Poincaré's Rendiconti Paper on Relativity. Part III

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This is the concluding part of a modernized rendition of Poincaré's Rendiconti paper on relativity, of which the first two parts appeared in the November 1971 and June 1972 issues of this Journal. It covers the last section of that paper, in which Poincaré develops in masterful, even if incomplete, fashion, a generalization of Newtonian gravitational theory, involving retarded action-at-a-distance interaction that is covariant under the Lorentz group. As the first such attempt it is of obvious historical significance. In addition, just as the first two parts, so this part, too, contains material of independent interest to the historian of the genesis of special relativity.

For the purpose and scope of the present modernized rendition of Poincaré's Rendiconti paper on relativity, and for the notation that is being employed, the reader is referred to the introductory remarks to Pt. 1 of this study [Amer. J. Phys. **39,** 1287 (1971)]. The additional remarks concerning notation made in the introduction to Pt. II [Amer. J. Phys. **40,** 862 (1972)] are also applicable here. Because this part of Poincaré's paper is of particular interest in connection with its methodological aspects, including an anticipation of the four-vector calculus, certain relevant portions of the original text are reproduced here more closely than would have been otherwise indicated.

As for the notes or comments—which, as in the earlier parts, are either given in footnotes or enclosed in braces in the text—these are intended in general, as previously, to serve only as explanation of the original text. In addition, there are included here a few footnotes that point out nontrivial misprints in existing French and English literal reproductions of Poincaré's paper and a footnote containing references to later work on the subject of Poincaré's pioneering investigation in relativistic gravitational theory.

9. HYPOTHESES CONCERNING GRAVITATION

[27] "Thus the impossibility of making evident the existence of absolute motion would be fully explained by Lorentz's theory, if all forces were of electromagnetic origin."

But there exist forces, such as gravitation, which are not of electromagnetic origin.

[28] "Lorentz was therefore obliged to complete his hypothesis by supposing that *forces of any origin, and in particular, gravitation, are affected by a translation* (or, if one prefers, by a Lorentz transformation) *in the same way as are the electromagnetic forces.*"

It follows from this assumption, as applied to gravitation, that we can no longer retain the Newtonian theory involving an attraction between two bodies that depends only on their relative position at each instant under consideration. The gravitational attraction must also depend on "the velocities of the two bodies." In addition, it is to be expected that "the force which acts on the attracted body at an instant t depends on the position and velocity of that body at the instant t; but also on the position and velocity of the *attracting* body, not at the instant t, but at an *earlier instant,* as if it took gravitation a certain time to propagate itself."

The equation for this propagation must therefore be of the form

$$\phi(t, \mathbf{x}, \mathbf{u}, \mathbf{u}_1) = 0, \quad [9.1] \quad (106)$$

where, $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_0$, $\mathbf{x}_0$ is the position vector of attracted body at time t_0, $\mathbf{x}_1$ is the position vector of attracting body at time $t_1 = t_0 + t$, and $\mathbf{u}$, $\mathbf{u}_1$ are the velocities of the attracted body at time t_0 and of the attracting body at time t_1.

Let now **F** represent the force exerted upon the attracted body at the time t_0.[1] It must be expressed in terms of

$$t,\ \mathbf{x},\ \mathbf{u},\ \mathbf{u}_1, \qquad [9.2] \quad (107)$$

and the following conditions must be satisfied:

(1) Equation (106) must be covariant under the Lorentz group.

(2) **F** must transform under the Lorentz transformations (9) in the same way as the electromagnetic force denoted in Sec. 1 by the same symbol, i.e., according to Eqs. (19).

(3) "When the two bodies are at rest one must regain the usual law of attraction." [Relation (106) becomes then, of course, irrelevant.]

These conditions obviously do not suffice. The following additional ones naturally come into consideration:

(4) "Since astronomical observations do not seem to disclose significant deviations from Newton's law, we shall choose the solution that deviates least from this law when the velocities of the two bodies are small."

