Wdv/v^2. \tag{4.15}$$

It is easiest to divide the contributions to momentum into two parts, that M_{+M} from Eq. (4.9) and M_{-M} from Eq. (4.10). This gives

$$M_{+M}=\frac{KQ^2}{4L}\frac{1}{(1-v/c)^2}, \tag{4.16}$$

$$M_{-M}=-\frac{KQ^2}{4L}\frac{1}{(1+v/c)^2}. \tag{4.17}$$

Whence

$$M=M_{+M}+M_{-M},$$
$$M=\frac{KQ}{L}\frac{(v/c)}{[1-(v/c)^2]^2}. \tag{4.18}$$

We note that this is an odd function of the velocity, as it should be.

5. The Force on the Charge

In Sec. 4 an electromagnetic momentum M was deduced from a definition of momentum, relation (4.12), and a relation between energy change and work, relation (4.13).

Physically, we are dealing with a charge moving in certain fields. Any "electromagnetic momentum" must manifest itself as an electric field which acts in the direction of motion when the speed of the charge is changed.

Let us, then, consider the fields present in the vicinity of a charge after the velocity of the charge has been changed by an amount δv. Let us deal first with the fields of the wave components traveling in the $+z$-direction, as expressed by Eq. (4.3). We have

$$V_+=KI_+, \tag{2.11}$$

$$E_+=-\frac{\partial}{\partial z}(KI_+) \tag{5.1}$$

and from Eq. (4.3)

$$E_+=K\left(v+\frac{\delta v}{(1-v/c)}\right)\frac{q'[z-v_at+(v_a-v)t_a]}{2(1-v/c)}$$
$$-K\frac{\delta v}{2(1-v/c)^2}q'[(z-vt)+(c-v)(t_a-t)]. \tag{5.2}$$

Now, the impulse H_+ given to the charge due to the change in velocity will be given by the integral

$$H_+=\int_{t=t_a}^{\infty}\int_{z=\ \infty}^{\infty}E_+q[z-v_at+(v_a-v)t_a]dzdt. \tag{5.3}$$

Here we have multiplied the field by an element of charge and integrated over all the charge, and then we have integrated this force with respect to time in order to obtain the impulse.

From Eq. (5.2) we see that there will be two terms in the integral. The first gives zero

$$\int_{t=t_a}^{\infty}\int_{z=-\infty}^{\infty}q'[z-v_at+(v_a-v)t_a]q[z-v_at+(v_a-v)t_a]dzdt$$
$$=\int_{t-t_a}^{\infty}dt\left|_{z=-\infty}^{\infty}\frac{q^2[z-v_at+(v_a-v)t_a]}{2}=0.\right. \tag{5.4}$$

The second term has a factor

$$\int_{t=t_a}^{\infty}\int_{z=-\infty}^{\infty} q'[(z-vt)+(c-v)(t_a-t)] \times q[z-v_at+(v_a-v)t_a]dzdt. \quad (5.5)$$

This can be integrated by the following substitution of variables:

$$t=u_1, \quad (5.6)$$

$$z-v_at+(v_a-v)t_a=u_2. \quad (5.7)$$

The element of area is obtained by multiplying du_1du_2 by the Jacobian

$$\begin{vmatrix} \frac{\partial u_1}{\partial t} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial t} & \frac{\partial u_2}{\partial z} \end{vmatrix} du_1du_2 = \begin{vmatrix} 1 & 0 \\ -v & 1 \end{vmatrix} du_1du_2 = du_1du_2. \quad (5.8)$$

The integral then becomes

$$\int_{u_1=t_a}^{\infty}\int_{u_2=-\infty}^{\infty} q'[u_2+(c-v)(t_a-u_1)]q(u_2)du_2du_1. \quad (5.9)$$

We integrate first with respect to u_1, and obtain

$$\int_{u_2=-\infty}^{\infty} q(u_2)du_2 \Big|_{u_1=t_a}^{\infty} -\frac{q[u_2+(c-v)(t_a-u_1)]}{(c-v)}$$

$$=\frac{1}{(c-v)}\int_{u_2=-\infty}^{\infty}[q(u_2)]^2du_2=\frac{Q^2}{(c-v)L}. \quad (5.10)$$

We note that we have used the symbol Q^2/L for the integral before.