(5) "We shall attempt to arrange for t to be always negative; for if, in fact, one conceives of the gravitational effect as requiring a certain time for its propagation, it is hard to understand how this effect could depend on the position which has *not yet been attained* by the attracting body."

[29] "There is one case when the indeterminacy of the problem disappears; this is the case of *relative* rest of the two bodies; i.e., when $\mathbf{u}=\mathbf{u}_1$; this then is the case which we shall examine first, on the assumption that these velocities are constant, so that the two bodies are involved in a common state of rectilinear, uniform, translational motion."

By choosing for the direction of our x_1 axis that of the common velocity of our bodies, so that $u_j=0$ $(j=2,3)$, then taking $\beta=u_1$ in Eqs. (9), the transformed reference frame S' becomes the rest frame of the bodies,[2] and consequently by condition (3), we have *to within a constant factor*,

$$\mathbf{F}'=-\mathbf{x}'/r'^3, \qquad r'^2=\mathbf{x}'^2. \quad [9.3] \quad (108)$$

In applying Eqs. (19) (where it is now being tacitly assumed that pertinent equations in Pt. I are taken with $l=1$), we note that Eq. (11) now gives

$$\rho'/\rho=\gamma(1-\beta u_1)=\gamma(1-\beta^2)=\gamma^{-1},$$

and that $\mathbf{F}\cdot\mathbf{u}=\beta F_1$, so that the transformation equations for **F** reduce to

$$F_1'=F_1, \qquad F_j'=\gamma F_j \qquad (j=2,3).$$

Hence, using Eqs. (108) and (9), we find

$$F_1=-\gamma(x_1-u_1t)/r'^3, \qquad F_j=-x_j/\gamma r'^3 \qquad (j=2,3) \quad [9.4] \quad (109)$$

or

$$\mathbf{F}=\nabla V, \qquad V=1/\gamma r', \quad [9.4\text{bis}] \quad (109')$$

where

$$r'^2=\gamma^2(x_1-u_1t)^2+x_2^2+x_3^2. \qquad (109'')$$

This result would appear to depend upon our choice of an hypothesis concerning t, "but it is easy to see that x_i-u_it, which alone appear in our formulas, do not depend on t."[3]

We also see that the force acting on the attracted body is normal to an ellipsoid whose center is at the position of the attracting body.

"In order to make further progress, it is necessary to look for the *invariants of the Lorentz group*."

"We know that the transformations of this group (taking $l=1$) are the linear transformations which do not change the quadratic form $\mathbf{x}^2-t^2$."[4] But this form can be written as $x_kx_k+(it)^2=x_\alpha x_\alpha$ $(\alpha=1,2,3,4)$, introducing the notation $x_4\equiv it$.[5] It can therefore be seen, since the quadruplets (x_α), (dx_α), $(dx_{1\alpha})$[6] transform in the same way under Lorentz transformations, that these quadruplets may be considered as "the coordinates of three points P, P', P'' in four-dimensional space," and that "the Lorentz transformation is but a rotation of this space about a fixed origin. It follows that the only independent invariants are "the six distances of the three points P, P', P'' from each other and from the origin"; in other words,[7] the six scalar products $x_\alpha x_\alpha$, $x_\alpha dx_\alpha$, etc., that can be formed from the four-vectors corresponding to P, P', and P''.

But what we actually need are not these invariants themselves but the invariant combinations which are homogeneous of degree zero with respect to the dx_α and the $dx_{1\alpha}$, since what we must find are suitable "invariant functions of the variables" (107). There are only four such combinations, namely,[8]

$$x_\alpha x_\alpha, \qquad (t-\mathbf{x}\cdot\mathbf{u})(1-\mathbf{u}^2)^{-1/2},$$
$$(t-\mathbf{x}\cdot\mathbf{u}_1)(1-\mathbf{u}_1^2)^{-1/2},$$
$$(1-\mathbf{u}\cdot\mathbf{u}_1)[(1-\mathbf{u}^2)(1-\mathbf{u}_1^2)]^{-1/2}. \quad [9.5] \quad (110)$$

Turning now our attention to the transformation properties of the force components, we are guided first by Eqs. (18), which show that if we write $\mathbf{f}\cdot\mathbf{u}\equiv f_0$, then

$$f_1'=\gamma(f_1-\beta f_0), \qquad f_j'=f_j \qquad (j=2, 3)$$
$$f_0'=\gamma(f_0-\beta f_1), \qquad [9.6] \quad (111)$$

so that f_ν ($\nu=0, 1, 2, 3$) are the components of a (real) four-vector. On the other hand,

$$F_\nu=f_\nu/\rho \qquad (\nu=0, 1, 2, 3; F_0\equiv\mathbf{F}\cdot\mathbf{u}), \quad (112)$$

and by Eq. (11) ($l=1$),

$$\rho/\rho'=1/\gamma(1-\beta u_1)=dt/dt'. \qquad (112')$$

Hence $(1-\mathbf{u}^2)^{-1/2}F_\nu$ are the components of a four-vector, and by reasoning similar to that used previously we find the additional four invariants[9]

$$(\mathbf{F}^2-F_0^2)(1-\mathbf{u}^2)^{-1}, \qquad (\mathbf{F}\cdot\mathbf{x}-F_0 t)(1-\mathbf{u}^2)^{-1/2},$$
$$(\mathbf{F}\cdot\mathbf{u}-F_0)[(1-\mathbf{u}^2)(1-\mathbf{u}_1^2)]^{-1/2},$$
$$(\mathbf{F}\cdot\mathbf{u}-F_0)(1-\mathbf{u}^2)^{-1}, \quad [9.7] \quad (113)$$

of which the last vanishes identically by virtue of the definition of F_0 [in Eq. (112)].

We now have to satisfy the following conditions:

(a) The left-hand side of Eq. (106) must be a function of the four invariants (110).

(b) The invariants (113) must be functions of the invariants (110).

(c) "When the two bodies are in a state of absolute rest, $\mathbf{F}$ must have the value deduced from Newton's law, and when they are in a state of relative rest, it must have the value deduced from Eqs. (109)."

As to condition (a), "many hypotheses can obviously be made, of which we shall only examine two" [given by the vanishing of the first two invariants in Eq. (110)]:

$$\text{(A)} \quad \mathbf{x}^2\equiv r^2=t^2, \qquad \text{(B)} \quad \mathbf{x}\cdot\mathbf{u}=t.$$

At first sight it might appear that (A) has to be rejected on the basis of Laplace's proof that the propagation speed of gravitation, if not infinite, must exceed that of light—

[30] "But Laplace has examined the hypothesis of a finite propagation velocity *ceteris non mutatis*; here, on the contrary, this hypothesis is entangled with many others, and it can transpire that there exists between them a more or less perfect mutual compensation of the kind for which the applications of the Lorentz transformation have already provided us with so many examples."

At the same time, hypothesis (B) must be rejected because although it agrees with Laplace's result, it can in some instances conflict with condition (5).[10] Hypothesis (A), on the other hand, always agrees with that condition, upon our choice of the solution

$$t=-r. \qquad (114)$$

We therefore adopt hypothesis (A).

Combining now conditions (b) and (c) "for the case of absolute rest, the first two invariants (113) must reduce to $\mathbf{F}^2$ and $\mathbf{F}\cdot\mathbf{x}$, or, by Newton's law to

$$r^{-4}, \qquad -r^{-1}; \qquad (115)$$

on the other hand, by hypothesis (A) [i.e., by Eq. (114)] the second and third invariants (110) become

$$(-r-\mathbf{x}\cdot\mathbf{u})(1-\mathbf{u}^2)^{-1/2}, \qquad (-r-\mathbf{x}\cdot\mathbf{u}_1)(1-\mathbf{u}_1^2)^{-1/2}, \qquad (116)$$

i.e., for absolute rest

$$-r, \qquad -r. \qquad (117)$$

We can therefore assume, *for example*, that the first two invariants (113) reduce to[11]

$$(1-\mathbf{u}_1^2)(r+\mathbf{x}\cdot\mathbf{u}_1)^{-4},$$
$$-(1-\mathbf{u}_1^2)^{1/2}(r+\mathbf{x}\cdot\mathbf{u}_1)^{-1};$$

but other combinations are possible."