We have now evaluated the integral occurring in Eq. (5.3) and from Eqs. (5.2), (5.3), and from this work we see that

$$H_+=-\frac{KQ^2\delta v}{2c(1-v/c)^3L}. \quad (5.11)$$

This H is the integral of the force experienced by the charge following a velocity change δv, times the time. If we express it as the result of a change in a momentum M, the relation should be of the form

$$H=\int Fdt$$

$$F=-dM/dt,$$

$$\int Fdt=H=-\delta M,$$

or

$$\frac{\delta M}{\delta v}=\frac{dM}{dv}=-\frac{KQ^2}{2c(1-v/c)^3L}. \quad (5.12)$$

If we compare this with Eq. (4.16) we see that Eqs. (5.12) and (4.16) differentiated are in agreement. In Eq. (4.14) the momentum was defined in terms of change with velocity of stored electromagnetic energy. We have seen that the force which resists (or aids) slow acceleration of the charge is actually the force exerted on the charge by the field of the wave which is radiated when the charge is accelerated. Although the energy radiated approaches zero as the acceleration between two particular velocities is made slower and slower, the impulse due to the radiated field does not approach zero but approaches a constant value instead. This is because the energy depends on the square of the radiated field while the impulse depends on the first power of the radiated field.

In other words, for a given change in velocity, for the radiated field the integral $\int_{-\infty}^{\infty} E^2dz$ approaches zero as the acceleration is made slow, but the integral $\int_{-\infty}^{\infty} |E|dz$ approaches a constant which is not zero.

Because the impulse depends on the first power of the radiated field, the calculations which have been made concerning it hold even when pulses representing the radiated field overlap. Hence, we may regard the momentum M as correct for all accelerations which are gradual enough so that the velocity does not change substantially in the time it takes the radiated pulse due to the velocity change to pass the charge.

6. Velocities Greater Than the Velocity of the Natural Wave

The queer feature about the total momentum as defined by Eq. (4.18) is that for $v>c$, $\partial M/\partial v<0$. Thus, if the velocity of the charge in the $+z$-direction is increased, the charge experiences a force in the $+z$-direction. Contrarily, if the velocity of the charge in the $+z$-direction is reduced, the particle experiences a retarding force. Thus, motion with a uniform velocity greater than c would seem to be unstable; it seems that the charge would tend either to speed up or to slow down. Physical considerations would suggest that it might slow down.

Suppose that the charge has a mechanical mass m. If we assume its acceleration to be small, the following equation should be satisfied

$$(dM/dt)+(mdv/dt)=0, \quad (6.1)$$

$$M+mv=\text{const.} \quad (6.2)$$

Thus, the charge cannot spontaneously change its velocity with a small acceleration. This need not embarrass us, for Eq. (6.1) holds for small accelerations only.

In principle, motion under large accelerations can be computed numerically. Suppose that an abrupt change is made in the velocity of the charge. The radiated fields acting during a short interval following the change can be computed using Eqs. (4.2) or (4.3), and an impulse can be obtained by integrating charge times mass over the interval. At the end of the first short interval another abrupt change in velocity, equal to the impulse divided by the mass, can be assumed, and new radiated fields can be computed. Over the next interval the radiated fields of the first two intervals act on the charge. The calculation can be continued thus, step by step. Alternatively, the equation of motion can be written in the form of an integral equation.

It is interesting to note that because of the electromagnetic mass we cannot gradually accelerate a charge past the velocity c, but this does not forbid shooting it into the system with a velocity greater than c, or even rapidly accelerating it to a velocity greater than c.

It seems likely that even when $v>c$ and $|\partial M/\partial v|$ is smaller than m, the motion will be stable. An approximate calculation was made for $v>c$ and for $|\partial M/\partial v|>m$. It showed a rapid deceleration of the moving charge. On the other hand, a calculation for $v>c$ and $|\partial M/\partial v|<m$ showed the charge settling down to a constant velocity.

If a charge is shot into a line with a velocity $v>c$, the field corresponding to the second term of Eq. (2.5) will cause an initial deceleration of the charge, and the fields radiated subsequently will decelerate the charge further.

Thus, a charge shot into a line with a speed $v>c$ tends to slow down and to radiate *provided that* $|\partial M/\partial v|$ *is greater than the mechanical mass* m.

7. Charge Separated from the Circuit

So far our picture has been much like that of a physically idealized transmission line with a charge very close to it. What happens if we separate the charge from the line? In the case of slow waves we can use an electrostatic approach in which an element of charge is regarded as exciting the line over a distance, with some appropriate weighting. Perhaps a better approach is to regard the actual transmission system as having many modes, active and passive. From this point of view, Part I can be regarded as having dealt with one mode of transmission of a realistic multimode transmission system.