"It is necessary to make a choice between these combinations, and we require also a third equation in order to determine $\mathbf{F}$." We take now into account condition (4). First we note that if we neglect the squares of u_i and u_{1i}, and use Eq. (114), then the invariants (110) and (113) become, respectively,

$$0, \qquad -r-\mathbf{x}\cdot\mathbf{u}, \qquad -r-\mathbf{x}\cdot\mathbf{u}_1, \qquad 1, \quad (118)$$

and

$$\mathbf{F}^2, \qquad \mathbf{F}\cdot(\mathbf{x}+r\mathbf{u}), \qquad \mathbf{F}\cdot(\mathbf{u}_1-\mathbf{u}), \qquad 0. \quad (119)$$

But we must also bear in mind that in the *Newtonian* theory we have $t=0$, where t is defined in connection with Eq. (106), so that in the present approximation we can neglect higher powers of t than the first, and we may thus "proceed as if the motion were uniform." Consequently,[12]

$$\mathbf{x}=\mathbf{x}_1(0)+\mathbf{u}_1t, \qquad r(r-r_1)=\mathbf{x}\cdot\mathbf{u}_1t,$$

where $\mathbf{x}_1(0)$ is the position vector of the attracting body relative to the attracted body at the time t_0, $r_1=|\mathbf{x}_1(0)|$, and $r=|\mathbf{x}|$ [See (A)]; or by Eq. (114),

$$\mathbf{x}=\mathbf{x}_1(0)-\mathbf{u}_1r, \qquad r=r_1-\mathbf{x}\cdot\mathbf{u}_1. \quad (120)$$

Eqs. (118) and (119) thus become

$$0, \qquad -r_1+\mathbf{x}\cdot(\mathbf{u}_1-\mathbf{u}), \qquad -r_1, \qquad 1 \quad (121)$$

and [writing now $\mathbf{x}_1$ for $\mathbf{x}_1(0)$]

$$\mathbf{F}^2, \qquad \mathbf{F}\cdot[\mathbf{x}_1+(\mathbf{u}-\mathbf{u}_1)r_1], \qquad \mathbf{F}\cdot(\mathbf{u}_1-\mathbf{u}), \qquad 0, \quad (122)$$

where in the second of expressions (122) we have replaced r by r_1, since it is multiplied by $\mathbf{u}-\mathbf{u}_1$.

On the other hand, with $\mathbf{F}$ given by Newton's law, the expressions (119) take the form

$$r_1^{-4}, \quad -r_1^{-1}-\mathbf{x}_1\cdot(\mathbf{u}-\mathbf{u}_1)r_1^{-2}, \quad \mathbf{x}_1\cdot(\mathbf{u}-\mathbf{u}_1)r_1^{-3}, \quad 0.$$

"If then we denote the second and third invariants (110) by A and B, and the first three invariants (113) by M, N, P, we shall satisfy Newton's law to within terms of the second order in the velocities by putting"[13]

$$M=B^{-4}, \qquad N=AB^{-2}, \qquad P=(A-B)B^{-3}. \quad [9.8] \quad (123)$$

This solution is, however, not unique: since $(A-B)^2$ and $C-1$, where C is the fourth invariant (110), are of the second order in the velocities, "we may add to the right-hand sides of each of Eqs. (123) a term"

$$(C-1)f_1(A,B,C)+(A-B)^2f_2(A,B,C), \quad (124)$$

where f_1 and f_2 are arbitrary functions. On the other hand, the solution (123) as it stands is not acceptable, because it can lead in some cases to nonreal values of the F_i, since the quantities M, N, P are functions of the F_i as well as of $F_0=\mathbf{F}\cdot\mathbf{u}$.