PART II

UNACCELERATED MOTION WITH A DISPERSIVE CIRCUIT

We have seen that when a charge moves with a constant velocity near a lossless-nondispersive circuit it gives up energy continually only when the velocity of the charge is equal to the wave velocity of the circuit. For dispersive circuits the phase velocity varies with frequency and thus over a wide range of charge velocity a wave of a particular frequency will have the charge velocity. This causes a continual transfer of energy from the charge to the circuit. Cherenkov radiation is an example of this. We will consider other simpler examples first and then discuss Cherenkov radiation.

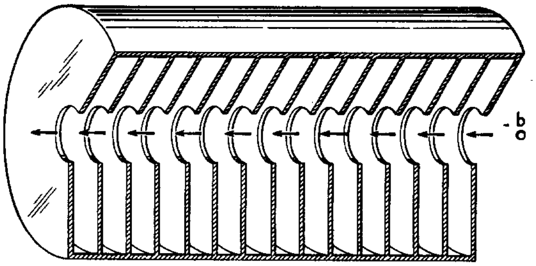

FIG. 3. A charge q flows through a series of resonators.

8. A Charge Passing Through a Series of Resonators

Imagine a charge q to travel with a velocity v through apertures in a series of thin, pillbox resonators, as shown in Fig. 3. Imagine that the holes are very small in diameter and that the resonators are so thin that the charge passes across one in a time small compared with a cycle of the natural frequency of a resonator.

Then, the passage of the charge across a given resonator corresponds to transferring a charge from one side of the resonator to the other, and if C_R is the effective capacitance of a given mode of the resonator referred to the point at which the electron passes through, an energy $W=q^2/2C_R$ is transferred to that mode of the resonator. The resonator will oscillate with this energy after the particle has passed. The resonators which the charge has passed oscillate independently, and they are phased so that the oscillations constitute a wave of phase velocity equal to the charge velocity v. The group velocity of the wave is of course zero; there is no energy flow from resonator to resonator.

If d is the distance between centers of resonators, the power P flowing from the electrons to the resonators is[5]

$$P=\frac{Wv}{d}\frac{q^2v}{2dC_R}. \tag{8.1}$$

As we make the resonators thinner and thinner, C_R tends to be proportional to $1/d$ and hence we may regard dC_R as a constant which does not depend on d.

Let us consider the retarding force F against which the electron works:

$$P=Fv=q^2v/2dC_R, \tag{8.2}$$

$$F=q^2/2dC_R. \tag{8.3}$$

Thus, the force is independent of velocity and proportional to the square of the charge.

9. Use of Fourier Transforms

It was possible to treat the case in Sec. 1 by very elementary means. In more complicated problems it is convenient to represent the charge by means of its Fourier transform. Let the charge per unit length q have

[5] Here we disregard all but one mode of oscillation of the resonator. Other modes will give other contributions to the power.

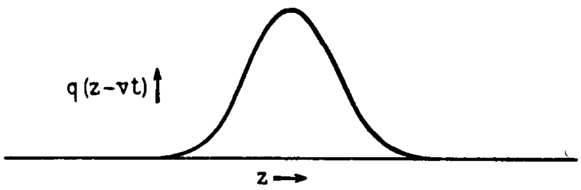

FIG. 4. The spatial distribution of the charge $q(z-vt)$.

some spatial distribution $q(z-vt)$ such that

$$\int_{-\infty}^{\infty} q(z-vt)dz=q. \tag{9.1}$$

Here t is regarded as a constant in the integration. We can also represent the charge as

$$q(z-vt)=\int_{-\infty}^{\infty} g(\beta)e^{-j\beta(z-vt)}d\beta. \tag{9.2}$$

The function $g(\beta)$ is given by

$$g(\beta)=\frac{1}{2\pi}\int_{-\infty}^{\infty} q(z-vt)e^{j\beta(z-vt)}dz. \tag{9.3}$$

Here we should note what sort of function of $\beta g(\beta)$ is. We will consider $q(z-vt)$ to represent a relatively narrow lump of charge, as shown in Fig. 4. From Eq. (9.3) we see that for small values of β,

$$e^{j\beta(z-vt)}$$

will be constant and nearly unity for all values of $(z-vt)$ for which $q(z-vt)$ is substantially different from zero. Thus, from Eq. (9.1) we see that for small values of β

$$g(\beta)=q/2\pi \tag{9.4}$$

and for larger values of β, $g(\beta)$ will decrease in value, perhaps as shown in Fig. 5.