"In order to avoid this inconvenience, we shall proceed in a different manner." We observe that the invariants (110) can be put [using Eq. (114)] in the form

$$0, \qquad A=-\gamma_0(r+\mathbf{x}\cdot\mathbf{u}), \qquad B=-\gamma_1(r+\mathbf{x}\cdot\mathbf{u}_1),$$
$$C=\gamma_0\gamma_1(1-\mathbf{u}\cdot\mathbf{u}_1),$$

where we have introduced the symbols

$$\gamma_0=(1-\mathbf{u}^2)^{-1/2}, \qquad \gamma_1=(1-\mathbf{u}_1^2)^{-1/2}, \quad (124')$$

"by analogy to the notation $\gamma=(1-\beta^2)^{-1/2}$ which appears in the Lorentz transformation," and that "the following systems of quantities

$$(\mathbf{x}, t=-r), \quad (\gamma_0\mathbf{F}, \gamma_0F_0), \quad (\gamma_0\mathbf{u}, \gamma_0), \quad (\gamma_1\mathbf{u}_1, \gamma_1)$$

undergo the *same* linear transformations when they are subjected to the transformations of the

Lorentz group. We are then led to put"

$$F_\nu = a\gamma_0^{-1}x_\nu + bu_\nu + c\gamma_0^{-1}\gamma_1 u_{1\nu}$$

$$(\nu = 0, 1, 2, 3;\, u_0 = u_{10} = 1), \quad [9.9] \quad (125)$$

which is obviously a four-vector provided a, b, c are four-scalars (i.e., Lorentz invariants).

"But for the compatibility of Eqs. (125) it is necessary {in order to agree with the *definition* of F_0 [see (112)]} that

$$\mathbf{F}\cdot\mathbf{u} - F_0 = 0,$$

which becomes upon replacing the F_ν by their values (125) and multiplying by γ_0^2"[14]:

$$aA + b + cC = 0. \quad [9.10] \quad (126)$$

"What we want is that if we neglect in comparison with the square of the velocity of light, the squares of the velocities u_i, as well as products of accelerations and distances, then the values of the F_ν remain in agreement with Newton's law."[15]

To this order of approximation, we have

$$\gamma_0 = \gamma_1 = 1, \qquad C = 1,$$

$$A = -r_1 + \mathbf{x}\cdot(\mathbf{u}_1 - \mathbf{u}), \qquad B = -r_1.$$

If we make then the simple choice [compatible with Eq. (126)]

$$b = 0, \qquad c = -aA/C,$$

we find, using Eqs. (120), that the three-vector part of Eq. (125) becomes

$$\mathbf{F} = a(\mathbf{x} - A\mathbf{u}_1) = a(\mathbf{x} + r\mathbf{u}_1) = a\mathbf{x}_1(0).$$

Since by Newton's law, $\mathbf{F} = -\mathbf{x}_1(0)/r_1^3$, "we must choose for the invariant a the quantity which reduces to $-r_1^{-3}$ within the adopted order of approximation, that is B^{-3}." Equations (125) assume then the form

$$F_\nu = \gamma_0^{-1}B^{-3}x_\nu - \gamma_0^{-1}\gamma_1 AB^{-3}C^{-1}u_{1\nu}. \quad [9.11] \quad (127)$$

[31] "We see at first that the corrected attraction consists of two components; one parallel to the vector joining the positions of the two bodies, the other parallel to the velocity of the attracting body."

"Let us recall that when we speak of the position or the velocity of the attracting body, we refer to its position or velocity at the instant the gravitational wave leaves it {i.e., at the *retarded* time, t_1}; whereas the position and velocity of the attracted body are referred to the instant when the gravitational wave reaches it, this wave being assumed to propagate with the velocity of light."

"I believe that it would be premature to wish to push the discussion of these formulas any further; I shall therefore confine myself to a few remarks."

1. "The solutions (127) are not unique" since we can add to the common factor B^{-3} the quantity (124); "or not take $b = 0$, but add arbitrary terms to a, b, c provided they satify condition (126) and are of second order in u_i as far as a is concerned, and of the first order as far as b and c are concerned."

2. The three-vector part of Eq. (127) can be written[16]

$$\mathbf{F} = \gamma_1 B^{-3}C^{-1}[(1 - \mathbf{u}\cdot\mathbf{u}_1)\mathbf{x} + (r + \mathbf{x}\cdot\mathbf{u})\mathbf{u}_1], \quad [9.11\text{bis}] \quad (128)$$

and the quantity in brackets can be written as

$$(\mathbf{x} + r\mathbf{u}_1) + [\mathbf{u}\times(\mathbf{u}_1\times\mathbf{x})], \quad [9.12] \quad (129)$$

so that $\mathbf{F}$ appears to consist of two components, the first having "a vague analogy to the mechanical force due to the electric field," and the second to "the mechanical force due to the magnetic field." This analogy can be improved by getting rid of the factor C^{-1} in Eq. (128), the resulting expression depending then only linearly on $\mathbf{u}$. This can be done by applying remark 1 "to replace B^{-3} by CB^{-3} in Eqs. (127)."[17]

"Setting now

$$\gamma_1(\mathbf{x} + r\mathbf{u}_1) = \boldsymbol{\lambda}, \qquad \gamma_1(\mathbf{u}_1\times\mathbf{x}) = \boldsymbol{\lambda}', \quad [9.13]$$

it follows, since C has disappeared from the denominator of (128), that

$$\mathbf{F} = B^{-3}\boldsymbol{\lambda} + B^{-3}(\mathbf{u}\times\boldsymbol{\lambda}'), \quad [9.14] \quad (130)$$

and one also has (as is easily checked)

$$B^2 = \lambda^2 - \lambda'^2. \quad [9.15]$$

Then λ or $B^{-3}\lambda$ is a kind of electric field, while λ' or, rather $B^{-3}\lambda'$, is a kind of magnetic field."

3. "The postulate of relativity would force us to adopt either the solution (127) or the solution (130), or any one of the solutions that can be deduced from them by using remark 1. But the primary question is whether they are consistent with astronomical observations. The deviation from Newton's law is of the order of u^2, that is, 10 000 times smaller than if it were of the order of u, that is, if the velocity of propagation were equal to that of light, *ceteris non mutatis*.[18] It is therefore permissible to hope that it will not be too great; however, only a more penetrating discussion could tell us that."[19]

[1] The original text contains here the phrase "at the instant t," a misprint which has not been corrected in either the French or English (partly edited) reproductions of the original paper, namely, those contained in H. Poincaré, *La Mécanique Nouvelle* [*conférence, mémoire et notes sur la théorie de relativité*. Introduction de m. Eduward Guillaume] (Gauthier–Villars, Paris, 1924), and in C. W. Kilmister, *Special Theory of Relativity* (Pergamon, New York, 1970). These will be referred to by the respective symbols (G) and (K).

[2] The original statement here reads: "the two bodies will be at rest after the transformation," i.e., the bodies will be in a state of *absolute* rest, this being clearly implicit in the wording of condition (3).

[3] Since assumption (3) is being used, we could simply set $t=0$ in Eqs. (109), or (109′) and (109″).

[4] See the discussion following the third paragraph after Eq. (40).

[5] No such symbol is introduced in the original text.

[6] The original symbols $\delta_1 x$, $\delta_1 y$, $\delta_1 z$ and $\delta_1 t$ are replaced here by $d\mathbf{x}_1$ and dt_1. The convenient symbol t_1 to represent t_0+t is not introduced in the original text, but from its context it is clear that $\delta_1 x \equiv \delta x_1$, etc., in Poincaré's notation for differentials. The reproduction of this notation in the present connection in (G) and (K) (see Ref. 1) is inconsistent with the editing of Poincaré's notation for derivatives found elsewhere in these references.