From Eq. (9.2) we can obtain the current I in the z-direction either by multiplying the charge by v, or by using the relation

$$\partial I/\partial z=-\partial q/\partial t, \tag{9.5}$$

$$I=v\int_{-\infty}^{\infty} g(\beta)e^{-j\beta(z-vt)}d\beta. \tag{9.6}$$

In connection with the current expressed as in Eq. (9.6) we can obtain the field produced by the current through the use of an impedance per unit length $Z(\omega,\beta)$.

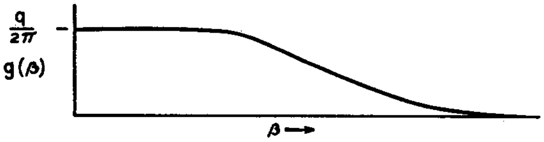

FIG. 5. The Fourier transform $g(\beta)$ of the charge $q(z-vt)$.

In terms of this impedance per unit length

$$E=-v\int_{-\infty}^{\infty} Z(\omega,\beta)g(\beta)e^{-j\beta(z-vt)}d\beta. \tag{9.7}$$

Here, we see that

$$\omega=v\beta. \tag{9.8}$$

We should note that for circuits of finite transverse extent involving no loss (resistance, conductance) Z will be purely imaginary. However, for actual circuits, which always have some loss, Z may be nearly imaginary over most of the range of ω and β but whatever real part Z has, that real part will be positive. This is important in connection with the poles of the idealized purely imaginary (reactive) impedances considered later.

The instantantaneous power flow *from the electrons to the field* (circuit) is

$$P=-\int_{-\infty}^{\infty} EIdz. \tag{9.9}$$

Now

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\gamma)e^{j\gamma z}h(\beta)e^{j\beta z}d\gamma d\beta dz$$
$$=2\pi\int_{-\infty}^{\infty} f(\beta)h^*(\beta)d\beta. \tag{9.10}$$

Whence, from Eqs. (9.6) and (9.7)

$$P=2\pi v^2\int_{-\infty}^{\infty} Z(\omega,\beta)g(\beta)g^*(\beta)d\beta. \tag{9.11}$$

10. Problem of Section 8 by Fourier Transform Method

Let us consider the problem of Sec. 8, disregarding all but one mode of the resonators. The impedance of one resonator Z_R will be

$$Z_R=\frac{-j}{C_R[\omega-(\omega_0^2/\omega)]}, \tag{10.1}$$

$$\omega_0^2=LC. \tag{10.2}$$

The impedance per unit length, Z, will be

$$Z=\frac{Z_R}{d}=\frac{-j\omega}{dC_R(\omega^2-\omega_0^2)}. \tag{10.3}$$

We remember that dC_R is independent of the resonator spacing d.

According to Eqs. (9.11) and (9.8), the power flow to the circuit will be

$$P=2\pi v^2\int_{-\infty}^{\infty}\left[\frac{-j\beta}{dC_Rv[\beta-(\omega_0/v)][\beta+(\omega_0/v)]}\right]$$
$$\times g(\beta)g^*(\beta)d\beta. \tag{10.4}$$

Now, because $q(z-vt)$ is real, $g(\beta)g^*(\beta)$ is an even function of β. The impedance (10.3), the bracketed part in the integrand, is an odd function of β with two poles, as shown in Fig. 6. Except for the contribution in passing the poles, the integral would be zero. How do we get by the poles?

We note that for an actual circuit the impedance will not pass through infinity but will assume a high, real positive value. Hence, ω_0 must [see Eq. (10.4)] have a small positive imaginary part which we will call $j\delta$. Near the pole at $\beta=\omega_0/v$, for instance, we can disregard the variation with respect to β except for that in the factor $\beta-\omega_0/v$ and write for the contribution to the integral from $\beta=\omega_0/v-\alpha$ to $\beta=\omega/v_0+\alpha$

$$\Delta P=\frac{-j\pi vg(\omega_0/v)g^*(\omega_0/v)}{dC_R}\times\int_{(\omega_0/v)-\alpha}^{(\omega_0/v)+\alpha}\frac{d\beta}{[\beta-(\omega_0/v)]-j\delta}, \quad (10.5)$$

$$\Delta P=\frac{-j\pi vg(\omega_0/v)g^*(\omega_0/v)}{dC_R}\ln\frac{\alpha-j\delta}{-\alpha-j\delta}. \quad (10.6)$$

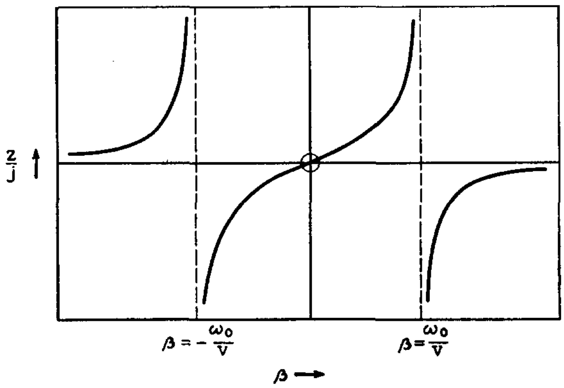

FIG. 6. The impedance Z of a circuit, plotted *vs* phase constant β, showing poles.

For $\alpha=0$ the logarithm is zero; for $\alpha\gg\delta$ it is $\pm j\pi$. We note that for intermediate values the argument has a positive imaginary part, and hence the value is $+j\pi$.

There is an equal contribution from the pole at $\beta=-\omega/v_0$, and hence the value of the total integral is

$$P=\frac{2\pi^2 vg(\omega_0/v)g^*(\omega_0/v)}{dC_R}. \quad (10.7)$$

If we assume that ω_0/v is small in the sense discussed in connection with Eq. (9.4), then Eq. (9.4) applies and

$$P=q^2v/2dC_R. \quad (10.8)$$

This agrees with Eq. (1.2), the assumption that ω_0/v is small merely saying, as we did in Sec. 9, that the charge passes through each resonator in a small part of a cycle of oscillation at the natural frequency of the resonator.

11. Nondispersive Line

Suppose that we have a transmission system consisting of a distributed series inductance L per unit length and distributed shunt capacitance C per unit length. Let us consider the interaction of the charge with the fields associated with the mode of propagation of this circuit. If we consider slow waves we can take the longitudinal field as $-\partial V/\partial z$. Under these circumstances the impedance Z per unit length is found to be

$$Z=\frac{-j\omega L\beta^2}{\beta^2-\omega^2 LC}. \quad (11.1)$$

From Eqs. (2.8) and (2.11) we obtain

$$P=\frac{-j2\pi v^3 L}{1-v^2LC}\int_{-\infty}^{\infty}\beta g(\beta)g^*(\beta)d\beta. \quad (11.2)$$

We see that the integrand is an odd function of β with no poles. Hence, the power is zero (except when $v^2LC=1$; this case must be treated differently, as, for instance, in Sec. 2.3).

12. Dispersive Circuits

When the circuit is dispersive and has for some frequency a phase velocity equal to the velocity of the charge, then there is a pole in the integrand of Eq. (9.11), as in the case of Eq. (10.4), and the power is not zero.

For instance, consider a wave guide filled with dielectric material, with a charge moving on the axis. There is an infinite number of transverse magnetic axially symmetrical modes of propagation which can be excited by the charge. Each is dispersive; the phase velocity varies from ∞ at a cutoff frequency to $1/(\mu\epsilon)^{\frac{1}{2}}$ at very high frequencies.

Thus, if the charge travels faster than $1/(\mu\epsilon)^{\frac{1}{2}}$, for each such mode of propagation there will be a pair of simple poles in the integral (9.11), and thus each mode will contribute something to the power given up by the moving charge.

Now, if we make the wave guide very large in diameter, the charge is to all intents in a large space of dielectric constant ϵ, and we should approach the usual case of Cherenkov radiation.

It seems worth while to examine this matter in detail and accordingly, the power flow from a moving charge distribution to *one axially symmetrical transverse magnetic* mode of a circular wave guide of radius a filled with a dielectric of dielectric constant ϵ will be evaluated. The relation between the field and the exciting current is[6]

$$E_z=\frac{-j(\Gamma^2+\beta_0^2)}{\omega\epsilon}e^{-\Gamma z}\Sigma_n\frac{\Pi_n(x,y)J_n}{\Gamma_n^2-\Gamma^2}, \quad (12.1)$$

[6] J. R. Pierce, *Traveling Wave Tubes* (D. Van Nostrand Company, Inc., New York, 1950), Chapter VI.

where

$$J_n=\frac{\iint J(x,y)\Pi_n(x,y)dxdy}{\iint [\Pi_n(x,y)]^2dxdy}. \quad (12.2)$$

We will take

$$\Pi_n(x,y)=J_0(\gamma_n r), \quad (12.3)$$

where

$$J_0(\gamma_n a)=0. \quad (12.4)$$

Then

$$\iint [\Pi_n(x,y)]^2dxdy$$

$$=\int_0^a 2\pi r[J_0(\gamma_n r)]^2dr=\pi a^2[J_1(\gamma_n a)]^2. \quad (12.5)$$

Here, use has been made of Eq. (12.4).