[7] We introduce at this point the convenient four-vector formalism. Had Poincaré adopted the ordinary vector calculus that was already in use by theoretical physicists—for example, Lorentz and Abraham—for some time, he would have in all likelihood introduced explicitly in the present connection the convenient four-dimensional vector calculus.

[8] The second expression in (110) can be written $\gamma_0(t-\mathbf{x}\cdot\mathbf{u})(\gamma_0^2-\gamma_0^2\mathbf{u}^2)^{-1/2}$, where γ_0 is defined later in (124′). Since (γ_0, $\gamma_0\mathbf{u}$) is a (real) four-vector (in fact, the velocity four-vector, in modern terminology), the invariance of this expression—and in a similar way, of the last two expressions—is apparent.

[9] Compare Ref. 8. The proof in the original text is based directly on Eq. (112′), which implies that F_ν and u_ν ($u_0=1$) transform in the same way under Lorentz transformations (as four-vectors except for the missing factor γ_0).

[10] There is a misprint here in the original text, which reads "but in certain cases, t could be negative" (instead of "positive"), which has been reproduced in (G) and (K).

[11] By choosing the second expression in (116) to compare with (117), and then taking account of (115). The misprints "invariants (4)" [for "invariant (7)"] is reproduced in (K) and changed to the wrong "invariants (5)" in (G).

[12] The second equation follows from the first, when higher powers than the first in the material velocities are neglected. We introduce here temporarily the symbol $x_1(0)$ to replace the symbol x_1 in the original text, because the latter symbol has been employed here previously in connection with Eq. (106).

[13] There is a misprint in (K) in the second equation of [9.8].

[14] With $a\cdot b \equiv a_0b_0 - a_ib_i$ ($\equiv a_\nu b^\nu$), we have $\gamma_0^2 F\cdot u = a\gamma_0 x\cdot u + b\gamma_0^2 u\cdot u + c\gamma_0\gamma_1 u_1\cdot u$, and by (110), $\gamma_0 x\cdot u = A$, $\gamma_0\gamma_1 u_1\cdot u = C$, while $\gamma_0^2 u\cdot u$ as the "square" of the "four-velocity" $\gamma_0 u$ is 1 (remembering that we are using units with $c=1$.)

[15] This is a more precise statement of condition (4) introduced in the beginning of this section.

[16] Recalling that $A=\gamma_0(t-\mathbf{x}\cdot\mathbf{u})$, $C=\gamma_0\gamma_1(1-\mathbf{u}\cdot\mathbf{u}_1)$, and using Eq. (114).

[17] By taking $f_1=B^{-3}$ and $f_2=0$ in expression (124).

[18] This sentence is rather obscure. Its meaning becomes clear when we read the corresponding part of the concluding paragraph in Poincaré's note on the subject of his Rendiconti article [Compt. Rend. **140**, 1504 (1905)]: "The deviation from the ordinary law of gravitation is, as I have said, of the order of ξ^2 {i.e., u^2}; if one only assumes, as was done by Laplace, that the velocity of propagation is that of light, this deviation would be of the order of ξ, that is, 10 000 times larger." (Cf. [30]).

[19] Such a discussion was presented a few years later by W. de Sitter [Monthly Notices Roy. Astron. Soc. **71**, 388 (1911)] and quite recently, as part of a general discussion of special relativistic theories of gravitation, by G. J. Whitrow and G. E. Morduch [Nature **118**, 790 (1960); also, "Relativistic Theories of Gravitation" in *Vistas in Astronomy*, edited by A. Beer, Editor, (Pergamon, New York, 1965), Vol. 6, pp. 1–68]. A brief summary of special relativistic theories of gravitation is contained in H. M. Schwartz, *Introduction to Special Relativity* (McGraw–Hill, New York, 1968), Appendix 7B (errata sheets can be obtained from the author).