We will assume a current I to flow at $r=0$, so that

$$\iint J(x,y)\Pi_n(x,y)dxdy=1. \quad (12.6)$$

Whence

$$J_n=\frac{I}{\pi a^2[J_1(\gamma_n a)]^2}. \quad (12.7)$$

Thus, from the relation

$$E=-IZ$$

we deduce that the impedance Z_n associated with the nth mode is

$$Z_n=\frac{j(\Gamma^2+\beta_0^2)}{\pi a^2\omega\epsilon[J_1(\gamma_n a)]^2(\Gamma_n^2-\Gamma^2)}. \quad (12.8)$$

Here

$$\omega=\beta v, \quad (12.9)$$

$$\beta_0=\omega(\mu\epsilon)^{\frac{1}{2}}=\frac{\omega}{u}=\frac{v}{u}\beta, \quad (12.10)$$

$$\Gamma=j\beta. \quad (12.11)$$

And, for the wave guide considered

$$\Gamma_n^2=-\beta_n^2,$$
$$\beta_n^2+\gamma_n^2=\beta_0^2. \quad (12.12)$$

So that

$$Z_n=\frac{-j\beta}{\pi a^2\epsilon v[J_1(\gamma_n a)]^2 \times\left[\beta-\frac{\gamma_n}{[(v/u)^2-1]^{\frac{1}{2}}}\right]\left[\beta+\frac{\gamma_n}{[(v/u)^2-1]^{\frac{1}{2}}}\right]}. \quad (12.13)$$

Thus, for this mode the power P_n will be

$$P_n=\frac{2v}{a^2\epsilon[J_1(\gamma_n a)]^2}\times\int_{-\infty}^{\infty}\frac{-j\beta g(\beta)g^*(\beta)d\beta}{\left[\beta-\frac{\gamma_n}{[(v/u)^2-1]^{\frac{1}{2}}}\right]\left[\beta+\frac{\gamma_n}{[(v/u)^2-1]^{\frac{1}{2}}}\right]}. \quad (12.14)$$

This is of exactly the same form as Eq. (10.4), and by the same process we obtain

$$P_n=\frac{2\pi vg\left(\frac{\gamma_n}{[(v/u)^2-1]^{\frac{1}{2}}}\right)g^*\left(\frac{\gamma_n}{[(v/u)^2-1]^{\frac{1}{2}}}\right)}{a^2\epsilon[J_1(\gamma_n a)]^2}. \quad (12.15)$$

If we assume that

$$\frac{\gamma_n}{[(v/u)^2-1]^{\frac{1}{2}}}$$

is sufficiently small so that we can use Eq. (9.4), we obtain

$$P_n=\frac{q^2v}{2\pi a^2\epsilon[J_1(\gamma_n a)]^2}. \quad (12.16)$$

To get the total power for all modes we would have to sum with respect to n.

Let us rather examine the power as a function of frequency. At a given point there is no radiation present in the wave guide prior to the passage of the charge. After the passage of the charge, each mode is excited at a particular frequency such that the phase velocity equals the velocity of the charge. Thus, the field has a line spectrum.

Let ω_n be the cutoff radian frequency of the nth mode. Then

$$\gamma_n=\omega_n(\mu\epsilon)^{\frac{1}{2}}=\frac{\omega_n}{u}. \quad (12.17)$$

The radian frequency ω_n' at which the phase velocity of the wave is equal to the velocity of the charge is such that

$$v(\mu\epsilon)^{\frac{1}{2}}=\frac{v}{u}=\left[1-\left(\frac{\omega_n}{\omega_n'}\right)^2\right]^{-\frac{1}{2}},$$
$$\frac{\omega_n'}{\omega_n}=\left[1-\left(\frac{u}{v}\right)^2\right]^{-\frac{1}{2}}. \quad (12.18)$$

Let us define

$$\Delta\omega=\omega_{n+1}'-\omega_n'. \quad (12.19)$$

Then

$$\Delta\omega=\frac{\omega_{n+1}-\omega_n}{[1-(u/v)^2]^{\frac{1}{2}}}. \quad (12.20)$$

For large values of n

$$\gamma_{n+1}a-\gamma_n a=\frac{\omega_{n+1}a}{u}-\frac{\omega_n a}{u}=\pi, \tag{12.21}$$

$$\omega_{n+1}-\omega_n=\pi u/a,$$

$$\Delta\omega=\frac{\pi u}{a[1-(u/v)^2]^{\frac{1}{2}}},$$

$$\frac{P_n}{\Delta\omega}=\frac{q^2[(v/u)^2-1]^{\frac{1}{2}}}{2\pi^2 a\epsilon[J_1(\omega_n a u)]^2}. \tag{12.22}$$

Now, for large values of $\omega_n a/u$ at the extrema where

$$J_0\left(\frac{\omega_n a}{u}\right)=0,$$

$$\left[J_1\left(\frac{\omega_n a}{u}\right)\right]^2=\left(\frac{2u}{\pi\omega_n a}\right)^2, \tag{12.23}$$

hence

$$\frac{P_n}{\Delta\omega}=\frac{q^2\omega_n[(v/u)^2-1]^{\frac{1}{2}}}{4\pi\epsilon u},$$

$$\frac{P_n}{\Delta\omega}=\frac{q^2\omega_n'[(v/u)^2-1]}{4\pi\epsilon u}. \tag{12.24}$$

This is in agreement with the power radiated per unit radian frequency in Cherenkov radiation.[2]

13. Comparison with Cherenkov Radiation

We have seen that the expression for the radiation per unit frequency of a charge moving along the axis of a large dielectric-filled guide is the same as the expression for the power of Cherenkov radiation. We feel that this is as it should be, and yet there are several matters which may appear puzzling.

First, we are inclined to regard an infinite, homogeneous, isotropic space as nondispersive, and yet we have seen that radiation into a wave guide depends on the dispersive nature of the modes of propagation.

As a matter of fact, an infinite homogeneous space will support all dispersive modes. It differs from a wave guide chiefly in having a continuous rather than a discrete spectrum of modes. If we wish, the characteristic pattern of Cherenkov radiation can be regarded as made up of a continuous spectrum of modes such as those used in Sec. 12.

The over-all pattern of Cherenkov radiation is simple. It is shown in Fig. 7. The pattern is characteristic of a shock wave. The wave front of the radiation is a vee extending back from the moving charge. The wave front advances with a speed u

$$u=\frac{1}{(\mu\epsilon)^{\frac{1}{2}}}$$

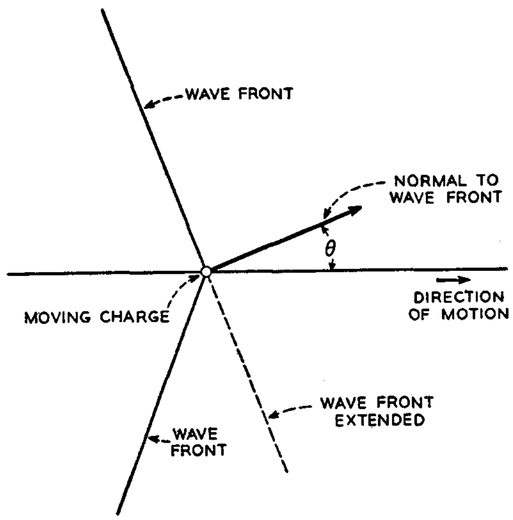

FIG. 7. The wave front of Cherenkov radiation.

normal to itself. The intersection of the wave front travels along the path of the particle with the particle speed v, which is greater than u. If θ is the angle between the normal to the wave front and the path of the particle, we see that

$$u=v\cos\theta. \tag{13.1}$$

The angle θ is the angle of the cone of radiation in Cherenkov radiation.

We see that the velocity of a *wave front* along a line may be greater than the velocity of a plane electromagnetic wave. In fact, the Cherenkov radiation clearly travels so as to make the velocity of the wave front along the path equal to the velocity of the moving charge.

We may note that the angle θ in Eq. (13.1) is characteristic of the resolution of wave-guide modes into sums of plane electromagnetic waves, in which case u is, as here, the velocity of a plane electromagnetic wave in the medium and v is the phase velocity of the wave in the guide.

Another thing which may at first seem puzzling is that Cherenkov radiation has a continuous spectrum, while the radiation in the wave guide has a line spectrum. The moving charge cannot be aware of the walls of the wave guide, because the charge is moving faster than the group velocity of any electromagnetic signal in the medium. We are unused to devices (in this case a wave guide) which seem to put all the power of a source with a continuous spectrum into a line spectrum.

This can be made clear by means of a physical picture. In order to make matters as simple as possible, let us consider a two-dimensional case, corresponding to a line charge moving between infinite parallel conducting planes, as shown in Fig. 8.

The reflection of the wave front from the conducting plane gives the lozenge-shaped wave front shown in Fig. 8. We may if we wish regard this pattern of wave

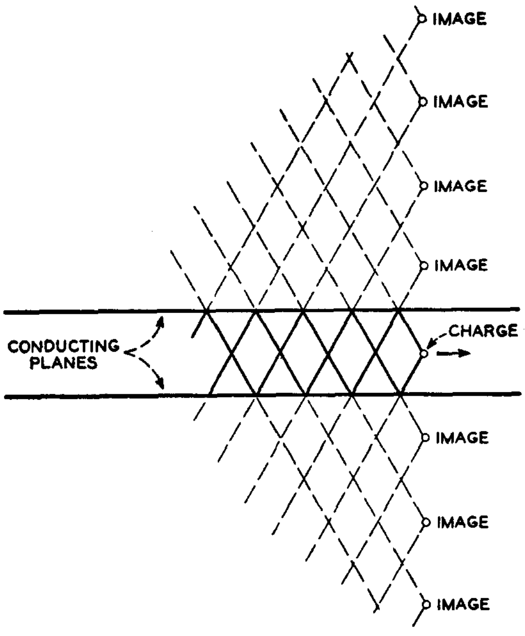

FIG. 8. Cherenkov radiation of a line charge between parallel planes—the wave fronts of the charge and of its images (dashed). The radiation has a line spectrum.

front as made up of the wave fronts of image charges, which are shown dotted outside of the planes in Fig. 8.

It is clear that the successive passages of the reflected wave fronts will result in a line spectrum rather than a continuous spectrum. However, the charge does not know that the reflecting walls are there. How, then does the presence of the walls convert the continuous spectrum of Cherenkov radiation into a line spectrum? Perhaps we should first note that in any case the charge itself sees a steady retarding field. We should also note that if we add the walls while the charge is moving, only the radiation which has not yet reached the position of the walls is reflected. The continuous spectrum of Cherenkov radiation can in principle be deduced only by an extended examination of the radiation pattern, while the line spectrum of the guide can be deduced only by an extended examination of the different radiation pattern in the guide. Now, one pattern which has been examined for a long period cannot be converted into the other.

There are other cases in which a signal which we think of as having a continuous spectrum can be converted into a signal which we think of as having a line spectrum. Let a very short pulse come along the transmission line of Fig. 9 from the left, and let the switch be closed after the pulse has passed but before the first reflection returns to the switch. All the electromagnetic energy of the pulse is now batting back and forth between the switch and the end of the line. The single pulse on a line, which we ordinarily think of as having a continuous spectrum, has been converted into a recurrent phenomenon with a line spectrum. This does not mean, however, that a linear, time-invariant, network can put all the energy of a signal source having a continuous spectrum into a number of discrete frequencies. All a linear network can do is reject or absorb some frequencies while passing others.

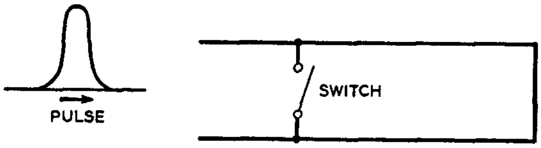

FIG. 9. A pulse trapped in a section of line assures a line spectrum.

A third question which might arise concerns the very fact of a charged particle traveling through a medium with a speed greater than that of an electromagnetic wave. In the case of Cherenkov radiation, this is possible because the particle is shot into the medium from outside. This does not show that it is possible to accelerate a particle within a homogeneous, isotropic dielectric to a speed greater than the speed of a plane electromagnetic wave in the medium, and indeed, if the medium is linear for all field levels this would seem to take an infinite energy under a variety of assumptions, provided that the acceleration is small.

ACKNOWLEDGMENT

The writer wishes to acknowledge the valuable comments and criticisms of Dr. P. L. Hartman.